General framework for topological Ramsey spaces, canonization theorems, and Tukey types of ultrafilters with partition properties

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References

[Dobrinen/Mijares/Trujillo]*General framework for topological Ramsey spaces, canonical equivalence relations, and initial structures in the Tukey types of p-points*, partial preprint.

References

[Dobrinen/Mijares/Trujillo] *General framework for topological Ramsey spaces, canonical equivalence relations, and initial structures in the Tukey types of p-points*, partial preprint.

[Dobrinen/Todorcevic 2011] *Tukey types of ultrafilters*, Illinois Journal of Mathematics, 2011.

[Dobrinen/Todorcevic 1,2] *Ramsey-Classification Theorems and their applications in the Tukey theory of ultrafilters, Parts 1 and 2*, Transactions of the American Mathematical Society, to appear.

[Laflamme 1989] *Forcing with filters and complete combinatorics*, Annals of Pure and Applied Logic, 1989.

[Raghavan/Todorcevic 2012] *Cofinal types of ultrafilters*, Annals of Pure and Applied Logic, 2012.

[Todorcevic 2010] *Introduction to Topological Ramsey Spaces*, Princeton University Press, 2010.

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3. Classify all isomorphism classes within the Tukey types of ultrafilters Tukey reducible to the associated ultrafilter.
4. Find initial structures in the Tukey types of p-points.

Motivation: What is the Tukey Structure of Ultrafilters?

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Def. $\mathcal{V}$ is *Tukey reducible* to $\mathcal{U}$ ($\mathcal{V} \le_T \mathcal{U}$) $\Leftrightarrow$ there is a *cofinal map* from $\mathcal{U}$ into $\mathcal{V}$: $\exists f : \mathcal{U} \to \mathcal{V}$ mapping each base for $\mathcal{U}$ to a base for $\mathcal{V}$.

The Top of Tukey: Isbell's Problem

Thm. [Isbell 65, Juhász 67] There is an ultrafilter $\mathcal{U}_{\text{top}}$ which has maximal Tukey type: $(\mathcal{U}_{\text{top}}, \supseteq) \equiv_T ([\mathfrak{c}]^{<\omega}, \subseteq)$.

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Question [Isbell 65]. Is it consistent with ZFC that all ultrafilters have top Tukey type?

Not Top of Tukey: P-Points

Def. $\mathcal{U}$ is a *p-point* if for each decreasing sequence $U_0 \supseteq U_1 \supseteq \dots$ in $\mathcal{U}$, there is a $Y \in \mathcal{U}$ such that for each $n < \omega$, $Y \subseteq^* U_n$ (i.e. $\forall n$, $|Y \setminus U_n| < \omega$).

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Cor. Every p-point is not Tukey top and has Tukey type of cardinality $\mathfrak{c}$.

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Def. The *Fubini product* of $\mathcal{W}$ and $\mathcal{V}_n$, $n < \omega$, is

$$\lim_{n \to \mathcal{W}} \mathcal{V}_n = \{X \subseteq \omega \times \omega : \mathcal{W} n \ (X)_n \in \mathcal{V}_n\}$$

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Proof uses: Continuous cofinal maps theorem of [D/T 11] and Pudlák-Rödl Canonization Theorem.

What can we say about the structure of non-Ramsey ultrafilters near the bottom of the Tukey hierarchy?

Background: Initial Structures of Descending Chains

Thm. [D/Todorcevic] For each $\alpha < \omega_1$, there are decreasing chains of p-points (satisfying weak partition properties) of order type $(\alpha + 1)^*$ as initial structures in the Tukey hierarchy.

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2. proving Ramsey-classification theorems for equivalence relations on barriers (extending the Pudlák-Rödl Theorem),
3. and applying them to decode the isomorphism types within the Tukey types of associated ultrafilters, thus also obtaining the Tukey structure.

Guiding Questions

1. Can topological Ramsey spaces provide a general framework for ultrafilters satisfying weak partition properties?

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2. What other structures (besides descending chains of order type $(\alpha + 1)^*$) appear as initial structures in Tukey types of ultrafilters?

Simplest Topological Ramsey Space: The Ellentuck Space

Example. Ellentuck space $[\omega]^\omega$. $Y \le X$ iff $Y \subseteq X$.
Basis for topology: $[s, X] = \{Y \in [\omega]^\omega : s \sqsubset Y \subseteq X\}$.

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Silver Theorem: All (metrically) Suslin sets are Ramsey.

Topological Ramsey spaces $(\mathcal{R}, \leq, r)$

r is a finite approximation function.
n-th Appproximations: $\mathcal{AR}_n = \{r_n(X) : X \in \mathcal{R}\}$.
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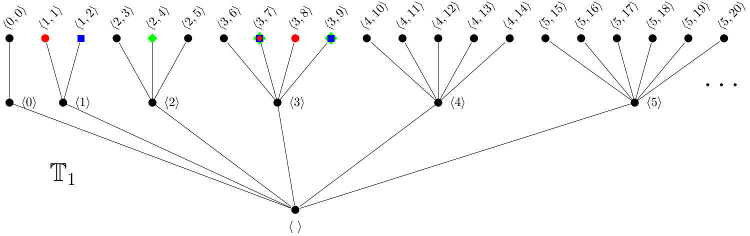
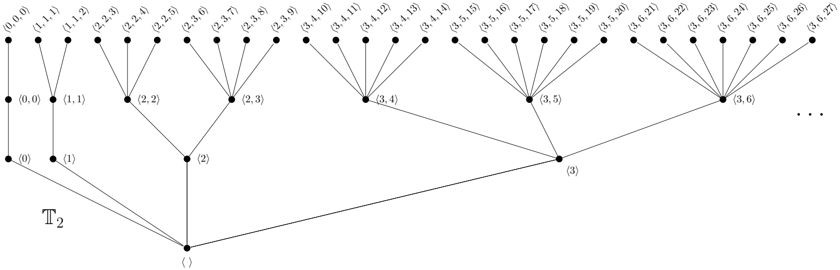
Topological Ramsey Space $\mathcal{R}_1$, [D/Todorcevic]



$X \in \mathcal{R}_1$ iff X is a subtree of $\mathbb{T}_1$ and $X \cong \mathbb{T}_1$.
For $X, Y \in \mathcal{R}_1$, $Y \leq X$ iff $Y \subseteq X$.
Associated Ultrafilter: *weakly Ramsey* $\omega \to [\mathcal{U}]^2_3$

The space $\mathcal{R}_2$, [D/Todorcevic]



$X \in \mathcal{R}_2$ iff X is a subtree of $\mathbb{T}_2$ and $X \cong \mathbb{T}_2$.
For $X, Y \in \mathcal{R}_2$, $Y \leq X$ iff $Y \subseteq X$.
Associated Ultrafilter: $\omega \to [\mathcal{U}]^2_4$

General Constructions of Topological Ramsey Spaces, [D/Mijares/Trujillo]

Axioms **A.1 - A.4** plus two more axioms **B.1 and B.2.**

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General: Unbounded Height Well-founded "Trees":
Subsequent levels formed by vertical gluing of finite products of finite ordered relational structures from a Fraïssé class with the Ramsey Property. (In progress.)

Examples: Ultrafilters from this tRs Construction Method

1. k-arrow not $k+1$-arrow ultrafilters of Baumgartner and Taylor, using finite ordered $k+1$-clique free graphs growing so as to have the Ramsey Property (possible by Nešetřil-Rödl Theorem).

   Partition Property of Associated Ultrafilter: $\omega \rightarrow (\mathcal{U}, k)^2$, and $\omega \not\rightarrow (\mathcal{U}, k+1)^2$.

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4. $\mathcal{R}_1 * \mathcal{H}^2$.

   Partition Property of Associated ultrafilter: $\omega \rightarrow [\mathcal{U}]^2_7$.

Applications: Initial Structures in Tukey Types

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The proof depends on new Ramsey-classification theorems for equivalence relations on fronts.

Ramsey Theory

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Erdős-Rado Canonization Theorem. For each $k \geq 1$ and each equivalence relation E on $[\omega]^k$, there is an infinite $M \subseteq \omega$ such that $\mathrm{E} \restriction [M]^k$ is *canonical*,

i.e. $\mathrm{E} \restriction [M]^k$ is given by E_I^k for some $I \subseteq k$.

$$\text{For } a, b \in [\omega]^k,\ a \ \mathrm{E}_I^k \ b \text{ iff } \forall i \in I,\ a_i = b_i.$$

Note. The Erdős-Rado Theorem is a canonization theorem for the fronts (barriers) of the form $[\omega]^k$ on the Ellentuck space.

Product Erdős-Rado Theorem

Thm. [D/Mijares/Trujillo] Suppose $\mathcal{R}$ is a topological Ramsey space of Trees of Height 2 (where blocks consist of a product of m many Fraïssé classes of ordered relational structures with the Ramsey property). Suppose E is an equivalence relation on the n-th blocks of members of $\mathcal{R}$ coming from within one block in the maximal tree.

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Then there is an $X \in \mathcal{R}$ and $I_k \subseteq |\mathbb{A}_k(n)|$, $k < m$, such that $E = E_{(I_0,\dots,I_{m-1})}$ when restricted to X.

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Remark. Only the order matters - the structure is irrelevant to the canonical equivalence relations.

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Thm. [D/Mijares/Trujillo] Suppose $\mathcal{R}$ is a topological Ramsey space of Trees of Height 2 (where blocks consist of a product of m many Fraïssé classes of ordered relational structures with the Ramsey property). Suppose E is an equivalence relation on the n-th blocks of members of $\mathcal{R}$ coming from within one block in the maximal tree.

Then there is an $X \in \mathcal{R}$ and $I_k \subseteq |\mathbb{A}_k(n)|$, $k < m$, such that $E = E_{(I_0,\dots,I_{m-1})}$ when restricted to X.

Remark. Only the order matters - the structure is irrelevant to the canonical equivalence relations.

Example. The 2-nd blocks of $\mathcal{H}^2$ consist of all 3×3 squares coming from within one square in the maximal tree. These are products of 2 linearly ordered sets. The canonical equivalence relations are given by all products $I_0 \times I_1$, where $I_0, I_1 \subseteq \{0, 1, 2\}$.

Fronts, Barriers, and Irreducible Functions on Ellentuck

Def. $\mathcal{F} \subseteq [\omega]^{<\omega}$ is a *front* on $[\omega]^{\omega}$ iff

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(a) *inner*, i.e. $\varphi(a) \subseteq a$ for all $a \in \mathcal{F}$, and

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Generalization of Erdős-Rado Theorem

Pudlak-Rödl Canonization Thm. For every front (barrier) $\mathcal{F}$ on ω and every equivalence relation E on $\mathcal{F}$, there is an infinite $M \subseteq \omega$ such that $\mathrm{E} \restriction (\mathcal{F}|M)$ is represented by an irreducible mapping defined on $\mathcal{F}|M$.

Def. $\mathcal{F}|M = \{a \in \mathcal{F} : a \subseteq M\}$.

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Extensions of the Pudlak-Rödl Theorem

Ramsey-Classification Theorems [D/Mijares/Trujillo] for a general class of topological Ramsey spaces; ([D/Todorcevic] for $\mathcal{R}_\alpha$, $\alpha < \omega_1$):

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Rem. Ramsey-classification theorems, along with continuous cofinal maps, are used to decode the isomorphism types within Tukey types of related ultrafilters; hence also the Tukey structure.