Sheaf representations of MV-algebras via Stone duality

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History of MV-algebras

- MV-algebras were introduced by C.C. Chang as algebraic semantics of Łukasiewicz infinite-valued logic (a propositional calculus with truth values in $[0, 1]$)
- MV-algebras have been studied extensively, mainly by Mundici and co-workers. They have developed the structure theory as well as links with other areas and applications
- The category of MV-algebras is equivalent to the category of unital lattice-ordered abelian groups

MV-algebras

An MV-algebra is an algebra $(A, \oplus, \neg, 0)$ such that

- $(A, \oplus, 0)$ is a commutative monoid,
- $\neg\neg x = x$, that is, $\neg$ is an involution,
- $x \oplus 1 = 1$ where $1 := \neg 0$,
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

- MV-algebras are bounded distributive lattices in the term-definable operations:

$$a \vee b := \neg(\neg a \oplus b) \oplus b,$$
$$a \wedge b := \neg(\neg a \vee \neg b).$$

A variety of residuated lattices with the property that $x \oplus x = x$ implies A is a BA

Examples of MV-algebras

- Boolean algebras
- The unit interval $[0,1]$ in its natural order is an MV-algebra with
$$a \oplus b = \min\{a+b, 1\}, \quad \neg a = \min\{1-a\}$$
- The MV-algebra $C(X,[0,1])$ of continuous functions from a space X to $[0,1]$ with point-wise operations
- Free MV-algebras, consisting of the McNaughton functions on the unit cube
- Nonstandard extensions of $[0,1]$ (ultrapowers) are examples of MV-algebras with infinitesimal elements

Representation theory for MV-algebras

- Duality between finitely presented MV-algebras and rational polyhedra [Marra and Spada 2012]
- Priestley duality for MV-algebras [N. G. Martinez 1994; Martinez- H. Priestley 1998; G-Priestley 2007-08]
  (Is it useful for MV-algebraists?)
- Sheaf representations over the maximum MV-spectrum [Keimel 1968; Filipoiu-Georgescu 1995] and over the MV-spectrum [Kennison 1976; Cornish 1980; Yang 2006; Dubuc-Poveda 2010]

The lattice spectrum

Let D be a distributive lattice, the points, $x \in X_D$, of the lattice spectrum of D are in one-to-one correspondence with each of the following:

- $h_x : D \to 2$ a lattice homomorphism
- I_x a prime ideal of D $(= h^{-1}(0))$
- F_x a prime filter of D $(= h^{-1}(1))$

If, in addition, D is finite, then the above are equivalent to

- $F_x = {\uparrow} p$ where $p \in J(D)$ and $I_x = {\downarrow} m$ where $m \in M(D)$

Stone's representation theorem and duality

Let D be a distributive lattice, X_D the spectrum of D, then

$$\begin{aligned} \eta : D &\to \mathcal{P}(X_D) \\ a &\mapsto \widehat{a} = \{x \in X_D \mid h_x(a) = 1\} \end{aligned}$$

is an injective lattice homomorphism

Topologies on X_D:

- $\tau_D = <\widehat{a} \mid a \in D>$ (spectral or Stone topology)
- $\tau_D^\partial = <(\widehat{a})^c \mid a \in D>$ (dual spectral topology)
- $\pi_D = <\widehat{a}, (\widehat{a})^c \mid a \in D>$ (Priestley topology)

In all three $Im(\widehat{\ })$ can be characterized (order-)topologically

The MV-spectrum

A simple but important fact in the theory of MV-algebras is that

$$\begin{aligned} \theta : A &\longrightarrow Con(A) \\ a &\longmapsto \theta(a) = <(0,a)>_{Con(A)} \end{aligned}$$

is a bounded lattice homomorphism.

The image of this map is the lattice $Con_{fin}(A)$ of finitely generated MV-algebra congruences of A (and thus these congruences are pairwise permuting).

The MV-spectrum of A, is the dual space, Y, of $Con_{fin}(A)$

The MV-spectrum

Since $A \twoheadrightarrow Con_{fin}(A)$ is a bounded distributive lattice quotient, by duality, $Y \hookrightarrow X$ may be seen as a closed subspace of X

The MV-spectrum may also be seen as the set of those MV-ideals (non-empty downsets closed under $\oplus$) that are prime in the sense that one of $a \ominus b (:= \neg(\neg a \oplus b))$ and $b \ominus a$ is a member for all $a, b \in A$. This is the same set $Y \subseteq X$.

The spectral topology on Y is also the hull-kernel or spectral topology corresponding to the MV-ideals of A.

Relative normality of the dual of the MV-spectrum

- The following are equivalent:
  - A bounded distributive lattice D is normal: For all $a, b \in D$, if $a \vee b = 1$ then there are $c, d \in A$ with $c \wedge d = 0$ and $a \vee d = 1$ and $c \vee b = 1$
  - Each point in the dual space of D is below a unique maximal point
  - The inclusion of the maximal points of the dual space of D admits a continuous retraction
- For any MV-algebra A, the lattice $Con_{fin}(A)$ is relatively normal (that is, each interval $[a, b]$ is a normal lattice)

As a consequence Y is always a root-system, that is, $\uparrow y$ is a chain for each $y \in Y$

The maximal MV-spectrum

Given an MV-algebra, A, the subspace Z of Y of maximal MV-ideals of A is called the maximal MV-spectrum

Since $Con_{fin}(A)$ is relatively normal for any MV-algebra Y is a root-system and the map

$$m : Y \longrightarrow Z$$
$$y \mapsto \text{ unique maximal point above } y$$

is a continuous retraction. The maximal MV-spectrum is compact Hausdorff, but not in general spectral

Extended Priestley duality for MV-algebras

(from BLAST'08 on [G-Priestley'07-'08])

The dual spaces of MV-algebras are the spaces $(X, \leqslant, \tau, \cdot)$ satisfying:

- $(X, \leqslant, \tau)$ is a Priestley space with bounds
- $(X, \tau, \leqslant, \cdot, 1)$ is an ordered topological monoid
- The $\cdot$ is open, commutative, and has a lower adjoint
- The element 0 is absorbant
- $\forall x, y \quad [\, x \neq y \star x \;\Rightarrow\; \forall z \; (x \leqslant z$ or $y \cdot z \leqslant y \star x \,)]$ where $y \star x$ is the least $z \in X$ such that

$$\forall x' \quad [y \not\leqslant x' \;\Rightarrow\; x' \cdot x \leqslant z]$$

A bit of explanation

To best make sense of this we need canonical extension, but there is not enough time

- On an MV-algebra we have $\oplus$ and $\neg$, but also $\ominus$, and $\odot$ and $\rightarrow$, which are de Morgan duals of $\oplus$ and $\ominus$
  - $\ominus$ is lower adjoint to $\oplus$
  - $\odot$ is lower adjoint to $\rightarrow$
- In [G-Priestley] we took $\rightarrow$ as basic. It is witnessed dually by $\odot$, which by DQA restricts to $X \cup \{0\}$
- In order to get a First-Order characterization, we also needed $\star$ (which plays no role in this work)

What we need here

- For lattice ideals I and J

$$I \overline{\oplus} J = {\downarrow}\{a \oplus b \mid a \in I, b \in J\}$$

- For $x, x' \in X$ we have $I_x \overline{\oplus} I_{x'}$ is either prime or all of A
- Define a partial operation on X by $x + x' = z$ iff $I_x \overline{\oplus} I_{x'} = I_z$
- The partial operation $+$ has domain $\{(x, y) \mid i(x) \geqslant y\}$, where i is the involution dual to $\neg$

This partial operation $+$ witnesses fully the MV-algebra structure of A relative to its lattice reduct

Relation to the MV-spectrum

For each $x \in X$ we have that $x + x$ is defined and

$$x \in Y \iff x + x = x$$

For each $x \in X$, there is a largest $x' \in X$ with

$$x + x' \leqslant x$$

and this x' satisfies $x' + x' = x'$

This defines a retraction $k : X \to Y$, which is continuous with respect to the Priestley topology on X and the spectral topology on Y. A function named k (defined differently, but with the same action) was present in Martinez' work

Interpolation Lemma

NB! The map k is neither order preserving nor reversing

However it satisfies the following Interpolation Lemma:

If $x \leqslant x'$ then there is x'' with

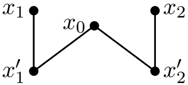
$$x \leqslant x'' \leqslant x' \quad \text{and} \quad k(x'') \geqslant k(x) \quad \text{and} \quad k(x'') \geqslant k(x')$$

From X to Z with $m \circ k$

Combining the two earlier retractions we get

$$m \circ k : (X, \pi) \longrightarrow (Z, \tau)$$

The kernel of this map is given by the relation $x_1 W x_2$ iff there are $x_1', x_2', x_0 \in X$ with



Proof: If $mk(x_1) = mk(x_2)$, then take $x_i' = k(x_i)$ and $x_0 = mk(x_i)$.

For the converse note that if $x \leqslant x'$, then by (Int) there is x'' between with greater k-image than both, but then $mk(x) = mk(x'') = mk(x')$. So all the elements of X in one order component have the same mk-image

Kaplansky's theorem

[Kaplansky 1947]
Let Z_1, Z_2 be compact Hausdorff spaces such that the lattices $C(Z_1, [0,1])$ and $C(Z_2, [0,1])$ are isomorphic. Then Z_1 and Z_2 are homeomorphic spaces.

Kaplansky theorem for arbitrary MV-algebras

Theorem

If A_1 and A_2 are MV-algebras having isomorphic lattice reducts, then the max MV-spectra of A_1 and A_2 are homeomorphic.

- Note that the max MV-spectrum of an MV-algebra of the form $C(Z, [0, 1])$ is Z so that our result generalizes Kaplansky's result.

Proof (sketch).

The maximal MV-spectrum can be constructed from the lattice spectrum by taking the topological quotient w.r.t. the relation W and W is definable from just the order structure of X □

Spectral sum

We have seen that there is a continuous retraction

$$k : (X, \pi) \longrightarrow (Y, \tau)$$

It follows that $k^{-1}(\uparrow y)$, call it X_y, is a closed subspace of X in the Priestley topology. In fact, X_y is a chain and

$$X_y = k^{-1}(\uparrow y) \text{ is the dual of the MV quotient } A_y = A/I_y$$

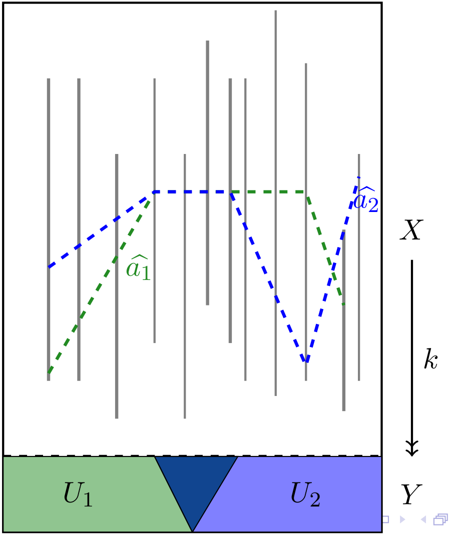
Moreover, one can show (Patch):

For any finite cover $(U_i)_{i=1}^n$ of Y by τ^∂-open sets, and any collection $(\widehat{a_i})_{i=1}^n$ of clopen downsets of X such that

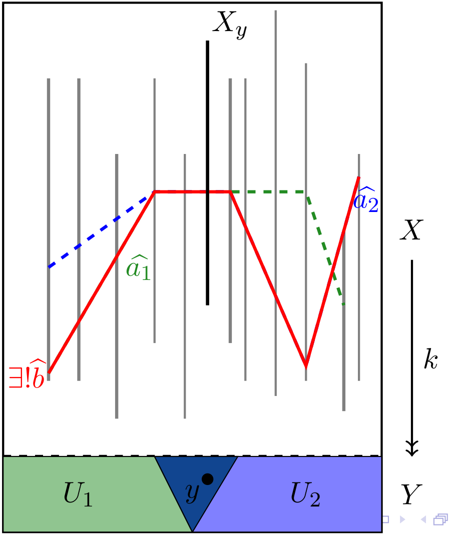
$$\widehat{a_i} \cap k^{-1}(U_i \cap U_j) = \widehat{a_j} \cap k^{-1}(U_i \cap U_j) \qquad (*)$$

holds for any $i, j \in \{1, \ldots, n\}$. Then the set $\bigcup_{i=1}^n (\widehat{a_i} \cap q^{-1}(U_i))$ is a clopen downset in X

Patching property



Patching property



Transparent solution via duality

- There is at most one $b \in A$,
- It must satisfy $\hat{b} = \bigcup_{i=1}^{n}(\hat{a}_i \cap k^{-1}(U_i)) =: K$, and K is closed
- Prove that K is a open: $K^c = \bigcup_{i=1}^{n}(\hat{a}_i{}^c \cap k^{-1}(U_i))$
- Prove that K is a downset: Interpolation Lemma!!!
- This also yields a formula for b by compact approximation:

$$b = \bigvee_{i=1}^{n}(a_i \odot \neg m u_i),$$

where $m, n \in \mathbb{N}$ and $u_i \in A$ such that $\widehat{u_i}^c = U_i$.

From spectral sums to sheaves

Theorem

Let D be a distributive lattice with dual space X. Suppose that $q\colon (X,\pi) \to (I,\rho)$ is a continuous surjection onto a stably compact space[1] which satisfies the property (Patch) as given earlier. Then the associated étale space $p\colon (E,\sigma) \to (I,\rho^{\partial})$ [1] is a sheaf representation of D over Y

- We saw above that $k\colon (X,\pi) \to (Y,\tau)$ satisfies the hypothesis of the above theorem
- One can also show that $m \circ k\colon (X,\pi) \to (Z,\tau)$ satisfies the hypothesis of the above theorem

[1] See Sam van Gool's talk tomorrow morning

Sheaf decompositions of MV-algebras

As a consequence of the sum decompositions we obtain:

Theorem

Any MV-algebra is representable as the global sections of a sheaf

1. *over the maximal MV-spectrum, which is a compact Hausdorff space, with stalks that are local MV-algebras (i.e. having a unique maximal ideal)*
   [Keimel 1968; Filipoiu-Georgescu 1995]

2. *over the MV-spectrum with the dual spectral topology with stalks that are MV-chains*
   [Kennison 1976; Cornish 1980; Yang 2006; Dubuc-Poveda 2010]

Conclusion

We have obtained the sheaf representations of MV-algebras over their prime and maximal MV-spectra from a corresponding decomposition of their Priestley dual spaces. This allows a direct treatment of all MV-algebras using only simple facts from the theory of MV-algebras

If sheaf representations are interesting to MV-algebraists, then so is Priestley duality