Prospects for a reverse analysis of topology

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Reverse mathematics

The program of reverse mathematics aims to figure out which axioms are necessary to prove theorems of everyday mathematics.

The axioms systems traditionally used are subsystems of second-order arithmetic. These are two-sorted systems with a number sort and a set sort. The number sort obeys the usual axioms for basic arithmetic (PA^-).

- RCA_0 is the base system it has just enough comprehension to show that sets are closed under relative computability
- ACA_0 adds comprehension for arithmetic formulas (without set quantifiers but maybe with set parameters)
- Π^1_1-CA_0 adds comprehension for Π^1_1-formulas (of the form $\forall X\, \phi(n, X)$ where ϕ is arithmetic)

All systems include induction for Σ^0_1-formulas

Fundamental problem

Second-order arithmetic has two layers of objects — numbers and sets — but topology usually works with three layers:

points

open sets, closed sets, etc.

covers, filters, etc.

Other approaches

- Complete separable metric spaces are well understood[1]
- Mummert studied maximal filter spaces as a more general notion of topological spaces[2]
- Hunter studied general topological spaces in systems of arithmetic with higher types and atoms[3]

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[1]S. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Cambridge University Press, Cambridge, 2009. DOI:10.1017/CBO9780511581007

[2]C. Mummert, *Reverse mathematics of MF spaces*, Journal of Mathematical Logic **6** (2007), 203–232. DOI:10.1142/S0219061306000578.

[3]J. Hunter, *Higher-order reverse topology*, Ph.D. Thesis, University of Wisconsin–Madison, 2008.

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Point-set approach

Idea:

- Points are a set of numbers
- Basic opens are an indexed sequence of esets of points
- Collections of indices are used to code higher order objects

Caveat:

- Limited to countable second-countable spaces

Bases

An **effective base** on a set X is a uniformly enumerable family $\mathcal{B} = (B_i)_{i \in \mathbf{N}}$ of esets for which there are partial functions $\alpha : X \to \mathbf{N}$ and $\beta : X \times \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ such that

$$x \in B_{\alpha(x)}$$

and

$$x \in B_i \cap B_j \Longrightarrow x \in B_{\beta(x,i,j)} \subseteq B_i \cap B_j.$$

Note: If $\mathcal{A} = (A_j)_{j \in \mathbf{N}}$ is any uniformly enumerable family then

$$B_s = \bigcap_{j \in s} A_j, \quad s \in \mathbf{N}^{[<\infty]},$$

is an effective base that generates the same topology on X.

Opens

A **CSC space** $\mathcal{X}$ is a set X equipped with an effective base $\mathcal{B}^X$.

An **open** in $\mathcal{X}$ is an eset $U \subseteq X$ such that for each $x \in U$ there is a basic open B_i such that $x \in B_i \subseteq U$.

An **effective open** in $\mathcal{X}$ is an eset $U \subseteq X$ for which there is a partial function $\gamma : X \to \mathbf{N}$ such that

$$x \in U \Longrightarrow x \in B_{\gamma(x)} \subseteq U.$$

An (**effective**) **closed** in $\mathcal{X}$ is the complement of an (effective) open.

Continuity

Let $\mathcal{X}$ and $\mathcal{Y}$ be CSC spaces.

A function $f : X \to Y$ is **continuous** if any of the following equivalent conditions hold:

- $f^{-1}[G]$ is open in $\mathcal{X}$ for every open G in $\mathcal{Y}$.
- $f^{-1}[B_j^Y]$ is open in $\mathcal{X}$ for every basic open B_j^Y.
- when $f(x) \in B_j^Y$ there is a basic open B_i^X such that

$$x \in B_i^X \subseteq f^{-1}[B_j^Y].$$

A function $f : X \to Y$ is **effectively continuous** if there is a partial function $\phi : X \times \mathbf{N} \to \mathbf{N}$ such that

$$f(x) \in B_i^Y \implies x \in B_{\phi(x,i)}^X \subseteq f^{-1}[B_i^Y].$$

Compactness

An **open cover** of $\mathcal{X}$ is an uniformly enumerable family of open sets $(U_j)_{j\in\mathbf{N}}$ such that $X = \bigcup_{j\in\mathbf{N}} U_j$.

A CSC space $\mathcal{X}$ is **compact** if every open cover of $\mathcal{X}$ has a finite subcover.

A CSC space $\mathcal{X}$ is **basically compact** if every basic open cover of $\mathcal{X}$ has a finite subcover.

The CSC space $\mathcal{X}$ with base $\mathcal{B} = (B_i)_{i\in\mathbf{N}}$ has a **finite cover relation** if

$$\{s \in \mathbf{N}^{[<\infty]} : \bigcup_{i\in s} B_i = X\}$$

is an internal set.

Discrete spaces

A CSC space $\mathcal{X}$ is **discrete** if every singleton $\{x\}$ is open in $\mathcal{X}$.

Theorem (RCA_0)

The following are equivalent:

- *Every basically compact discrete space is finite*
- *Arithmetic comprehension* (ACA_0)

basically compact $\not\Rightarrow$ compact

Sequential Compactness

A CSC space $\mathcal{X}$ is **sequentially compact** if every sequence $(x_n)_{n=0}^{\infty}$ of points has an accumulation point.

Theorem (RCA_0)

The following are equivalent:

- *Every finite CSC space is sequentially compact*
- *The infinite pigeonhole principle*
- Π^0_1*-bounding* ($\mathsf{B}\Sigma^0_2$)

$$\text{compact} \quad \not\Longrightarrow \quad \text{sequentially compact}$$

Product spaces

The product of two CSC spaces $\mathcal{X}$ and $\mathcal{Y}$ is the CSC space on $X \times Y$ with basis $(B_i^X \times B_j^Y)_{(i,j) \in I \times J}$.

Theorem (RCA_0)

The following are true:

- *The product of two sequentially compact CSC spaces is sequentially compact*
- *The product of two basically compact CSC spaces with finite cover relations is basically compact and has a finite cover relation*

Product spaces

Theorem ($\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$)

If there is a function $f : \mathbf{N} \times \mathbf{N} \to \{0, 1\}$ such that the map $x \mapsto \lim_{y\to\infty} f(x, y)$ is 1-generic, then there are two basically compact CSC spaces $\mathcal{X}$ and $\mathcal{Y}$ such that the product $\mathcal{X} \times \mathcal{Y}$ is not basically compact.

basic compactness is not always productive

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Point-free approach

Idea:

- Basic opens are represented by a poset of numbers
- Collections of basic opens exist
- Points are identifierd with their basic neighborhood filters

Caveat:

- Limited to a certain class of second-countable spaces

Bases

Let P be a poset. If $A, B \subseteq P$, we write

$$A \leq B \iff (\forall p \in A)(\exists q \in B)(p \leq q).$$

A **coverage system** $\mathcal{C}$ associates to each $p \in P$ a collection $\mathcal{C}_p$ of subsets of $P[\leq p]$ — **basic covers** of p — such that if $q \leq p$ and $C \in \mathcal{C}_p$ then there is a $C' \in \mathcal{C}_q$ such that $C' \leq C$.

A **countable coded coverage system** is a coverage system where each $\mathcal{C}$ is coded as a subset of $P \times P \times \mathbb{N}$.

A **countable coded posite** is a pair $(P, \mathcal{C})$ where $\mathcal{C}$ is a countable coded coverage system on P.

Points and opens

A $(P, \mathcal{C})$-**point** $F \subseteq P$ is a (nonempty) filter such that if $p \in F$ and $C \in \mathcal{C}_p$ then $F \cap C \neq \varnothing$.

A $(P, \mathcal{C})$-**open** is a lower set $I \subseteq P$ such that if $C \in \mathcal{C}_p$ and $C \subseteq I$ then $p \in I$.

Thus a $(P, \mathcal{C})$-point is a filter on P whose complement is a $(P, \mathcal{C})$-open.

Theorem (ACA_0)

If $I \subseteq P$ is a $(P, \mathcal{C})$-open and $p \notin I$ then there is a $(P, \mathcal{C})$-point F such that $p \in F$ and $F \cap I = \varnothing$.

Opens

Given a posite $(P, \mathcal{C})$ and $p \in P$ we write $\mathcal{X}_p$ for the class of all $(P, \mathcal{C})$-points containing p.

Theorem (ACA_0)

The following are equivalent:

- *If $(P, \mathcal{C})$ is a countable coded posite, then for every set $A \subseteq P$ there is a $(P, \mathcal{C})$-open I such that $\bigcup_{p \in A} \mathcal{X}_p = \bigcup_{q \in I} \mathcal{X}_q$*
- Π^1_1*-comprehension*

It is enough to consider the case where $(P, \mathcal{C})$ is the usual posite for Baire space.

Continuity

A **continuous map** $F : (Q, \mathcal{D}) \to (P, \mathcal{C})$ is a relation $F \subseteq P \times Q$ such that:

- For every $q \in Q$ there is a $p \in P$ such that $(p, q) \in F$
- If $(p, q) \in F$ and $p' \geq p, q \geq q'$ then $(p', q') \in F$
- If $(p_1, q), (p_2, q) \in F$ then there is a $p \leq p_1, p_2$ such that $(p, q) \in F$
- If $(p, q) \in F$ and $C \in \mathcal{C}_p$ then $(p', q) \in F$ for some $p' \in C$

If X is a $(Q, \mathcal{D})$-point then

$$F(X) = \{p \in P : (\exists q \in X)[(p, q) \in F]\}$$

is a $(P, \mathcal{C})$-point.

Regular spaces

Write $q \preccurlyeq p$ if $P[\leq p] \cup P[\perp q]$ is a $(P, \mathcal{C})$-cover. The posite $(P, \mathcal{C})$ is **regular** if

$$P[\preccurlyeq p]$$

covers p, for every $p \in P$.

We say that $(P, \mathcal{C})$ is **strongly regular** if there exists a relation $\triangleleft$ such that

- $q \triangleleft p \Longrightarrow q \preccurlyeq p$, and
- $P[\triangleleft p] \in \mathcal{C}_p$ for every p.

Metrizability

Theorem (ACA^+)

Every strongly regular countable coded posite is embeddable in $[0,1]^{\mathbf{N}}$.

Theorem (Π^1_1-CA_0)

Every regular countable coded posite is embeddable in $[0,1]^{\mathbf{N}}$.

Reversals are unclear. Mummert has shown that complete metrizability of regular maximal filter spaces may require up to Π^1_2-comprehension!

Choquet games

Theorem

A topological space is representable by a countable coded posite if and only if

- *X is T_0*
- *X is second-countable*
- *Nonempty has a weakly convergent winning strategy in the strong Choquet game on X.*

A winning strategy for Nonempty in the strong Choquet game is **weakly convergent** if the open sets played by Nonempty generate the neighborhood filter of some point.

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Arithmetic transfinite recursion

An **arithmetic operator** is of the form $\Phi(X) = \{n \in \mathbf{N} : \phi(n, X)\}$ where ϕ arithmetic. The **iteration** of Φ along $(A, \prec)$ is the set $X \subseteq \mathbf{N} \times A$

$$X_a = \Phi(X \restriction a)$$

where $X_a = \{n \in \mathbf{N} : (n, a) \in X\}$ and $X \restriction a = \{(n, b) \in X : b \prec a\}$.

- ATR_0 (Arithmetic Transfinite Recursion) states that every arithmetic operator can be iterated along any countable wellordering.
- ACA_0^+ states that every arithmetic operator can be iterated along $(\mathbf{N}, <)$.

Rudimentary functions

The **rudimentary functions** are generated by composition from the nine basic functions:

$$
\begin{array}{lll}
R_0(x,y) = \{x,y\} & R_3(x) = \operatorname{dom} x & R_6(x) = \{(v,u,w) : (u,v,w) \in x\} \\
R_1(x,y) = x \setminus y & R_4(x,y) = x \times y & R_7(x) = \{(v,w,u) : (u,v,w) \in x\} \\
R_2(x) = \bigcup x & R_5(x) = x \cap (\in) & R_8(x,y) = \{x``\{u\} : u \in y\}
\end{array}
$$

The **Jensen hierarchy** is defined by

$$J_\xi = \bigcup_{\zeta<\xi} \operatorname{rud}(J_\zeta).$$

There is a rudimentary function $\mathbb{T}$ such that $J_\xi = T_{\xi\omega}$ where $T_\xi = \bigcup_{\zeta<\xi} \mathbb{T}(T_\zeta)$.

Rudimentary recursive functions

The **rudimentary recursive functions** are solutions of equations of the form

$$F(x) = G(p, F \restriction x)$$

where G is rudimentary and p is a set parameter.

- $\text{rank}(x) = \bigcup\{\text{rank}(y) + 1 : y \in x\}$
- $\text{trcl}(x) = x \cup \bigcup\{\text{trcl}(y) : y \in x\}$
- $T_\xi = \bigcup_{\zeta<\xi} \mathbb{T}(T_\zeta)$
- $\check{x} = \{(1, \check{y}) : y \in x\}$
- $\text{val}_G(x) = \{\text{val}_G(y) : \exists p \in G\ (p, y) \in x\}$

Providence

Definition (Mathias)

A **provident set** is a transitive set A closed under pairing and rudimentary recursion (with parameters in A).

J_α is provident if and only if α is indecomposable.
PROVI_0 is the elementary theory of provident sets with infinity:

- Extensionality
- Infinity
- Rudimentary closure axioms
- Rudimentary recursion axioms

Mathias showed that PROVI_0 is finitely axiomatizable.

Arithmetical interpretation

Theorem (with Mathias)

The theory PROVI_0 *is mutually interpretable with* ACA_0^+.

1. The arithmetic part of a model of $\mathsf{PROVI}_0 + \mathsf{HC}$ is a model of ACA_0^+.
2. Every model of ACA_0^+ is the arithmetic part of a model of $\mathsf{PROVI}_0 + \mathsf{HC}$.
3. Every model of $\mathsf{PROVI}_0 + \mathsf{HC}$ is an initial segment of the model of $\mathsf{PROVI}_0 + \mathsf{HC}$ reconstructed from its arithmetic part as in 2.