Variety of orthomodular posets

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Support of the research of the first author by the Project CZ.1.07/2.3.00/20.0051 “Algebraic Methods in Quantum Logics” and of the second author by the Project CZ.1.07/2.3.00/20.0060 “International Center for Information and Uncertainty” is gratefully acknowledged.

Outline

By a **logical structure of a physical system** (see [1, 6, 8] or [13]) is meant a couple $(L; F)$, where L is a nonvoid set and F is a set of functions from L into the interval $[0,1]$ of real numbers satisfying the following axioms:

(I) If $p,q \in L$ and $f(p) = f(q)$ for every $f \in F$ then $p = q$.

(II) There exists an element $u \in L$ such that $f(u) = 1$ for each $f \in F$.

(III) For each $p \in L$, there exists an element $p' \in L$ such that $f(p) + f(p') = 1$ for every $f \in F$.

Let $\leq$ be the relation defined on L by

$$p \leq q \quad \text{if and only if} \quad f(p) \leq f(q) \text{ for every } f \in F.$$

Then $\leq$ is a partial order on L with the least and greatest element.
We say that $p,q \in L$ are **orthogonal** if $p \leq q'$ (which is equivalent to $q \leq p'$, see [1] for details).
We add one more axiom:

(IV) For every orthogonal elements $p,q \in L$ there exists supremum $s = \sup(p,q)$ and $f(s) = f(p) + f(q)$ for each $f \in F$.

It is well-known that the system $(L; \leq, ', 0, 1)$ is an orthomodular poset, the so-called **associated poset with the logical structure** $(L; F)$, see e.g. [1]. Hence, orthomodular posets serve as an axiomatic description of physical systems, see e.g. [7, 4]. If $\sup(p, q)$ exists for each couple p, q of elements of L, then $(L; \leq, ', 0, 1)$ becomes an orthomodular lattice. Hence, the theory of orthomodular posets includes the theory of orthomodular lattices and, simultaneously, serves as an axiomatization of the logic of physical systems. In particular, it axiomatizes the logic of quantum mechanics, see [6, 4, 8, 11] and [13].

Due to the above mentioned properties, orthomodular posets were and are studied by numerous authors for several decades see e.g. [7, 4, 9, 12, 13]. However, up to now, orthomodular posets were treated as partial algebras where the binary operation of supremum is ensured only for orthogonal or comparable elements. In this paper, we try another approach, namely to introduce a certain everywhere defined algebra which can be assigned to every orthomodular poset in the way that the underlying poset coincides with the original one but its axioms can be expressed as identities. Hence, the class of these so-called orthomodular directoids forms a variety of algebras having nice algebraic properties. Moreover, every orthomodular poset can be recovered by means of this assigned algebra despite the fact that the assignment need not be done in a unique way.

Outline

Recall by [10] (see also [3]) that a groupoid $(A;+)$ is called a **commutative directoid** if it satisfies the following axioms:

$$x+x=x$$

$$x+y=y+x$$

$$x+((x+y)+z)=(x+y)+z.$$

In what follows, we enrich the commutative directoid by a unary operation (orthocomplementation) and by two constants to get an algebra for our study. Since we need ask two more properties connected with orthomodular posets (namely the orthomodular law and the existence of suprema for orthogonal elements), we add two more axioms which caused that some other axioms for orthomodular directoids can follow from the remaining ones. Hence, we can define:

Definition 1

By an **orthomodular directoid** is called an algebra $\mathscr{D} = (D; +, ', 0, 1)$ of type $(2, 1, 0, 0)$ satisfying the following axioms:

(D1) $x + y = y + x$

(D2) $x + ((x + y) + z) = (x + y) + z$

(D3) $x + 0 = x$

(D4) $x + x' = 1$

(D5) $(((x + z) + (y + z)')' + (y + z)') + z' = z'$

(D6) $x + (x + (x + y)')' = x + y$.

Theorem 1

The axioms (D1)–(D6) are independent.

We can derive several more useful identities satisfied by orthomodular directoids.

Lemma 1

Every orthomodular directoid satisfies the following:

(a) $x'' = x$

(b) $x + 1 = 1$

(c) $x + x = x$

(d) $0' = 1$ and $1' = 0$

(e) $(x' + y)' + x = x$.

Lemma 2

Let $\mathscr{D} = (D; +, ', 0, 1)$ be an orthomodular directoid. Define a binary relation $\leq$ on D as follows

$$x \leq y \quad \text{if and only if} \quad x + y = y. \qquad (*)$$

Then $\leq$ is a partial order on D such that:

(a) $0 \leq x \leq 1$ for each $x \in D$

(b) $x \leq x + y$, $y \leq x + y$

(c) $x \leq y$ implies $y' \leq x'$

(d) if $x + y = 0$ then $x = 0 = y$

(e) if $x + (x + y)' = 1$ then $y \leq x$.

The partial order defined by $(*)$ will be referred to as the **induced order** of $\mathscr{D} = (D; +, ', 0, 1)$.

Now, we recall the concept of orthomodular poset (from [1]).

Definition 2

By an **orthomodular poset** is meant a structure $\mathscr{P} = (P; \leq, ', 0, 1)$, where $\leq$ is a partial order on P, $0 \leq x \leq 1$ for each $x \in P$, $x'' = x$, x' is a complement of x and $x \leq y$ implies $y' \leq x'$, and satisfying the following two conditions:

(i) if $x \leq y'$ then the set $\{x, y\}$ has the supremum $x \vee y$ in $(P; \leq)$

(ii) if $x \leq y$ then $x \vee (x \vee y')' = y$.

Remark.

(a) Since $x \leq x$ for each $x \in P$, $x \vee x'$ exists and $x \vee x' = 1$.

(b) Since $x \leq y$ implies $y' \leq x'$, the existence of $x \vee y$ yields the existence of $x' \wedge y' = (x \vee y)'$, the infimum of x', y', by De Morgan laws. In particular, $x \vee x' = 1$ and $x'' = x$, $1' = 0$ get immediately $x' \wedge x = 0$ and hence x' is a complement of x.

(c) If $x \leq y$ then, by (i), $x \vee y'$ exists. Since $x \leq x \vee y'$, also $x \vee (x \vee y')'$ exists thus (ii) is correctly defined. By using De Morgan laws, (ii) can be read as follows:

$$x \leq y \quad \Rightarrow \quad x \vee (x' \wedge y) = y \tag{OML}$$

which is the **orthomodular law**. Hence, if $x \vee y$ exists $\forall x, y \in P$ then $\mathscr{P} = (P; \leq, ', 0, 1)$ is an **orthomodular lattice** (see [1, 5]).

By (i), if x, y are orthogonal then $x \vee y$ exists. Of course, $x \vee y$ exists also for comparable elements since $x \leq y$ gets $x \vee y = y$.

If $\mathscr{P} = (P; \leq, ', 0, 1)$ is an orthomodular lattice then the orthomodular law (OML) can be expressed in the form of identity as follows:

$$x \vee (x' \wedge (x \vee y)) = x \vee y.$$

(d) If $x \leq a$ and $y \leq a'$ for some $a \in P$ then $x \vee y$ exists. Namely, $y \leq a'$ yields $a \leq y'$ thus $x \leq a \leq y'$ gets that x, y are orthogonal and, by (i), $x \vee y$ exists in $(P; \leq)$.

Example

See [1]. Let M be a finite set with an even number of elements. Let P be the set of all subsets of M which have even number of elements ordered by inclusion and let $A' = M \setminus A$, the set-theoretical complementation. Then $\mathscr{P} = (P; \subseteq, ', \emptyset, M)$ is an orthomodular poset. If $|M| \geq 6$ then $\mathscr{P}$ is not a lattice.

Now, we are going to show that every orthomodular directoid is an orthomodular poset. For this, we need the following lemma.

Lemma 3

Let $\mathscr{D} = (D; +, ', 0, 1)$ be an orthomodular directoid, $\leq$ its induced order.

$$\text{If } x \leq y' \text{ then } x + y = x \vee y.$$

Outline

Now, we are ready to state our first main theorem.

Theorem 2

Let $\mathscr{D} = (D; +, ', 0, 1)$ be an orthomodular directoid and $\leq$ be its induced order. Then $\mathscr{P}(D) = (D; \leq, ', 0, 1)$ is an orthomodular poset where for orthogonal elements $x, y \in D$ we have

$$x + y = x \vee y.$$

Moreover, we are able to prove the converse.

Theorem 3

Let $\mathscr{P} = (P; \leq, ', 0, 1)$ be an orthomodular poset. Define a binary operation $+$ on P as follows:

- $x + y = x \vee y$ if $x \vee y$ exists
- $x + y = y + x$ is an arbitrary element of $U(x, y) = \{z \in P; x, y \leq z\}$ otherwise.

Then $\mathscr{D}(P) = (P; +, ', 0, 1)$ is an orthomodular directoid.

By Theorem 3, to every orthomodular poset $\mathscr{P} = (P; \leq, ', 0, 1)$ can be assigned an everywhere defined algebra which is an orthomodular directoid $\mathscr{D}(P) = (P; +, ', 0, 1)$. By Theorem 2, to the orthomodular directoid $\mathscr{D}(P)$ can be assigned an orthomodular poset $\mathscr{P}(\mathscr{D}(P))$. Since the underlying posets $(P; \leq)$ coincide in all $\mathscr{P}$, $\mathscr{D}(P)$ and $\mathscr{P}(\mathscr{D}(P))$ and the complementation is also the same, we conclude that $\mathscr{P} = \mathscr{P}(\mathscr{D}(P))$. Hence, although the directoid $\mathscr{D}(P)$ need not be assigned in a unique way, it bears all the information on $\mathscr{P}$ because $\mathscr{P} = \mathscr{P}(\mathscr{D}(P))$ for every such a directoid.
On the contrary, if $\mathscr{D} = (D; +, ', 0, 1)$ is an orthomodular directoid, $\mathscr{P}(D)$ the assigned orthomodular poset and $\mathscr{D}(\mathscr{P}(D))$ the assigned orthomodular directoid then $\mathscr{D}$ and $\mathscr{D}(\mathscr{P}(D))$ need not be even isomorphic because the operation $+$ in $\mathscr{D}(\mathscr{P}(D))$ can be choosen differently than that in $\mathscr{D}$.

Theorem 4

Let $\mathscr{D} = (D; +, ', 0, 1)$ be an orthomodular directoid, $\leq$ its induced order and $a \in D$. Then $([a, 1]; +, ^a, a, 1)$ for $x^a = x' + a$ is an orthomodular directoid.

Outline

By Theorems 2 and 3, orthomodular posets can be represented by everywhere defined algebras, i.e. by orthomodular directoids. However, by Definition 1, these directoids are determined by the identities (D1)–(D6) and hence the class $\mathscr{K}$ of orthomodular directoids forms a variety of algebras. In what follows, we present several important properties of the variety $\mathscr{K}$.
Recall that an algebra $\mathscr{A}$ is **congruence distributive** if its congruence lattice $\mathrm{Con}\mathscr{A}$ is distributive. A variety $\mathscr{V}$ is congruence distributive if each $\mathscr{A} \in \mathscr{V}$ has this property. By a majority term is meant a ternary term $m(x,y,z)$ such that

$$m(x,x,y) = m(x,y,x) = m(y,x,x) = x.$$

It follows directly by the Jónsson characterization that a variety $\mathscr{V}$ having a majority term is congruence distributive, see e.g. [2].

Theorem 5

The variety $\mathscr{K}$ of orthomodular directoids is congruence distributive.

Proof.

Consider the ternary term

$$m(x,y,z) = ((x \sqcap y) + (y \sqcap z)) + (x \sqcap z),$$

where $a \sqcap b = (a' + b')'$. It is easy to see that m is a majority term of $\mathscr{K}$. $\square$

Let us recall (see e.g. [2]) that an algebra $\mathscr{A}$ is **congruence regular** if every congruence on $\mathscr{A}$ is determined by every its class, i.e. if for any $\Theta, \Phi \in \mathrm{Con}\mathscr{A}$ and each $a \in A$, if $[a]_\Theta = [a]_\Phi$ then $\Theta = \Phi$. A variety $\mathscr{V}$ is **congruence regular** if each $\mathscr{A} \in \mathscr{V}$ has this property. The following result was proved by B. Csákány, see [2].

Proposition

A variety $\mathscr{V}$ is congruence regular if and only if there exists $n \geq 1$ and ternary terms $t_1, \ldots, t_n$ such that

$$(t_1(x,y,z) = z \text{ and } \ldots \text{ and } t_n(x,y,z) = z) \text{ if and only if } x = y.$$

Using this, we prove the following result.

Theorem 6

The variety of orthomodular directoids is congruence regular.

Proof.

At first, consider the term

$$x \triangle y = (x + (x + y)')' + (y + (x + y)')'.$$

If $x = y$ then clearly $(x + (x + y)')' = (x + x')' = 1' = 0$ and hence $x \triangle x = 0$. Conversely, assume $x \triangle y = 0$. By (d) of Lemma 2, $(x + (x + y)')' = 0$ and $(y + (x + y)')' = 0$, i.e. $x + (x + y)' = 1$ and $y + (x + y)' = 1$. By (e) of Lemma 2, $y \leq x$ and $x \leq y$ giving $x = y$. Now, take $n = 2$ and consider the terms $t_1(x, y, z) = (x \triangle y) + z$, $t_2(x, y, z) = ((x \triangle y) + z')'$. It is elementary to check that t_1, t_2 are the terms of the Proposition proving congruence regularity. □

Recall that a variety $\mathscr{V}$ is **permutable** if $\Theta \circ \Phi = \Phi \circ \Theta$ for every $\mathscr{A} \in \mathscr{V}$ and each $\Theta, \Phi \in \mathrm{Con}\mathscr{A}$. As proved by A.I. Mal'cev (see e.g. [2]), a variety $\mathscr{V}$ is permutable if and only if there exists a ternary term $p(x, y, z)$ such that $p(x, x, z) = z$ and $p(x, z, z) = x$.

Theorem 7

The variety of orthomodular directoids is permutable.

Proof.

In Theorem 4, the involution x^a for $x \in [a,1]$ has been introduced. Since $y \leq x+y$, we have $x+y \in [y,1]$ and hence $(x+y)^y = (x+y)'+y$ is defined. As shown by Theorem 4, we have $(x+y)^{yy} = x+y$ and $(x+y)^y \geq y$ whence $(x+y)^y + y = (x+y)^y$. Define

$$p(x,y,z) = ((z+y)^y + x)^x \sqcap ((x+y)^y + z)^z,$$

where again $a \sqcap b = (a'+b')'$. Then

$$p(x,x,z) = ((z+x)^x + x)^x \sqcap ((x+x)^x + z)^z = (z+x) \sqcap z = z$$

and, analogously,

$$p(x,z,z) = ((z+z)^z + x)^x \sqcap ((x+z)^z + z)^z = x \sqcap (x+z) = x.$$

Hence, p is a Mal'cev term and thus the variety of orthomodular directoids is permutable. □

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Thank you for your attention.