Universal Coalgebra - An Introductory Tutorial

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Outline

Systems

Can we get a general theory for all of the following Systems?

- Deterministic
  - Black Boxes
    - ★ w/o errors
  - Automata
    - ★ Moore, Mealy
  - Objects
    - ★ Java, C++, ...
- Nondeterministic systems
  - Automata
  - Kripke Structures
- Fuzzy Systems
  - Probabilistics Systems
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  - Topologies
  - Neighbourhood systems
- and more ...

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State Based

- Behaviour
  - depends on
    - ★ input
    - ★ internal state
  - visible only
    - ★ outputs
- Implementation concerns
  - Observational equivalence
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- Logics
  - global logics
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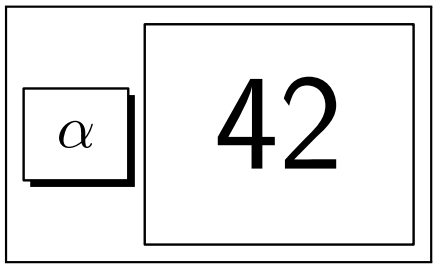

- Signature:

- Peeking inside

- Indistinguishability relation: largest $\sim$ with

$$\frac{s \sim s'}{h(s)=h(s'),\ t(s) \sim t(s')}$$

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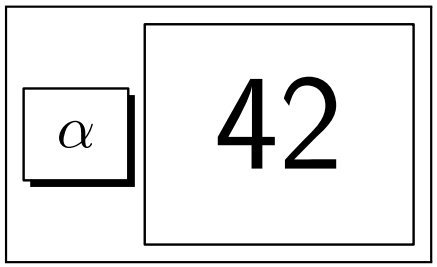

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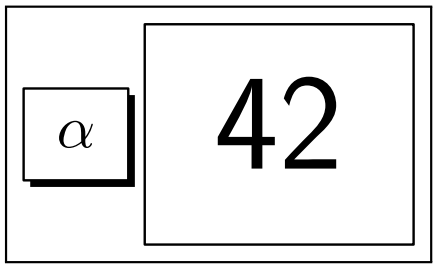

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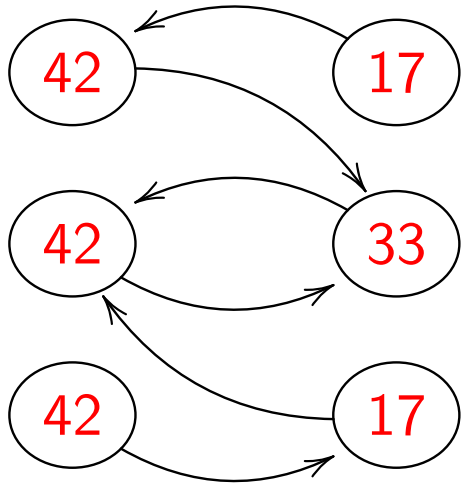

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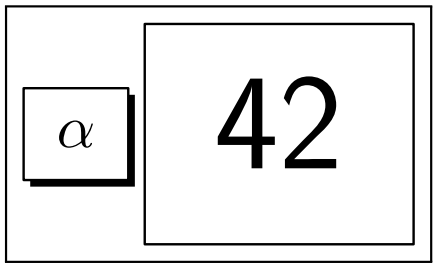

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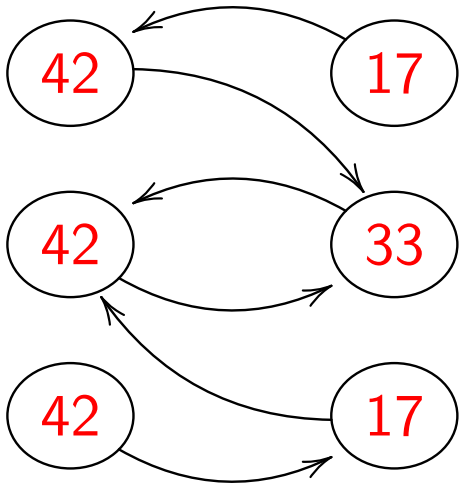

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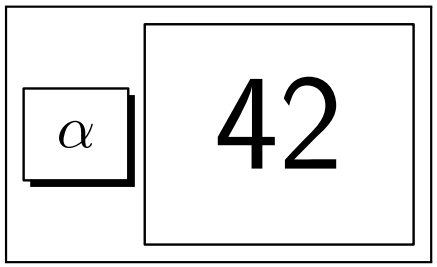

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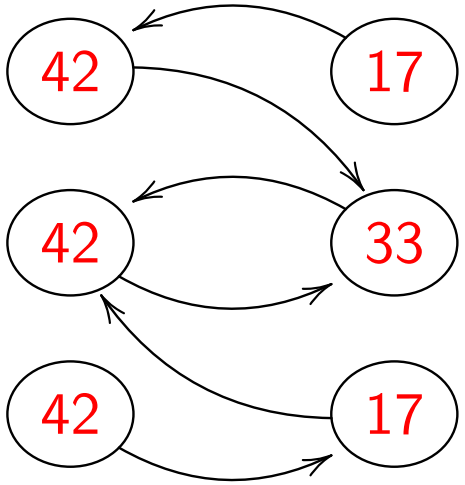

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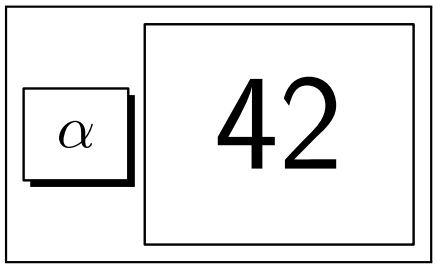

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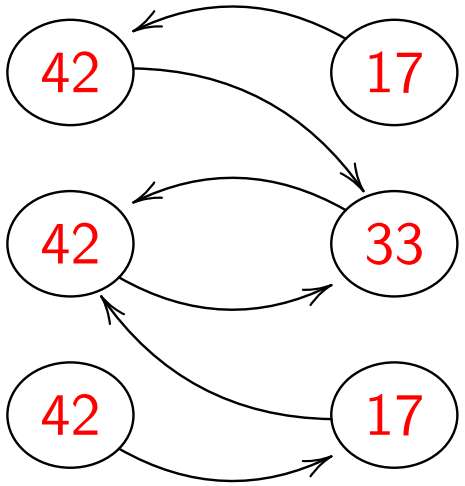

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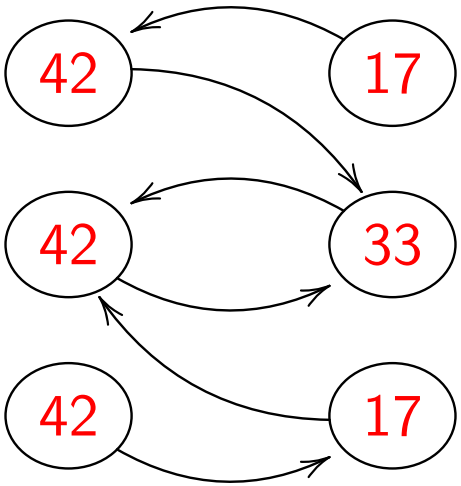

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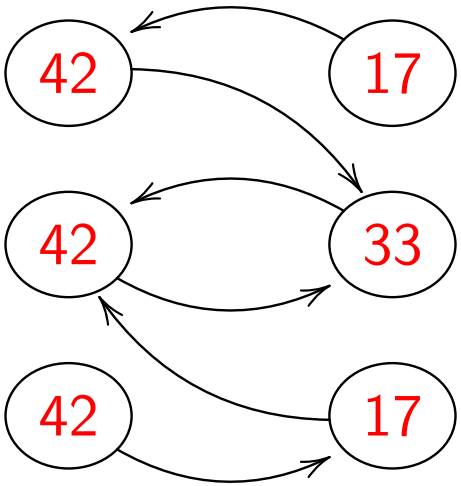

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$$[42, 33, 42, 33, \ldots]$$

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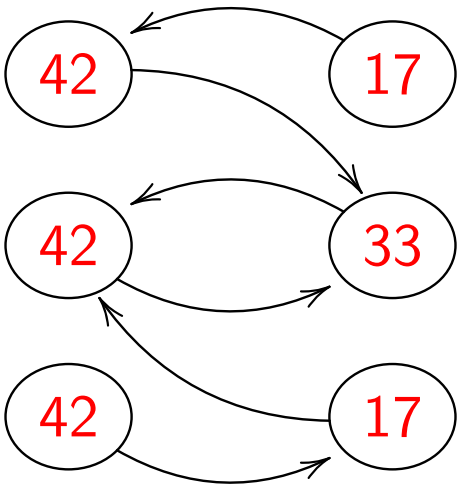

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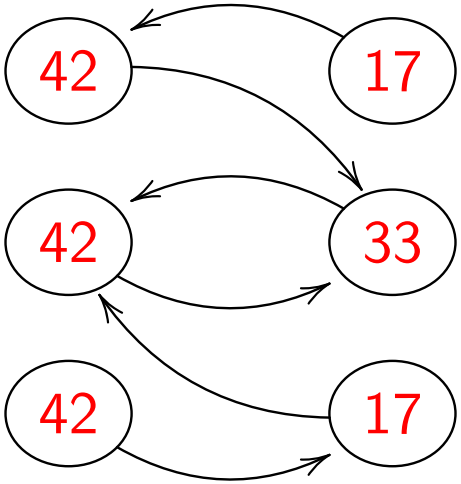

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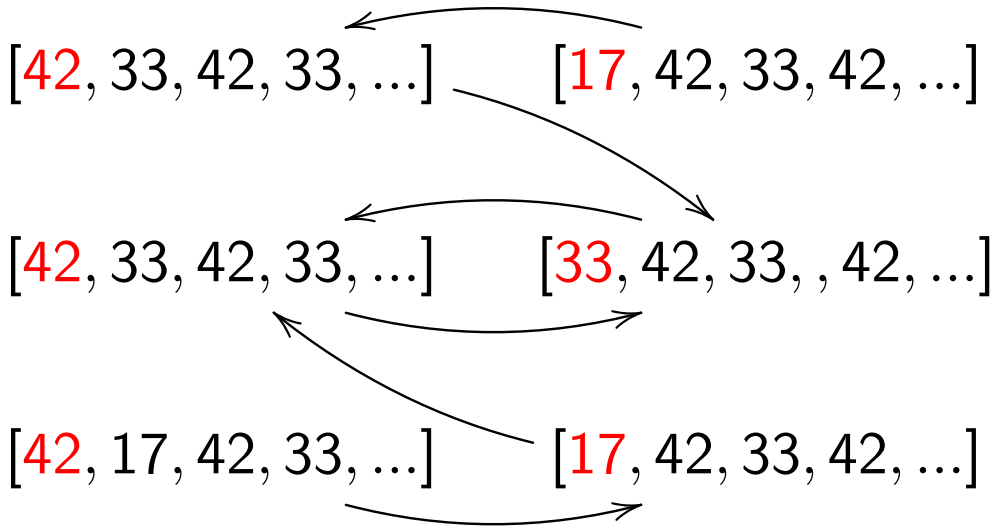

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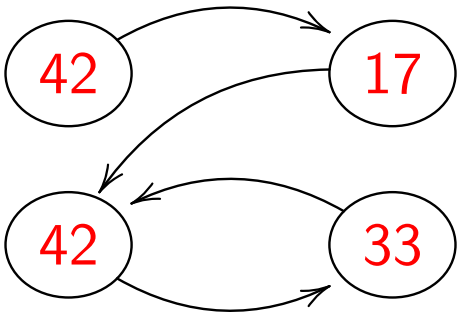

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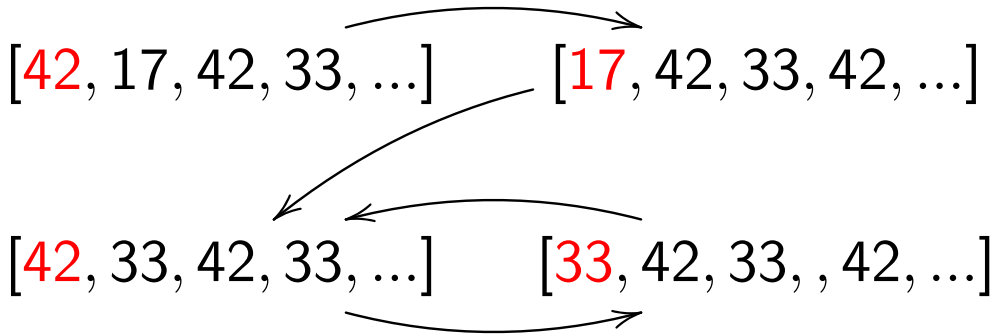

Streams are universal

D-streams $= D^\omega = D \times D^\omega$

- $hd : D^\omega \to D$
- $tl : D^\omega \to D^\omega$

For each Black Box S

- *Unique* map $\varphi : S \to D^\omega$
  - φ is *homomorphism*

$$hd(\varphi(s)) = h(s)$$
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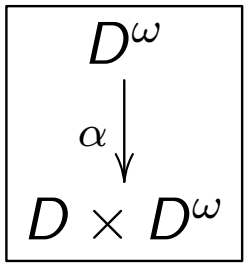

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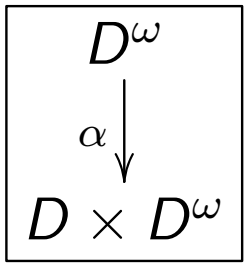

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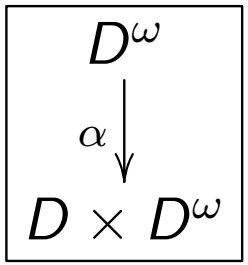

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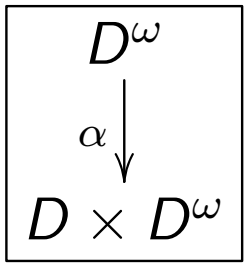

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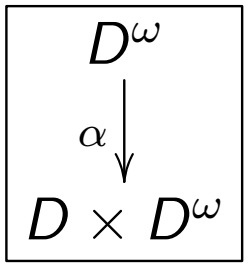

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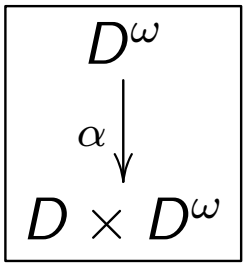

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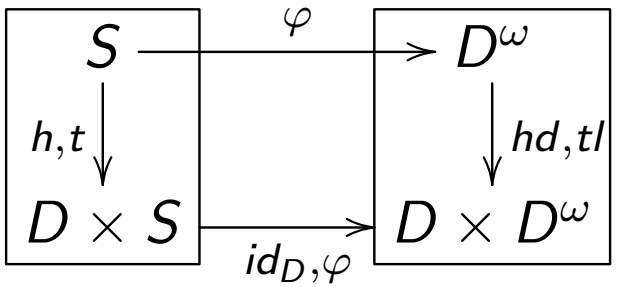

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Programming with streams

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[1,1,1,1,...]
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> from n = n : from(n+1);
> ints = from 0;
[0,1,2,3,...]
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> add u v = h(u)+h(v) : add t(u) t(v);
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- Claim: *add ones ints* $=$ *from* 1

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A Coinductive Proof

Generalize: $\mathbf{add\ ones\ (from\ n)} \sim \mathbf{from(n+1)}$

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$$\begin{aligned} hd(\mathbf{add\ ones\ (from\ n)}) &= hd(ones) + hd(from\ n) \\ &= 1 + hd(from\ n) \\ &= 1 + n \\ &= hd(\mathbf{from(n+1)}) \end{aligned}$$

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Local logic

What can be said about a state

- Formulas

$$\phi ::= \mathit{true} \mid < d > \phi, \text{ for each } d \in D$$

- semantics s

$$s \models \mathit{true}$$

$$s \models < d > \phi \quad :\Longleftrightarrow \quad h(s) = d \wedge t(s) \models \phi$$

- logical equivalence

$$s \approx s' :\Longleftrightarrow (s \models \phi \iff s' \models \phi)$$

- completeness:

$$s \sim s' \iff s \approx s'$$

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Black Box with error

- Type

$$\boxed{\begin{array}{c} S \\ \downarrow \alpha \\ D \times S + 1 \end{array}}$$

- observational equivalence

$$\frac{s \sim s'}{(\alpha(s)=1 \iff \alpha(s')=1) \vee (hd(s)=hd(s'),\ tl(s) \sim tl(s'))}$$

- Observations

$$D^\omega + D^\star$$

- ... form a black box with error:

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- Logic - correct and complete:

$$\phi ::= true \mid < d > \phi \mid error$$

Black Box with error

- Type

$$\boxed{\begin{array}{c} S \\ \downarrow \alpha \\ D \times S + 1 \end{array}}$$

- observational equivalence

$$\boxed{\frac{s \sim s'}{(\alpha(s)=1 \iff \alpha(s')=1) \vee (hd(s)=hd(s'),\ tl(s) \sim tl(s'))}}$$

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Automata (Acceptor)

- S set of states, Σ alphabet
  - $T \subseteq S$
    - $\chi_T : S \to 2$
  - $\delta : S \times \Sigma \to S$
    - $\hat{\delta} : S \to S^\Sigma$
- Type
- Homomorphisms $\varphi : \mathcal{A} \to \mathcal{B}$
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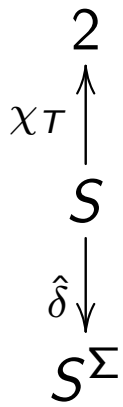

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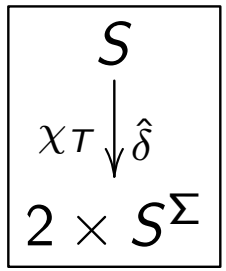

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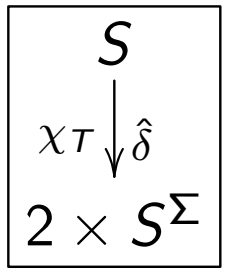

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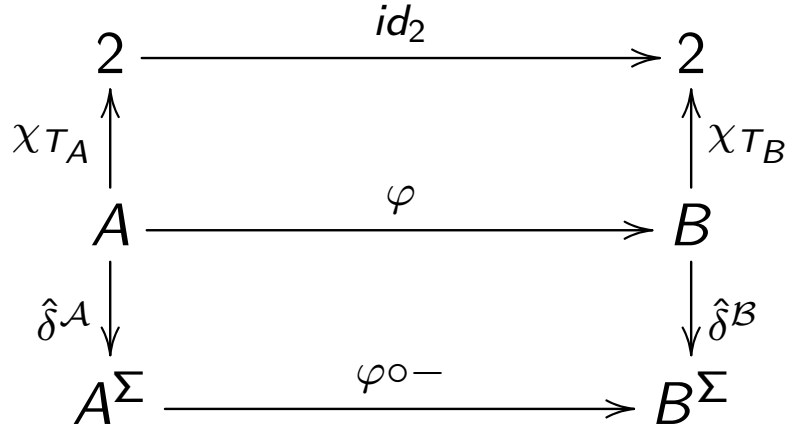

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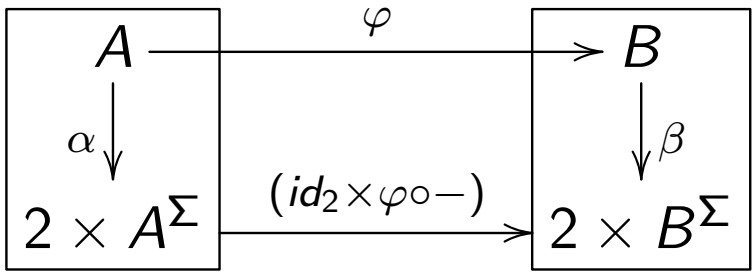

Observations are languages

- Observation at s:

$$L(s) := \{w \in \Sigma^\star \mid \delta^\star(s, w) \in T\}$$

- $s \sim s' \iff L(s) = L(s')$
- Indistinguishability relation $\sim$

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Languages are universal

- $\mathbb{P}(\Sigma^\star)$ = Set of all *languages* with alphabeth Σ
  - $\delta : \mathbb{P}(\Sigma^\star) \times \Sigma \to \mathbb{P}(\Sigma^\star)$
    - $\delta(L, e) := L_e = \{u \in \Sigma^\star \mid e \cdot u \in L\}$ *derivative*
    - $T = \{L \in \mathbb{P}(\Sigma^\star) \mid \varepsilon \in L\}$
- For every Σ-automaton $\mathcal{A}$ there is a unique automata homomorphism $\varphi : \mathcal{A} \to \mathbb{P}(\Sigma^\star)$
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Rule of coinduction

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- Can be used to prove language equalities, e.g.

$$L^* \cdot M = L \cdot (L^* \cdot M) + M$$

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Object oriented programs

- State encapsulated in objects
  - Mutators, Observers

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class Account{
   private int balance = 0;
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- Do'nt care whether
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  - not observable
- Insist that
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Moore-Automata

- S set of states, Σ alphabet (input), D set of outputs
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- in state s word $w = e_1 \cdot e_2 \cdot \ldots \cdot e_n$ gives output $\gamma(\delta^\star(s, w))$
  - observations at s: $D - \mathit{labelled}, \Sigma - \mathit{branching}$ tree:

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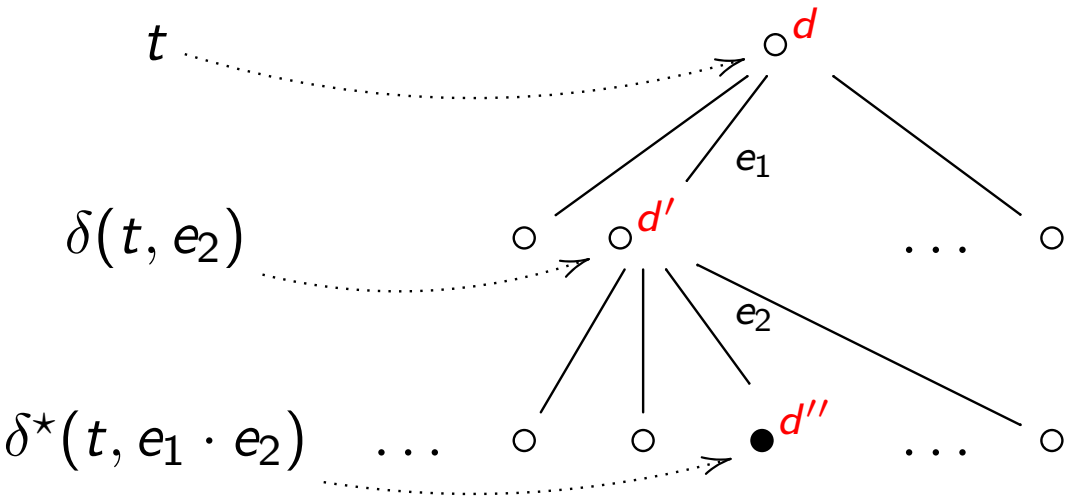
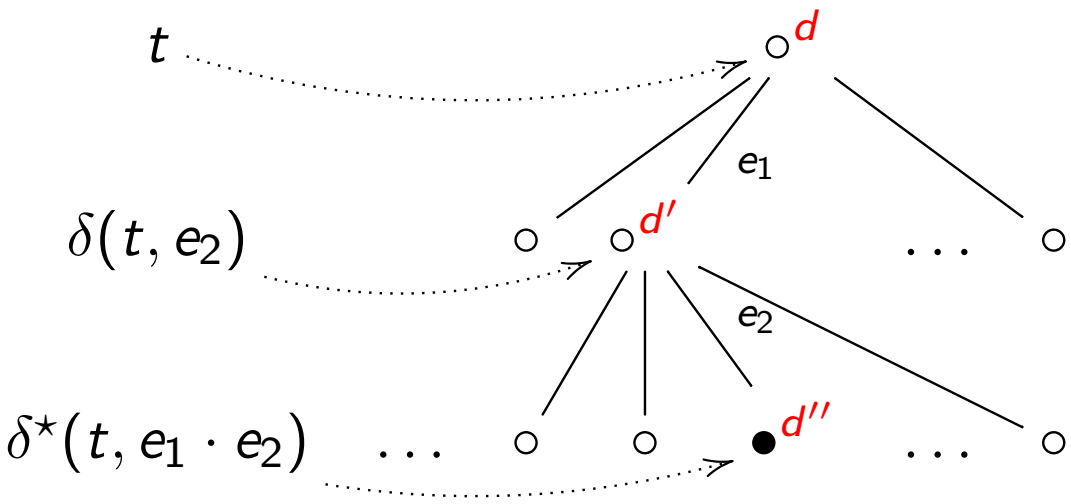


D-labelled Σ-branching trees

- Automaton of all $a\Sigma - branching\ D - labeled$ trees: $D^{\Sigma^\star}$



- Each node uniquely determined by its "*address*" $w \in \Sigma^\star$
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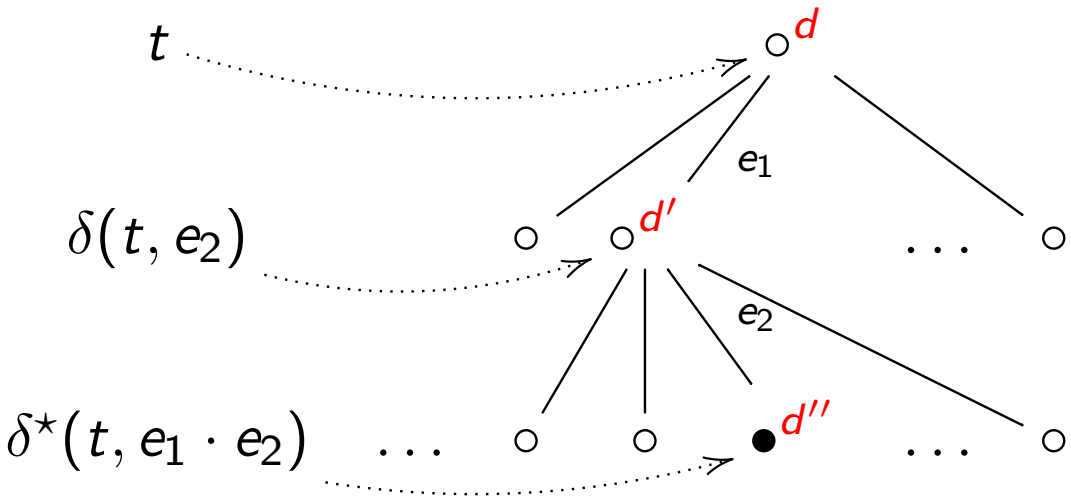

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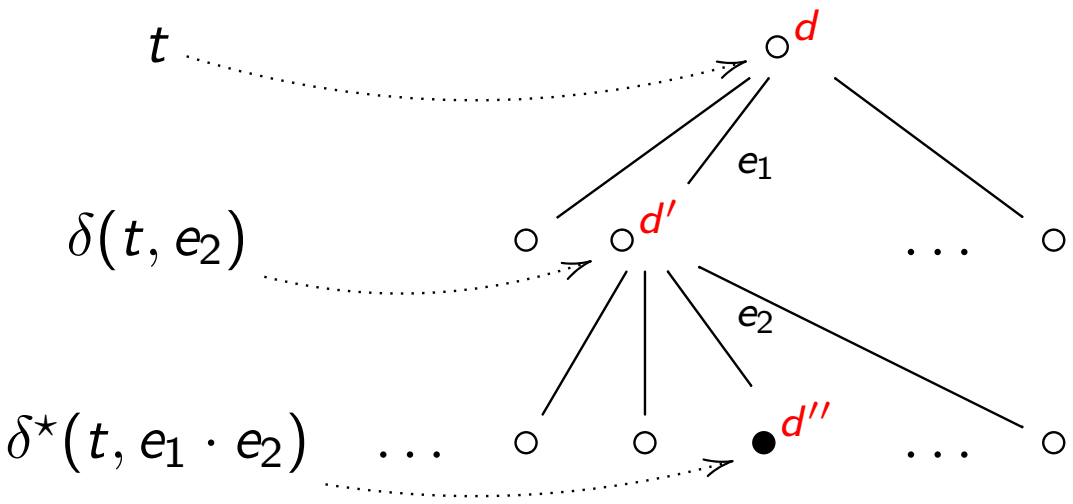

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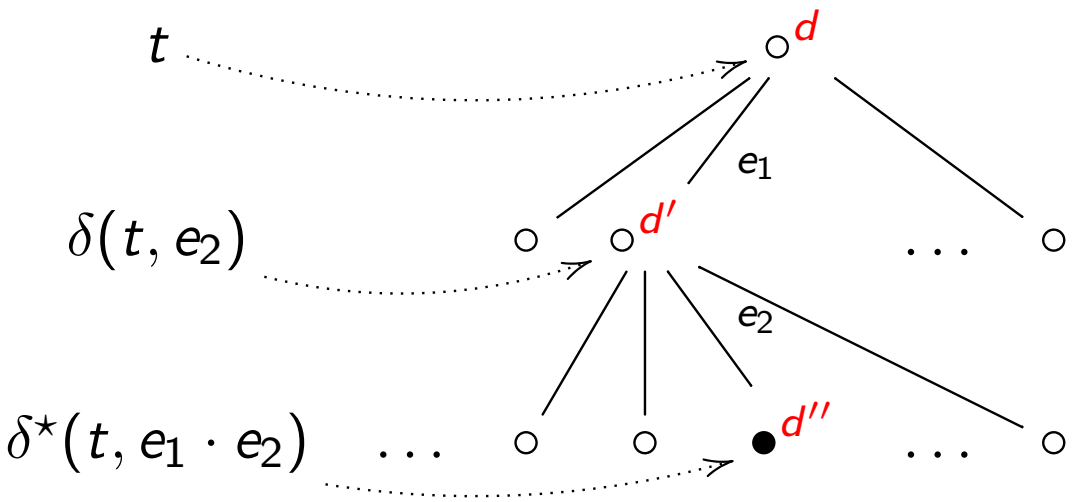

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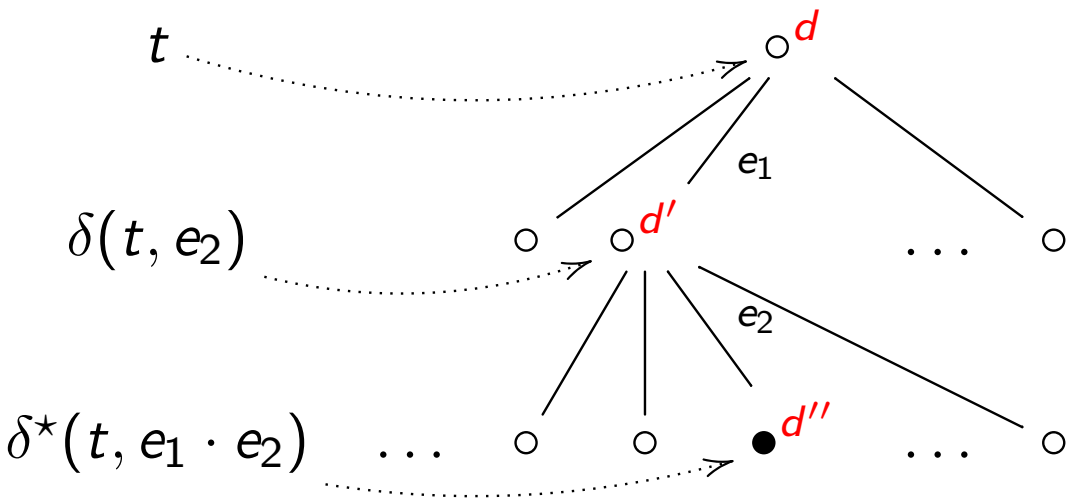

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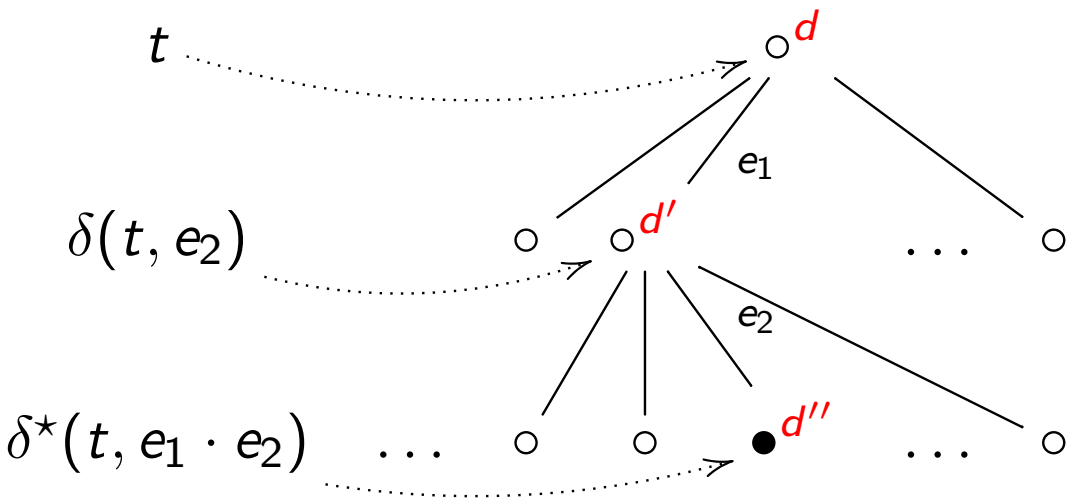

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  - $\varphi(a)(\varepsilon) = \gamma(\tau(a)) = \gamma(a)$
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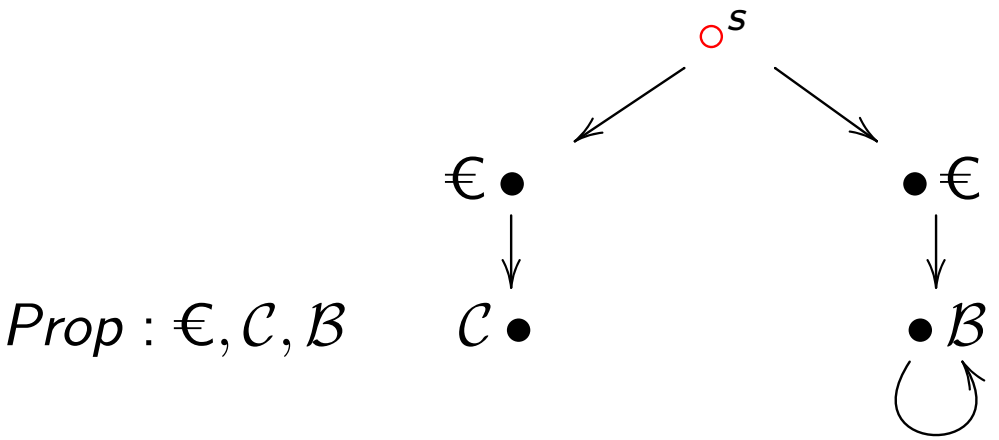

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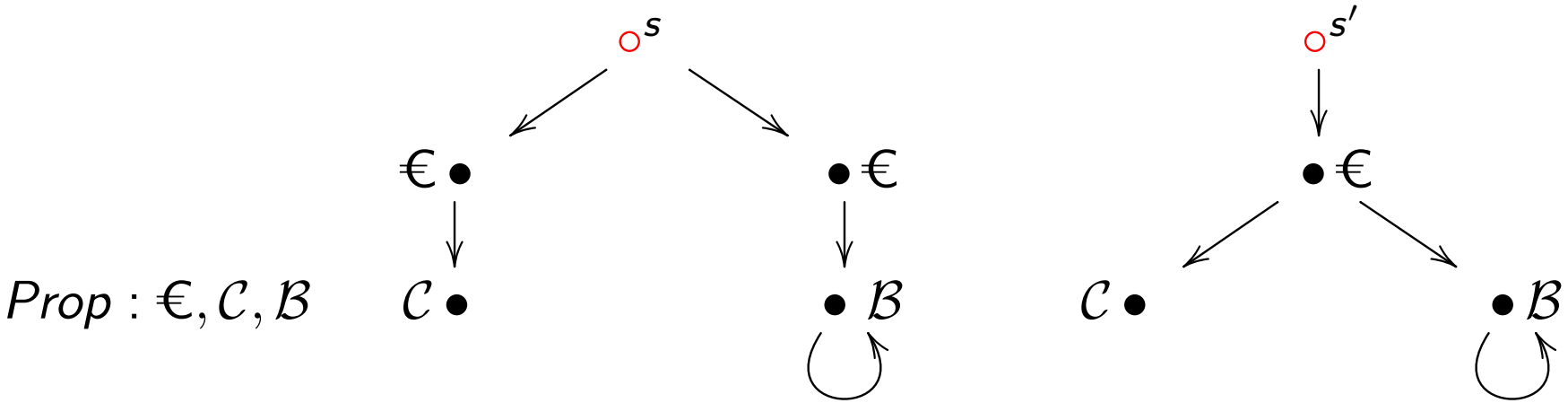

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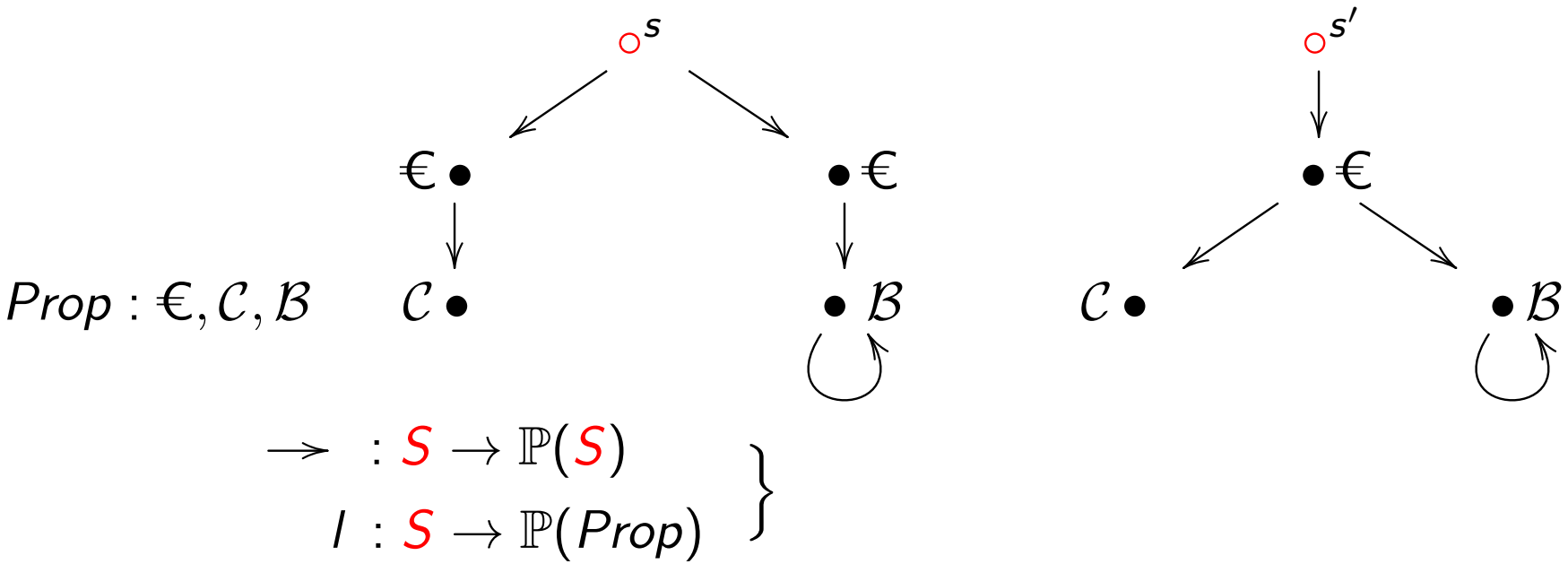

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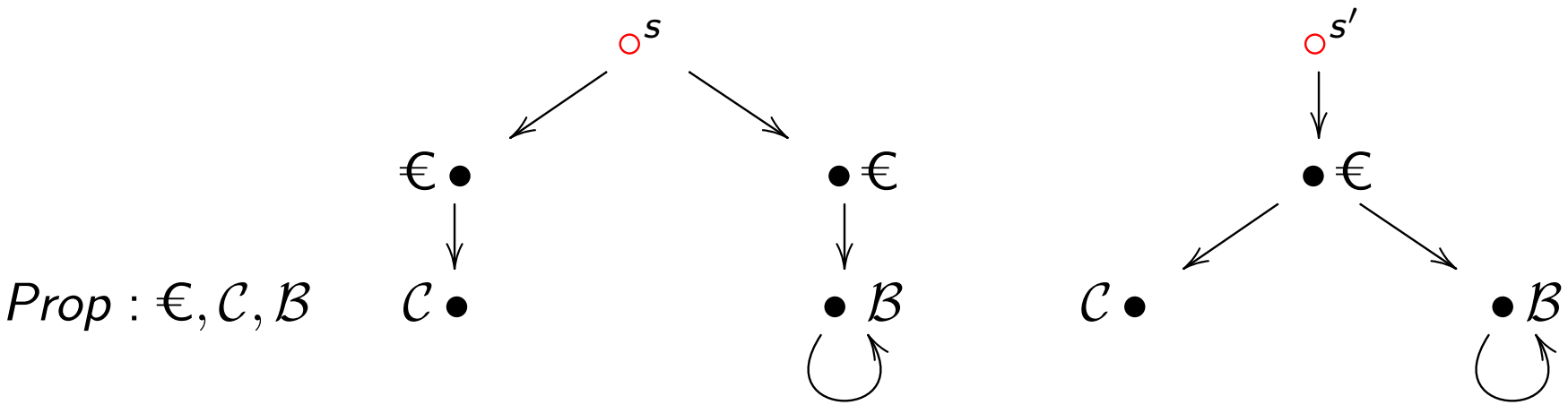

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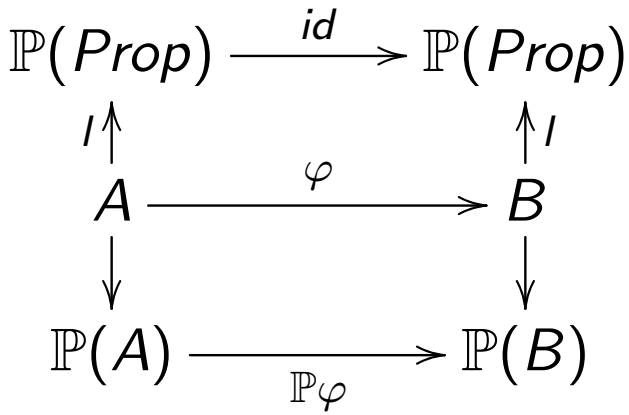

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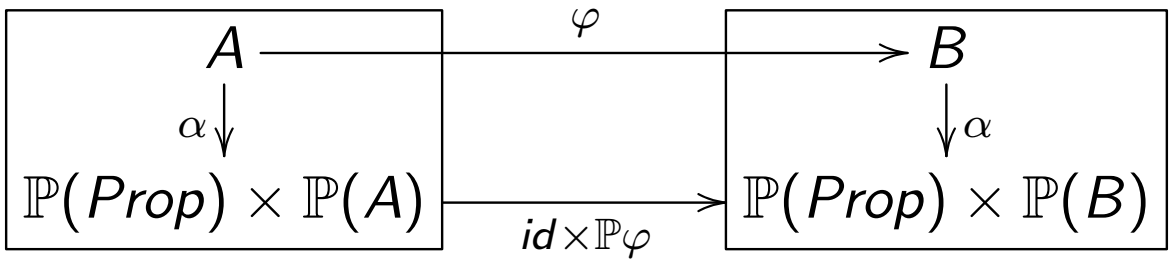

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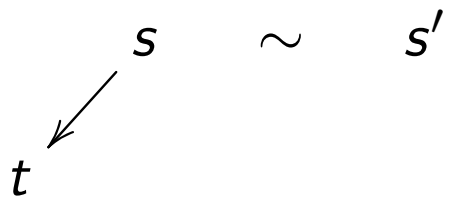

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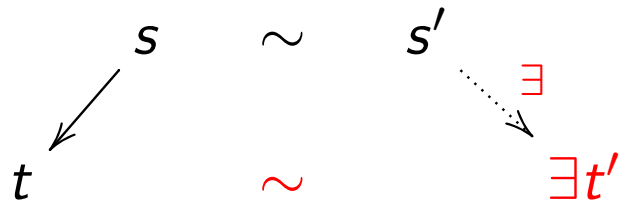

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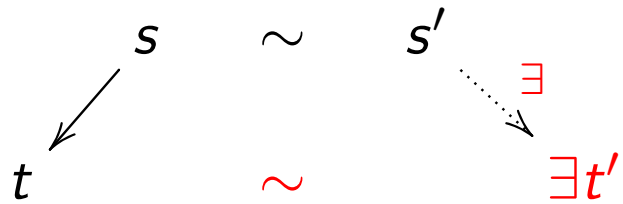

...and symmetrically

$$\frac{s \sim s'}{s' \sim s \wedge l(s) = l(s') \wedge \forall t.\, s \to t \implies \exists t'.\, s' \to t' \wedge t \sim t'}$$

Modal Kripke Logic

- Formulae

$$\begin{array}{rcl} \phi & :: & p \in AP \\ & | & \mathit{true} \mid \neg\phi \mid \phi_1 \vee \phi_2 \\ & | & \Box\phi \mid \Diamond\phi \end{array}$$

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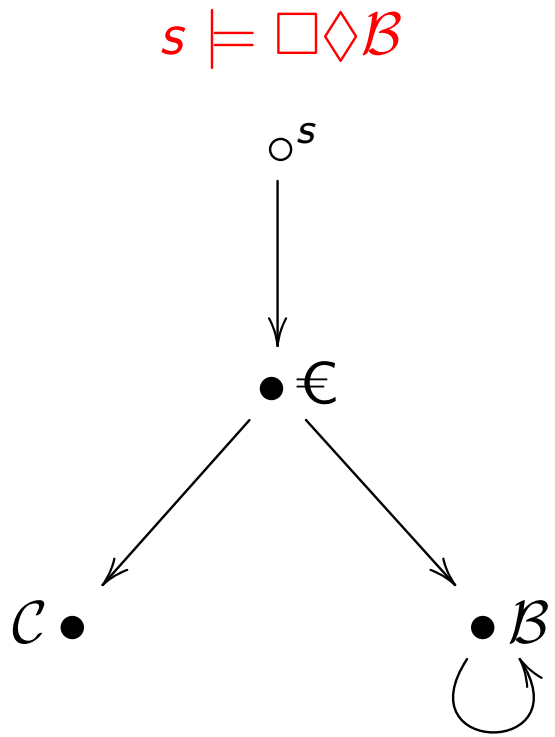

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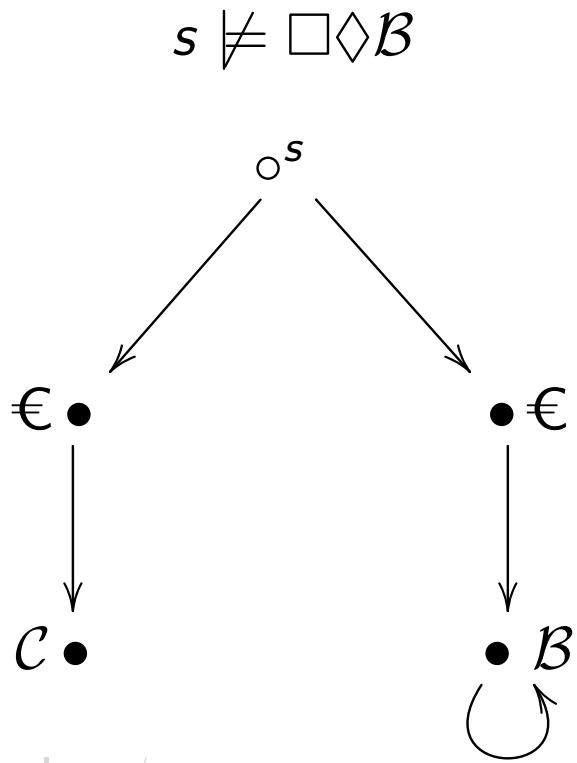

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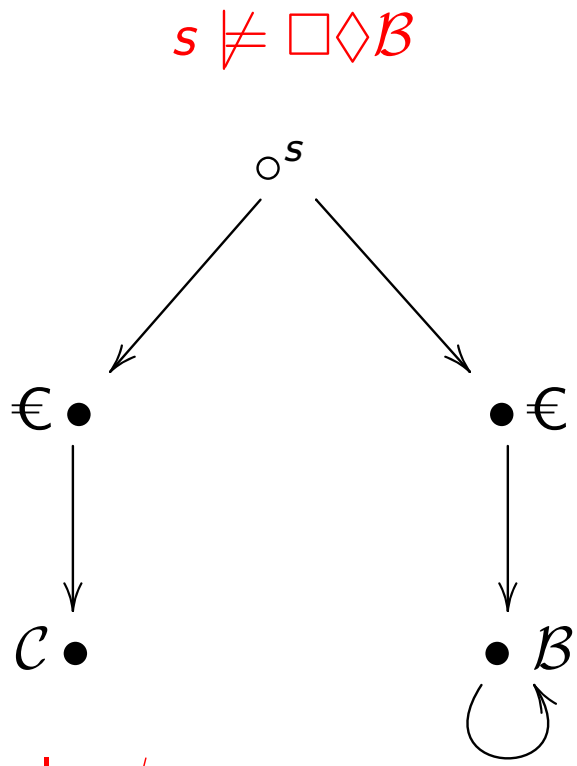

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Hennessy-Milner Theorem

- Adequacy: $s \sim t \implies s \approx t$
- Expressiveness: $s \approx t \stackrel{?}{\implies} s \sim t$
  - needs bounded nondeterminism

Theorem (Hennessy,Milner)

If $\mathcal{K}$ is image − finite then $s \approx t \iff s \sim t$.

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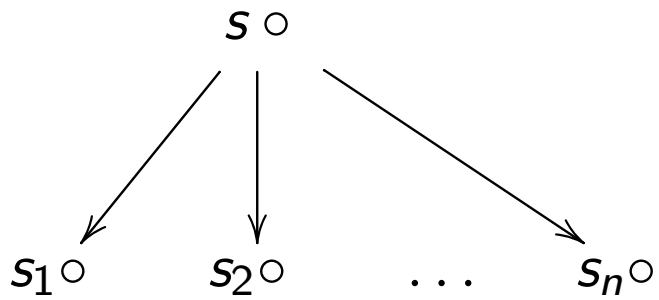

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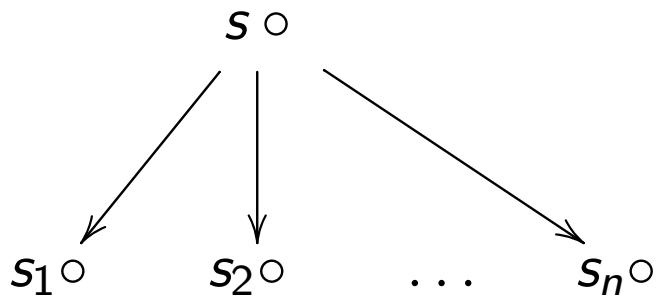

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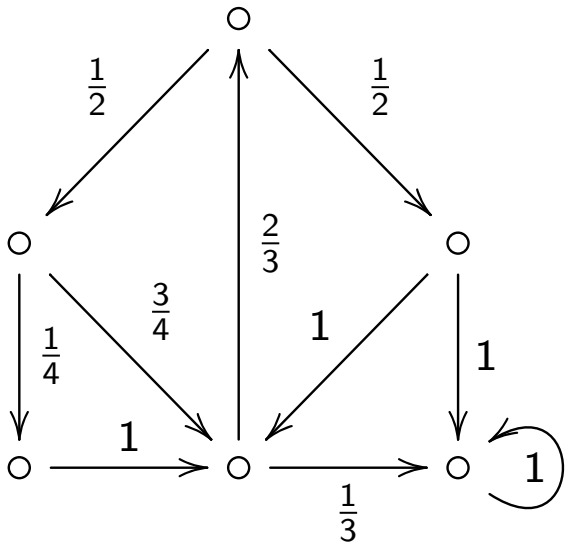

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- $f : S \times S \to [0,1]$
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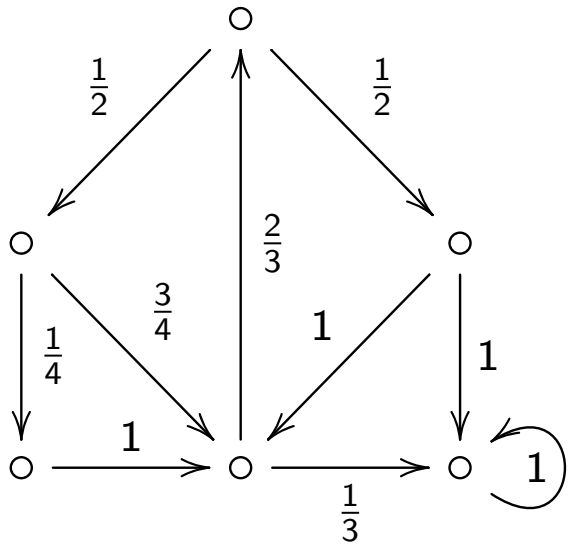

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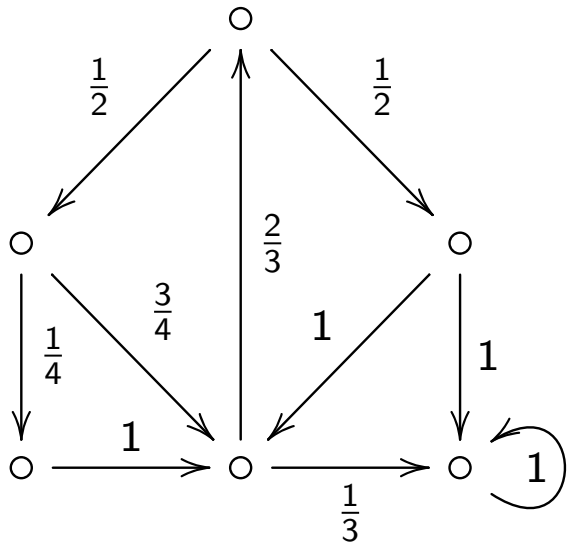

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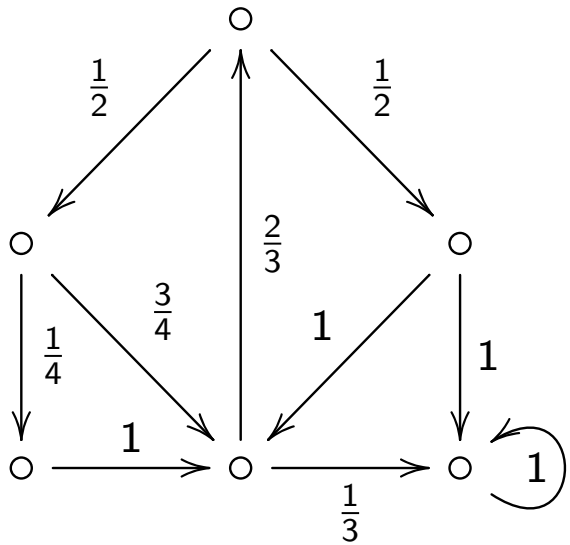

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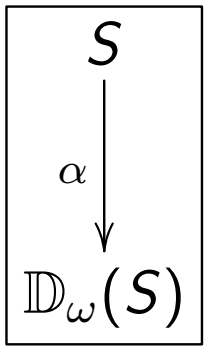
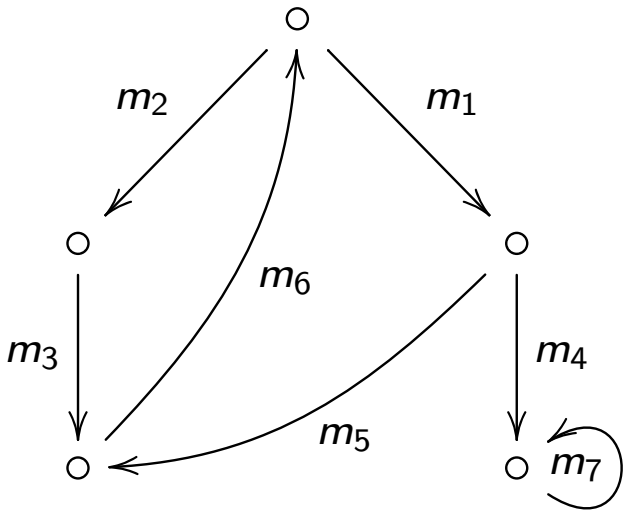


Monoid labeled systems

$\mathcal{M}$ any commutative monoid



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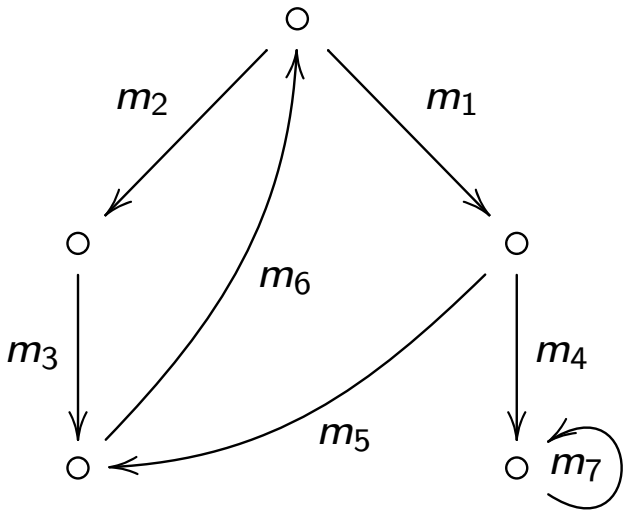
- $\forall s \in S. \sum_{s' \in S} f(s, s')$ exists

$$S \xrightarrow{\alpha} \mathcal{M}_\omega^S$$

Monoid labeled systems

$\mathcal{M}$ any commutative monoid



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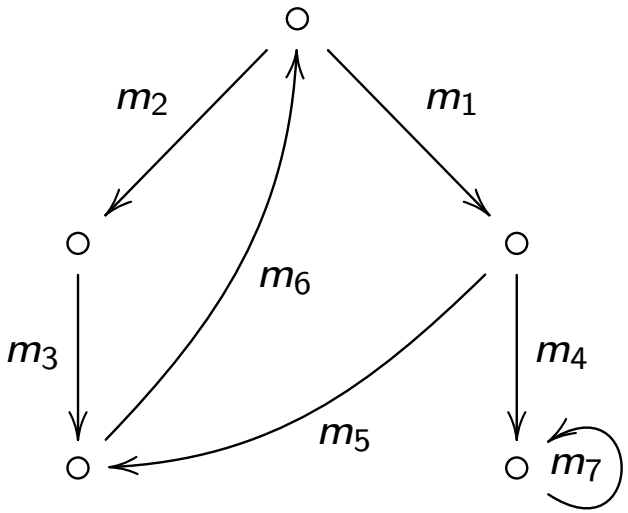

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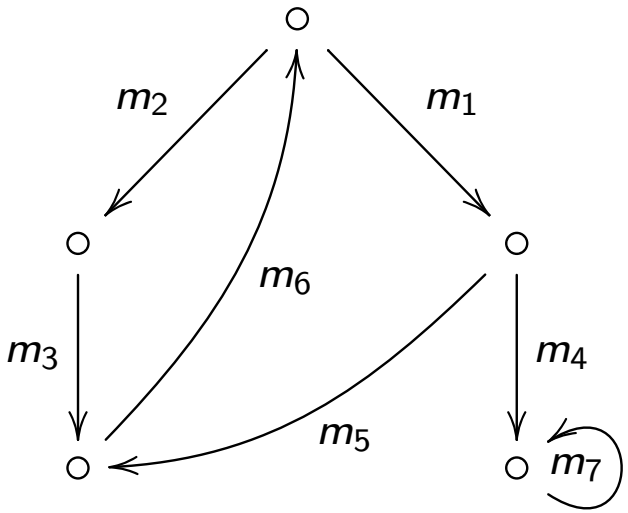

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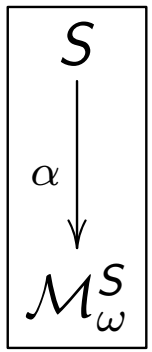
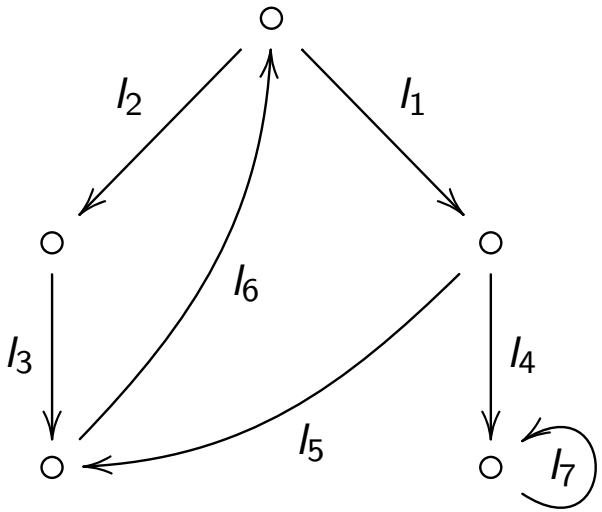


Lattice labeled systems

$\mathcal{L}$ any complete lattice



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$$\begin{array}{c} S \\ \downarrow \alpha \\ \mathcal{L}^S \end{array}$$

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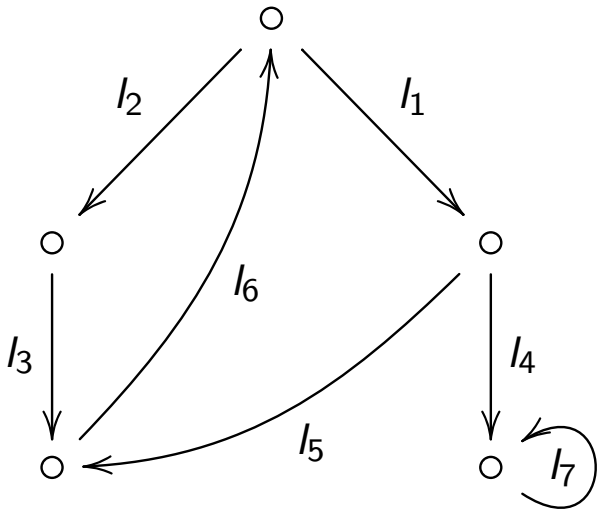

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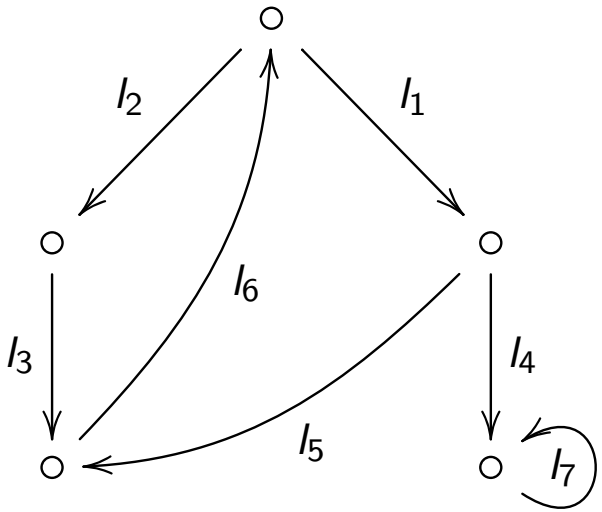

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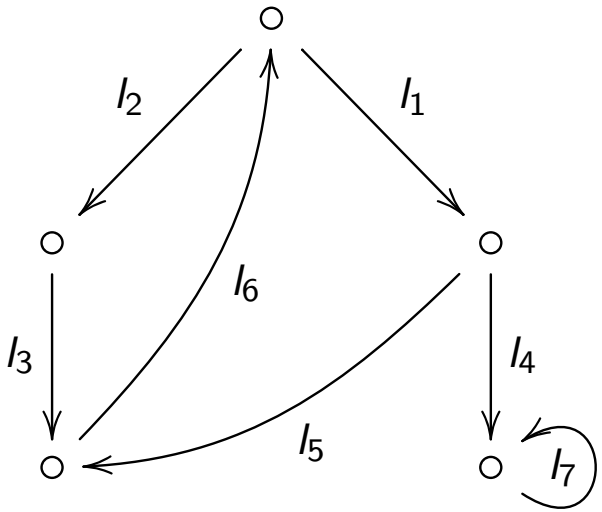

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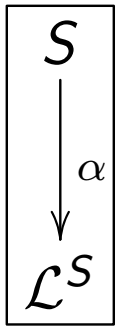

Systems to coalgebras

Name	**Structure**	**Coalgebra**
Black Box	$h : S \to D$ $t : S \to S$	
Acceptor		
Bank Account		
Transition System		
Probabilistic system		
topological space		
F-coalgebra		

Systems to coalgebras

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Systems to coalgebras

Name	**Structure**	**Coalgebra**
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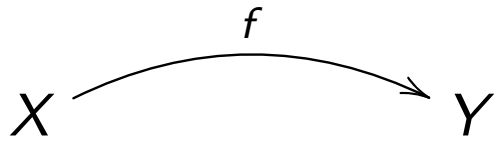
F-coalgebra		$S \to F(S)$

Relevant Properties of $\mathcal{S}et$

- epi-regular mono factorization



- $E - M$-property

- currying-uncurrying
- Axiom of choice

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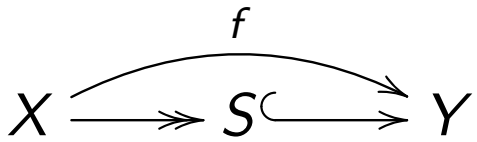

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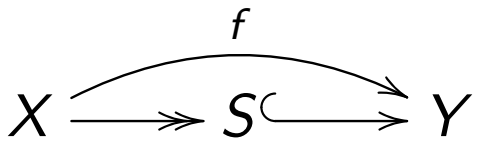

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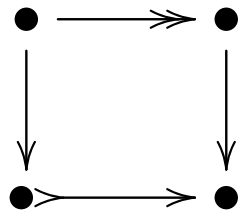

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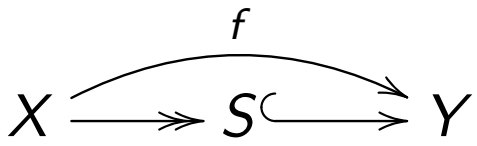

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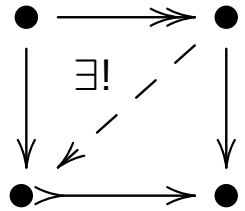

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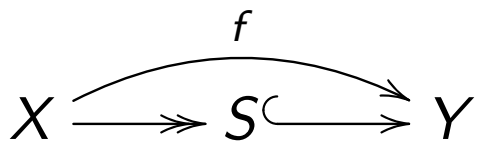

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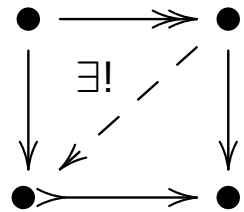

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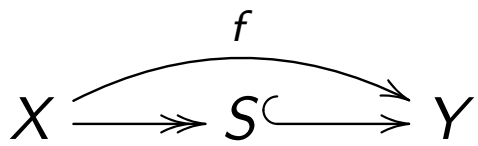

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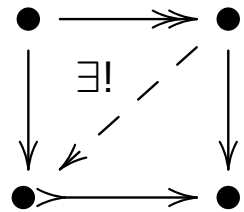

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The notion of F-coalgebra

- Type (signature) F:
  - $F : \mathcal{S}et \to \mathcal{S}et$
- F-Coalgebra
  - $\alpha : A \to F(A)$
- Homomorphism
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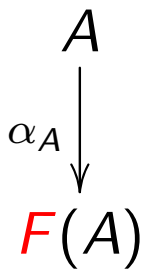

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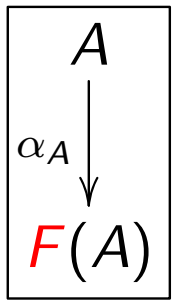

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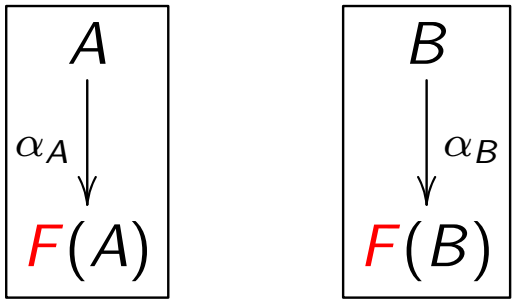

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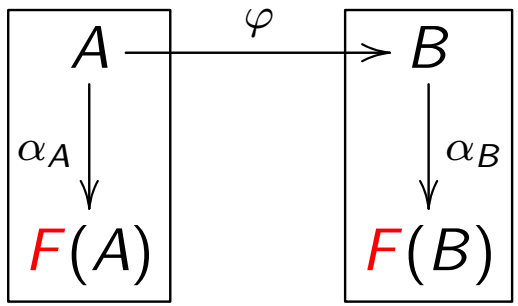

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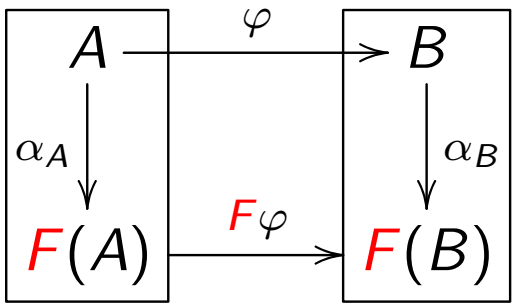

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$$\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$$

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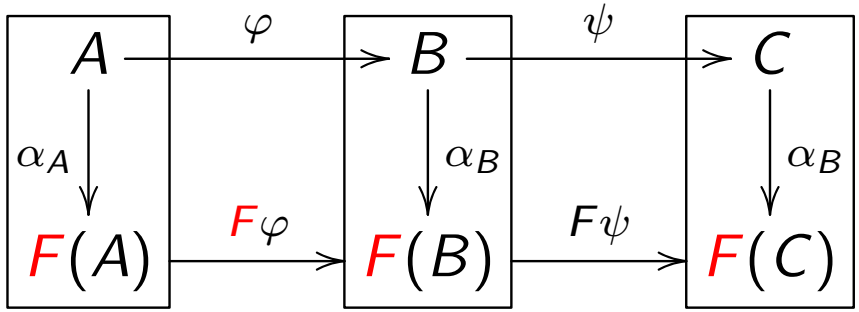

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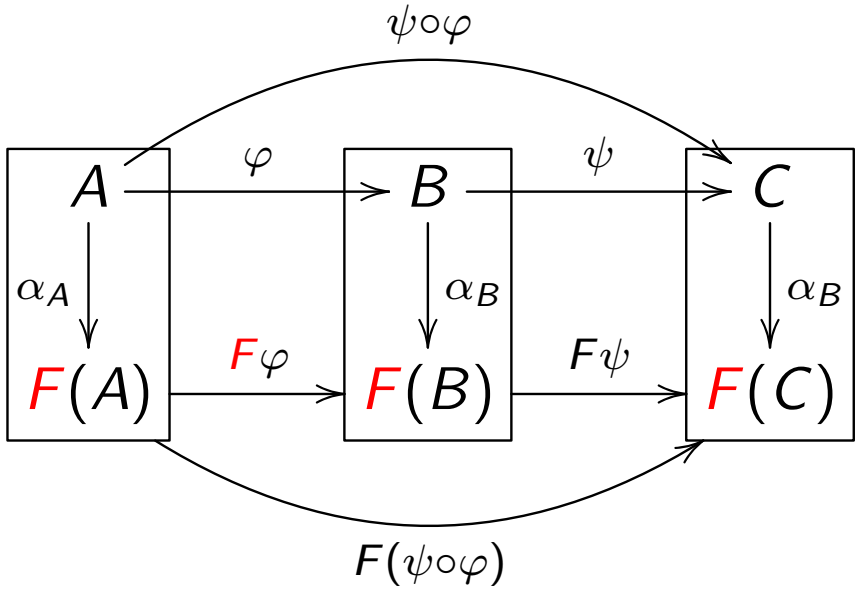

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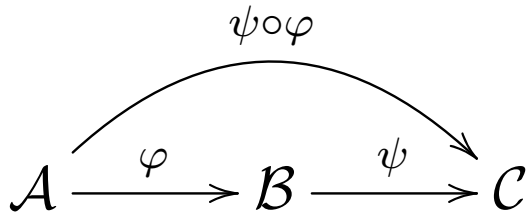

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- $\varphi : \mathcal{A} \to \mathcal{B}$ homomorphism.
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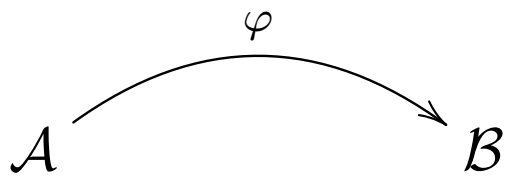

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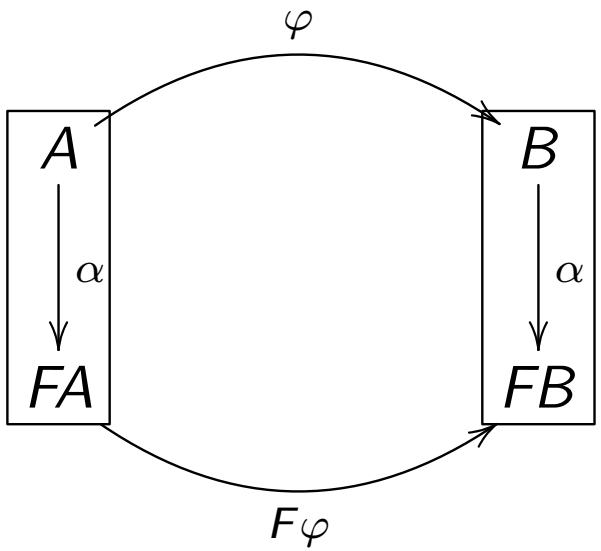

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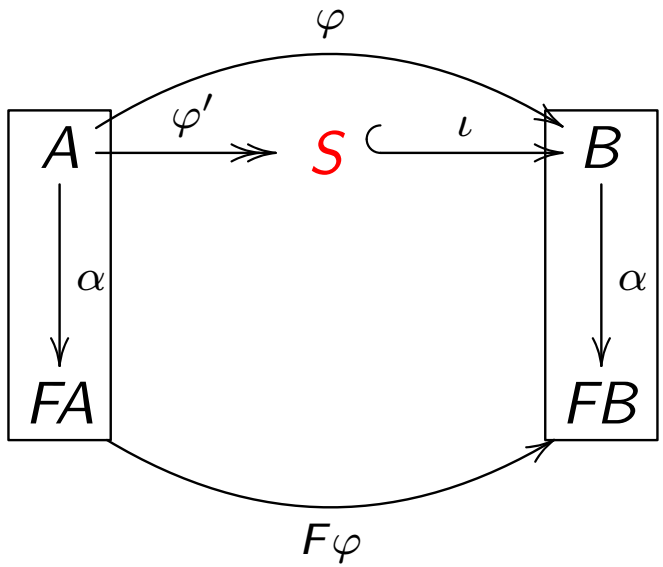

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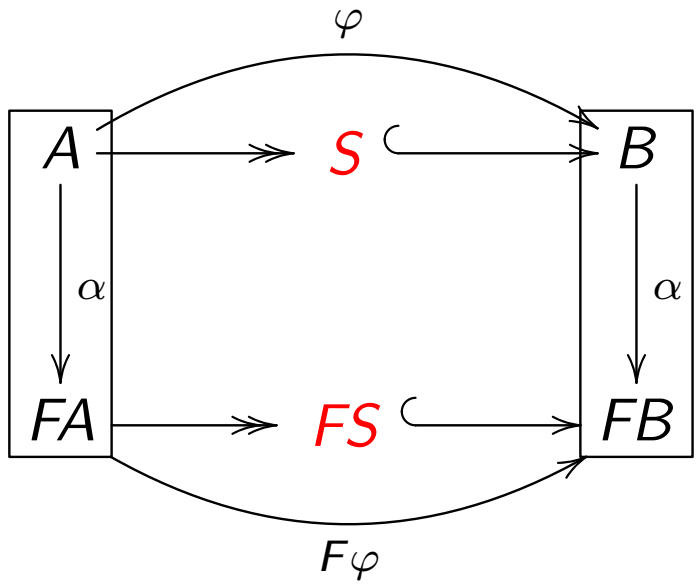

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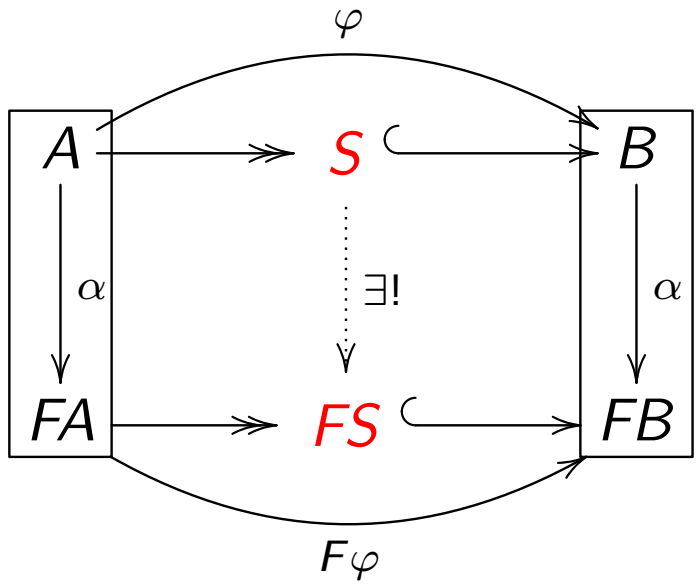

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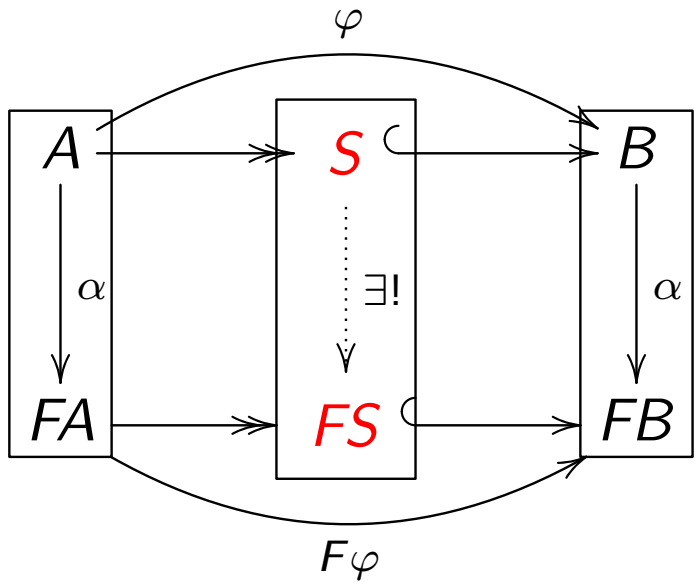

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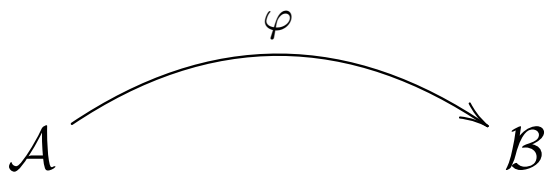

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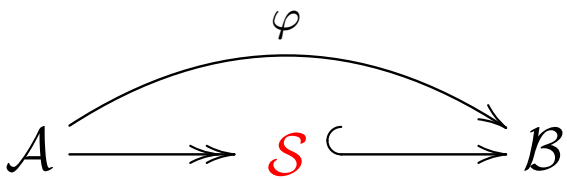

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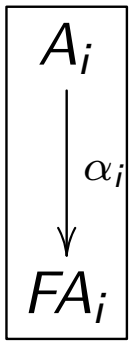

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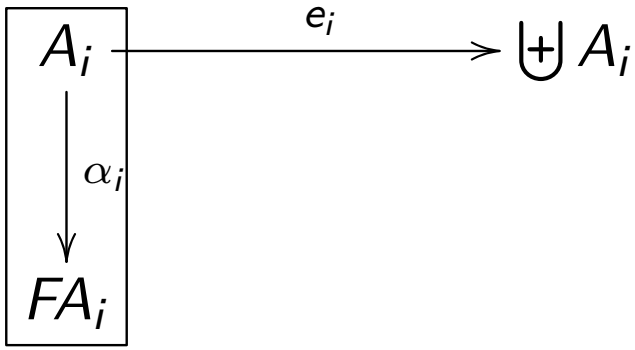

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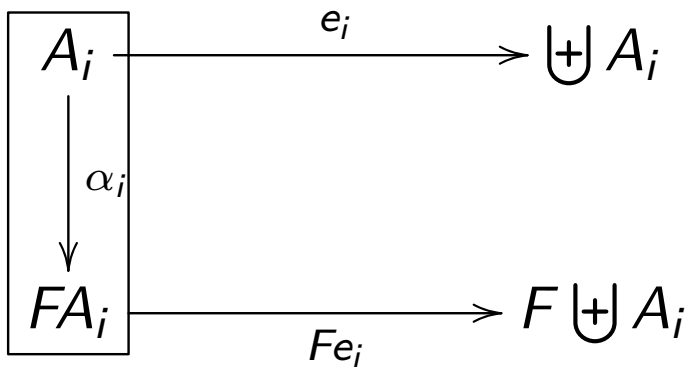

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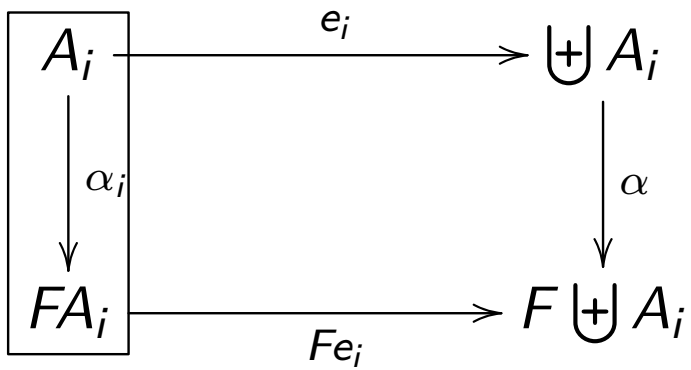

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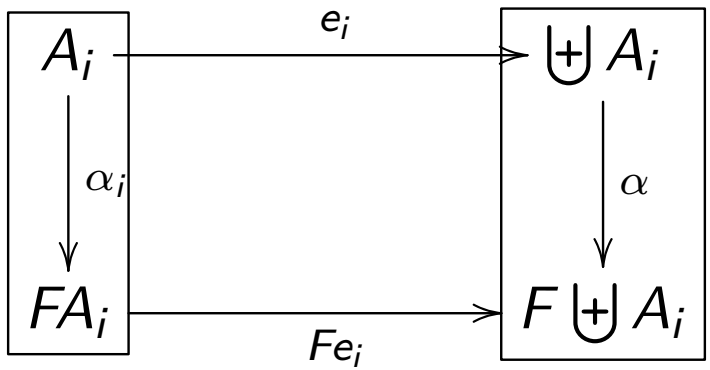

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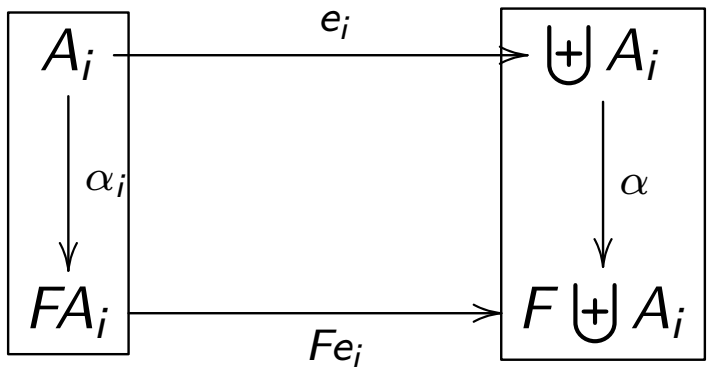

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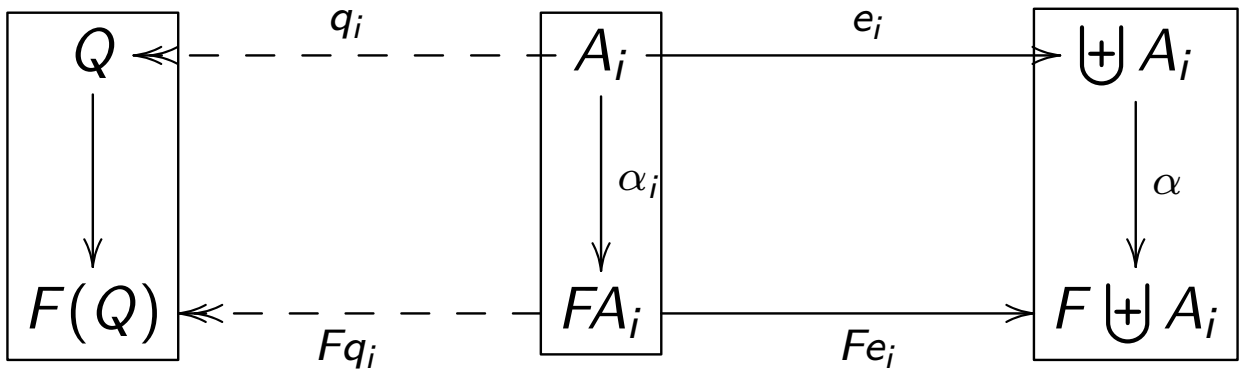

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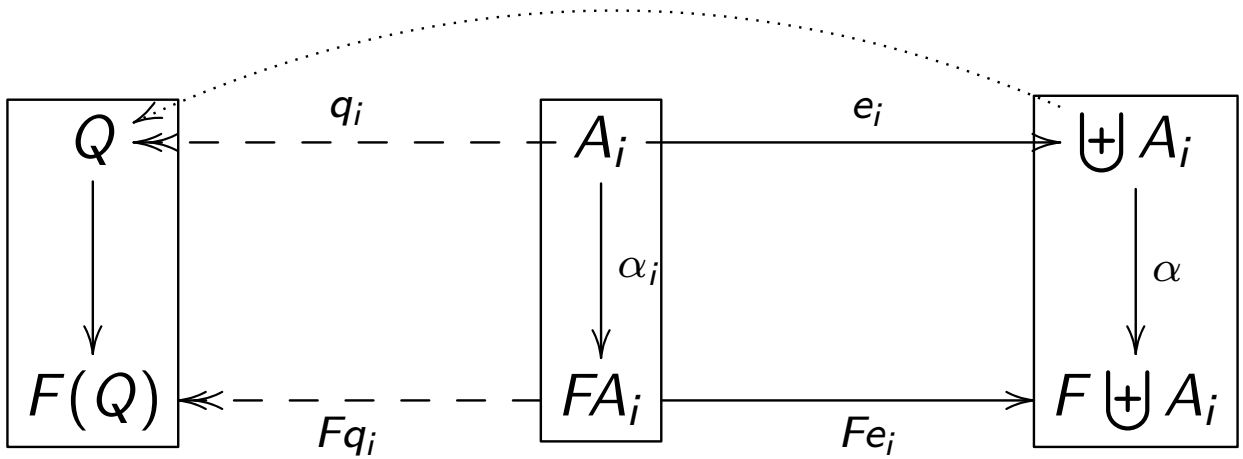

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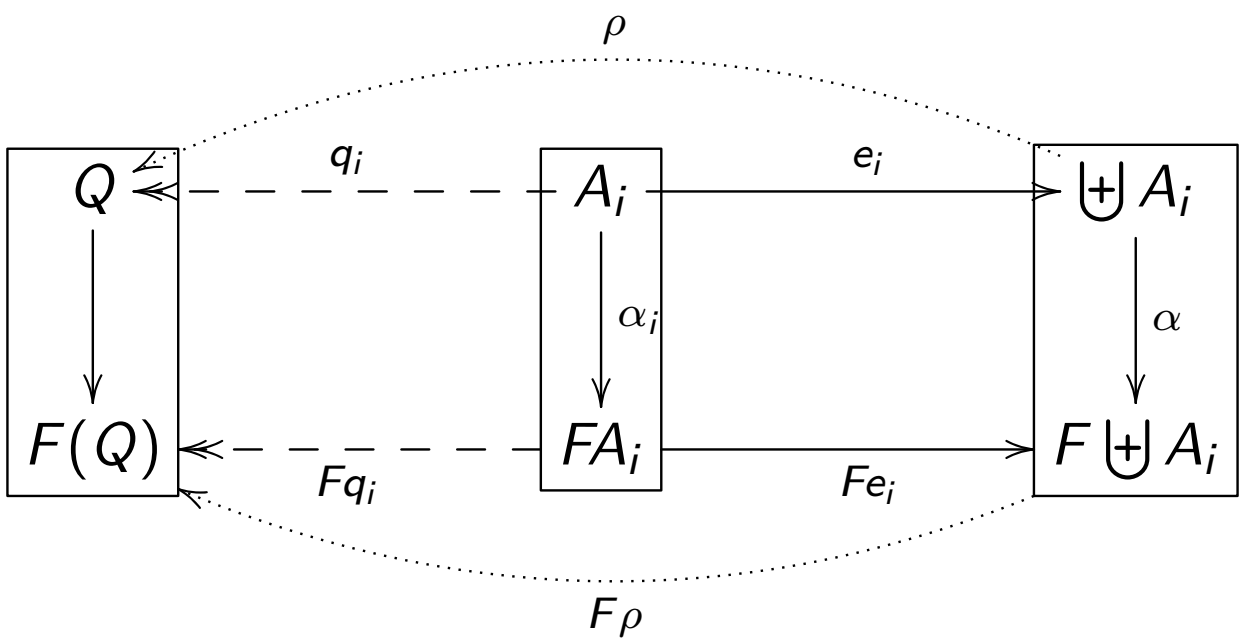

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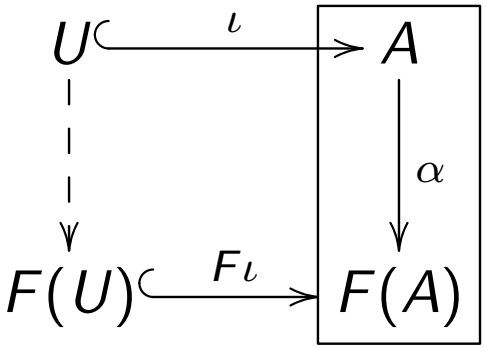
$\mathcal{A} = (A, \alpha)$ coalgebra

- U subset with coalgebra structure and inclusion homomorphism



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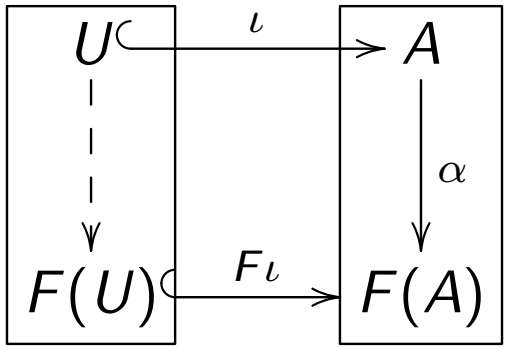

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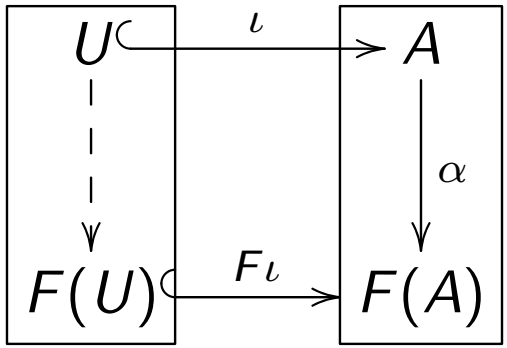

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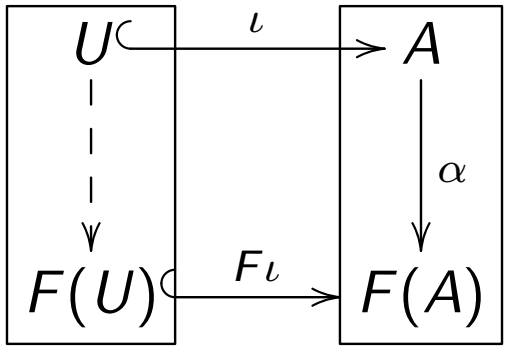

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- Epimorphism $\varphi : \mathcal{A} \twoheadrightarrow \mathcal{B}$
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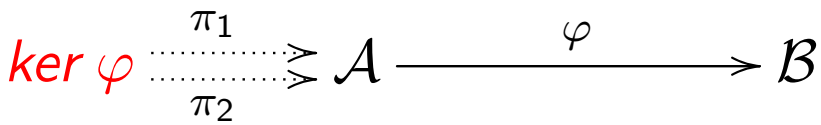

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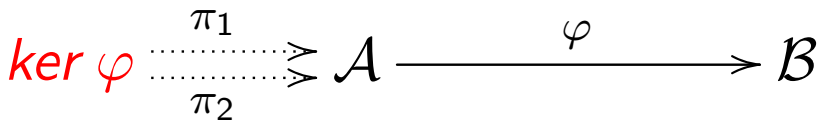

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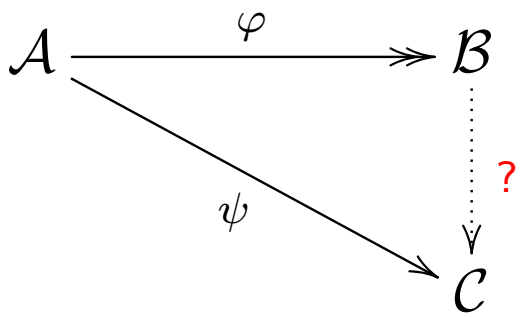

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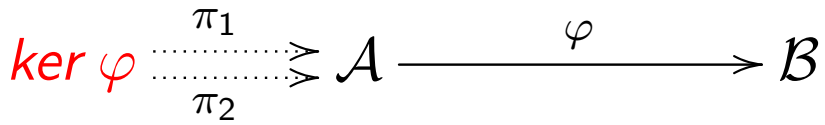

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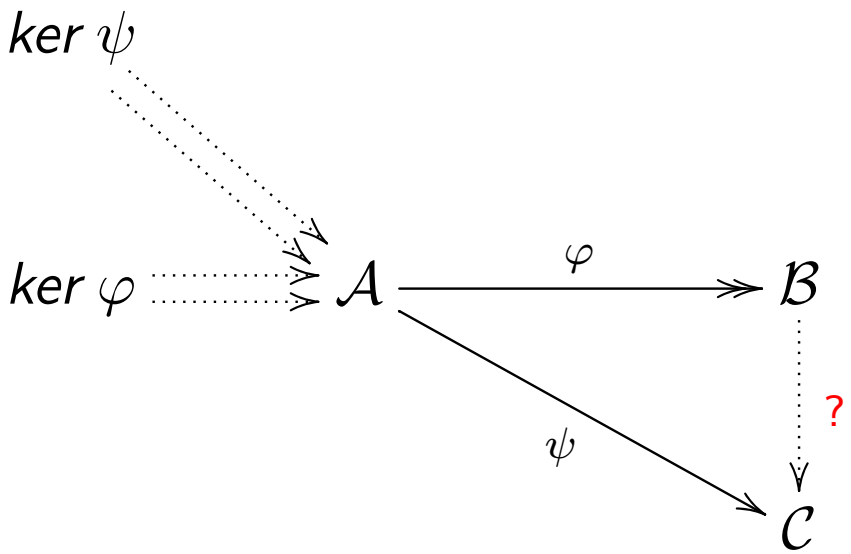

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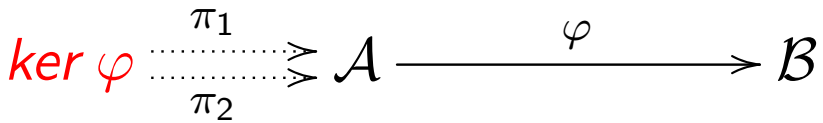

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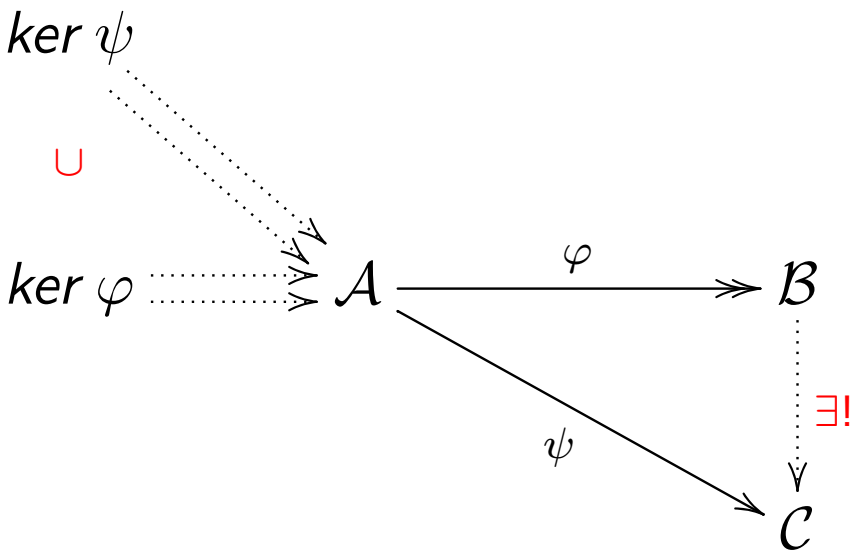

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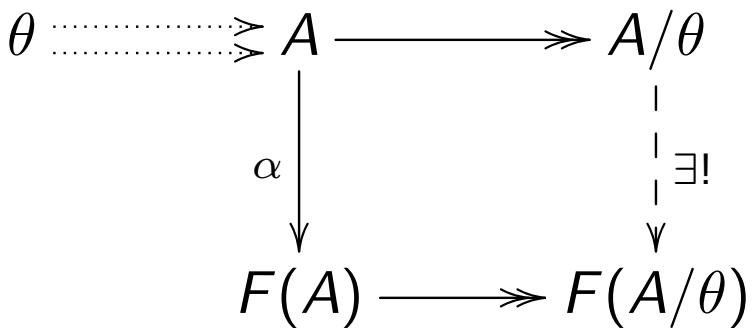

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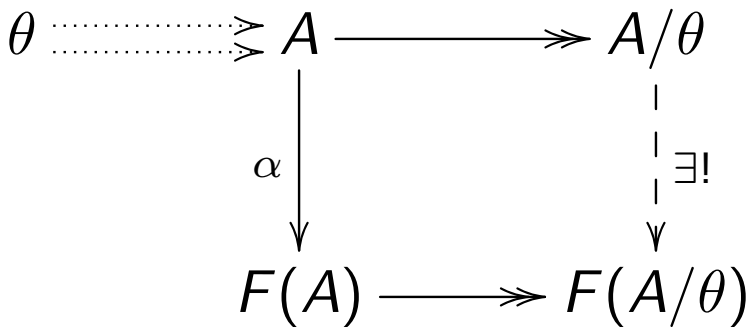

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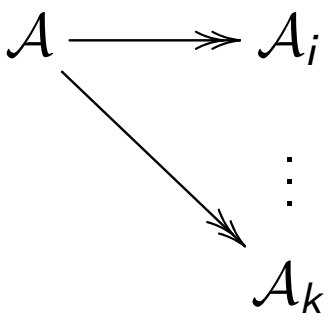

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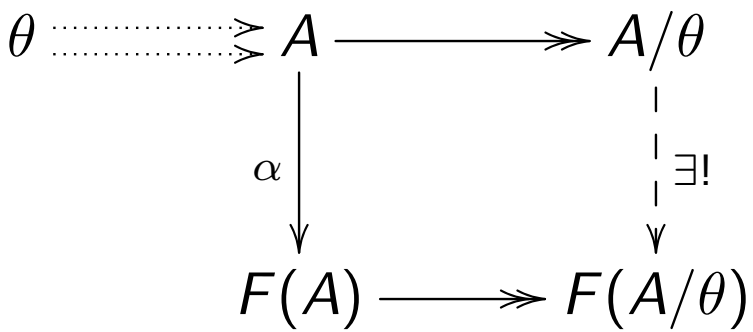

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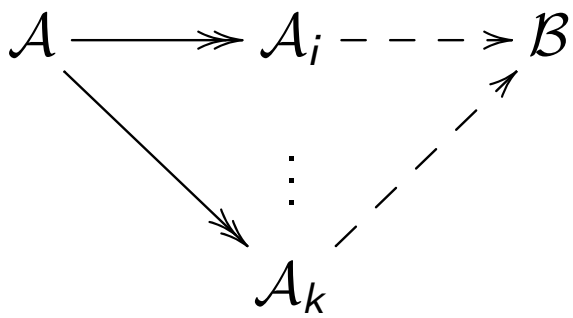

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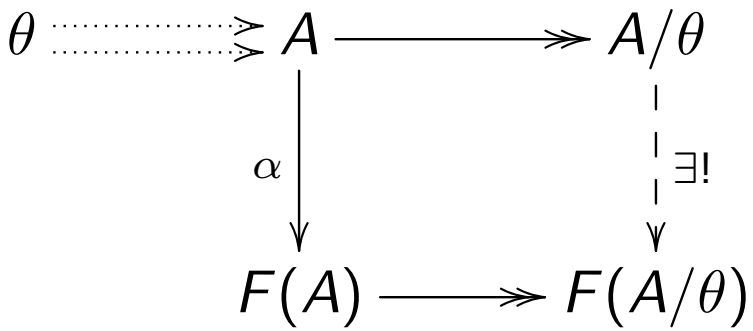

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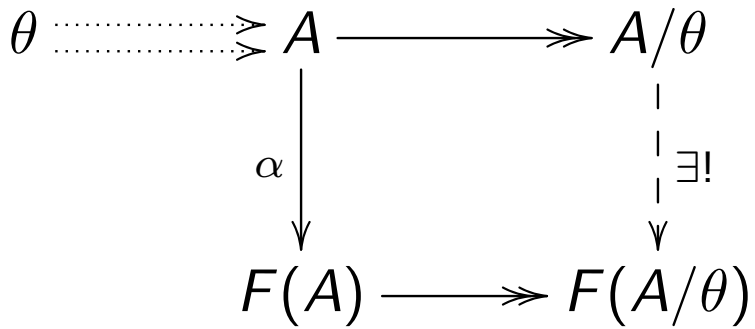
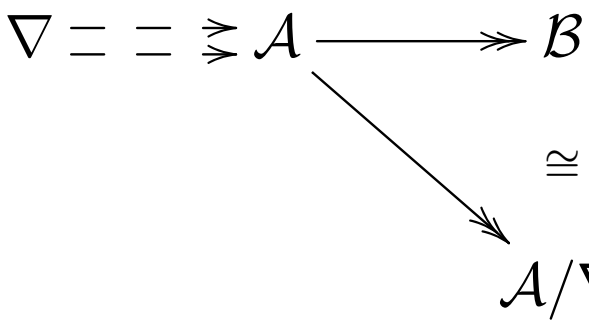


- Smallest factor = Colimit of all factors



minimal

- largest congruence ∇

Some Limits exist

- Equalizers
  ... start with set $E := \{a \in A \mid \varphi_1(a) = \varphi_2(a)\}$ then ...

$$E \hookrightarrow \mathcal{A} \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} \mathcal{B}$$

- Preimages

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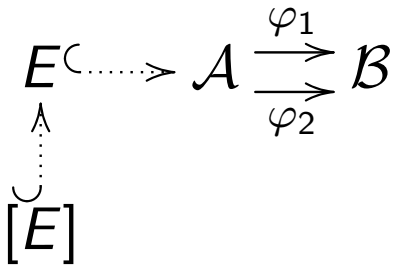

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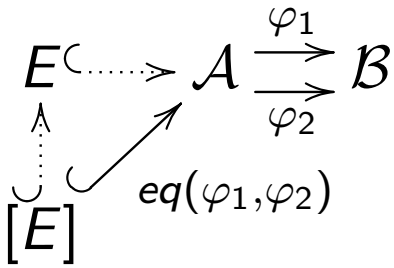

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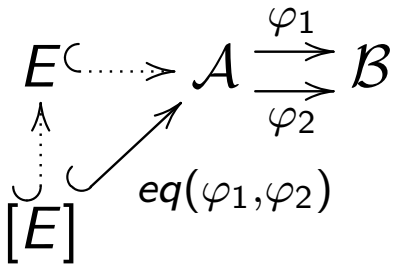

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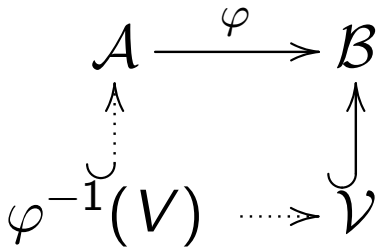

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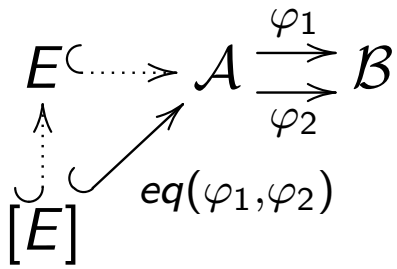

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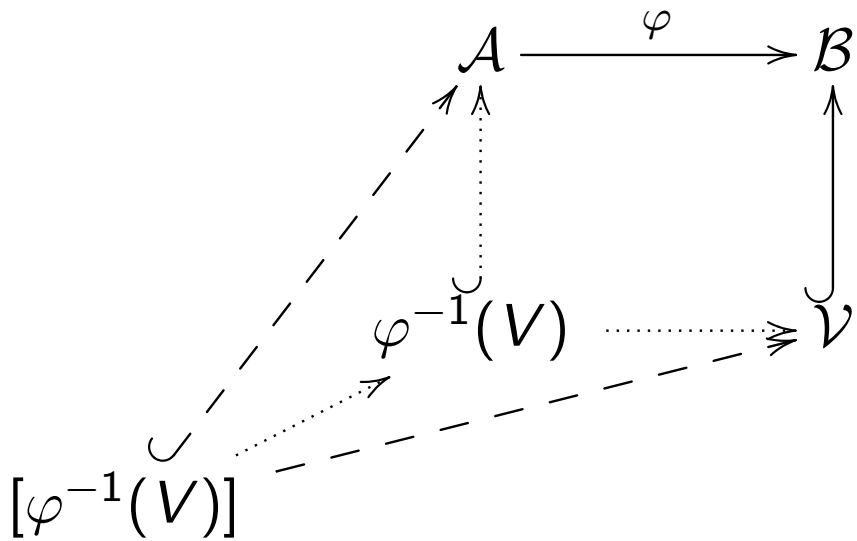

Products

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$\mathcal{A} =$ 

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- $\mathcal{A} \times \mathcal{B} \cong \emptyset$
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- $\mathcal{B} \times \mathcal{B}$ infinite in $Set_{\mathbb{P}_\omega}$
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Lambek's lemma

- The structure map $\alpha : T \to F(T)$ on a terminal coalgebra is an isomorphism.

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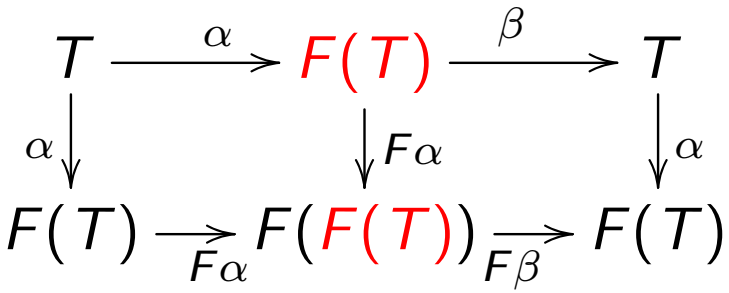

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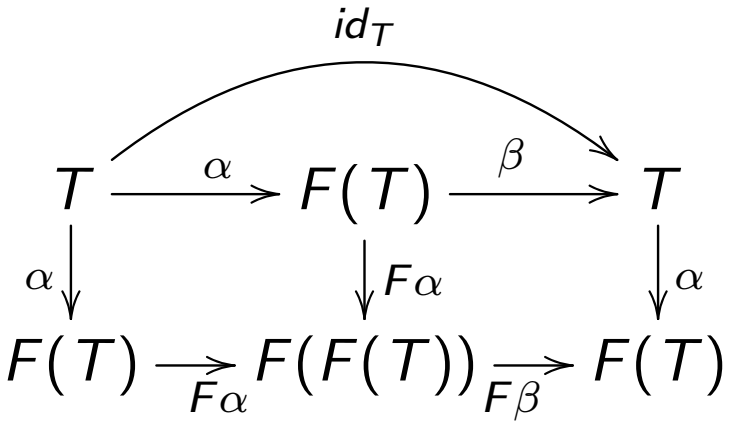

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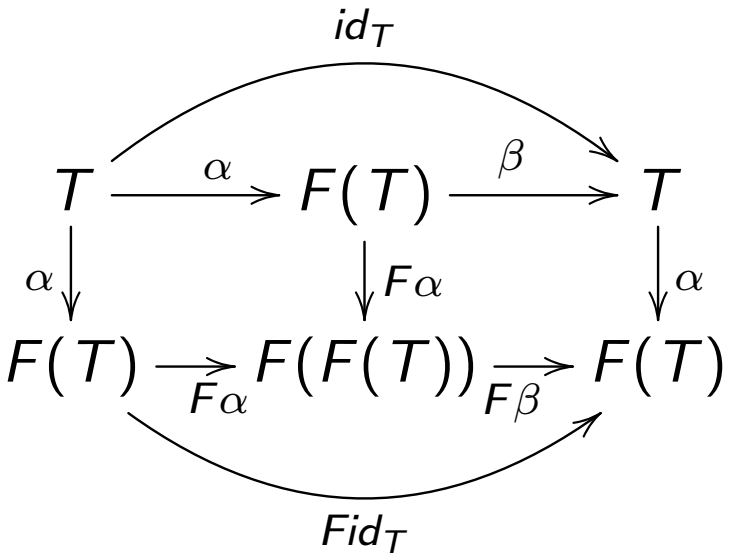

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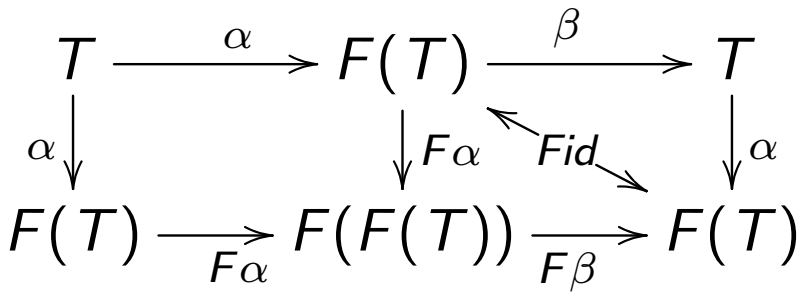

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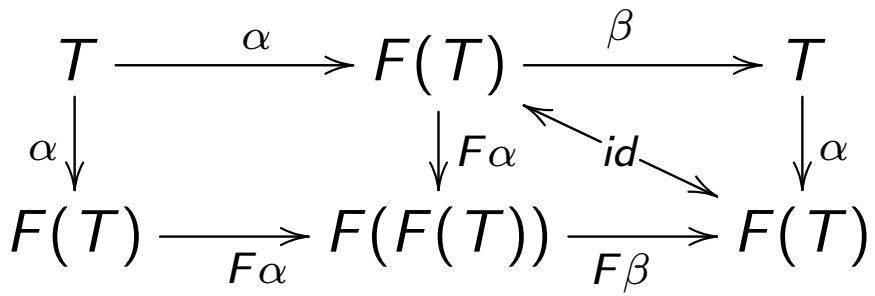

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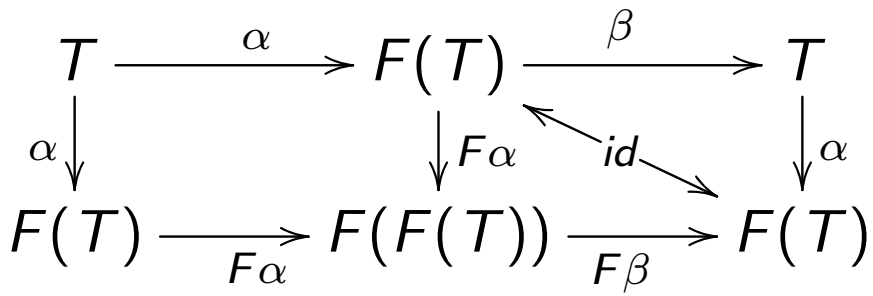

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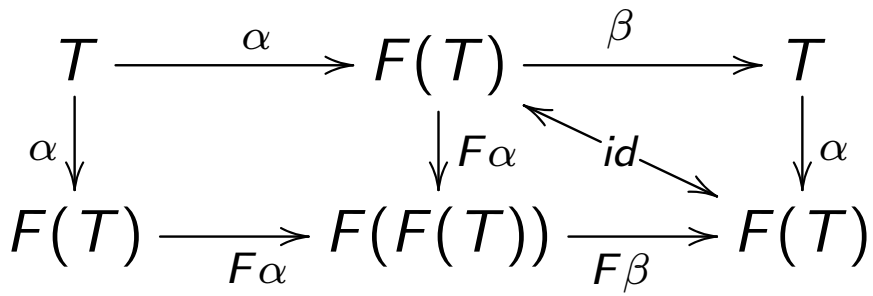

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F is $\kappa - \textit{bounded}$: $\iff$ $\forall u \in F(X).\exists_{|X_0|<\kappa}.u \in F(X_0)$

- Every functor F has a $\kappa - \textit{bounded}$ subfunctor:

Examples

- $\mathbb{P}_\omega(X)$ - all finite subsets
- $X^*_\omega = X^*$
- X^ω_ω - streams with finitely different elements
- X^3_3 - $\{(x_1, x_2, x_3) \mid |\{x_1, x_2, x_3\}| < 3\}$

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The following are equivalent

1. F is $\kappa-$bounded
2. Each F-coalgebra has a subcoalgebra of size $\leq \kappa$
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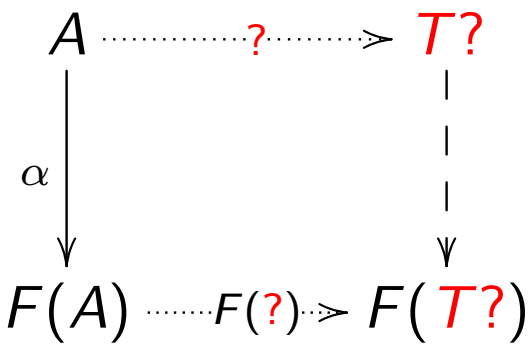
- ... choose $D = F(\kappa)$ and $\Sigma = \kappa$



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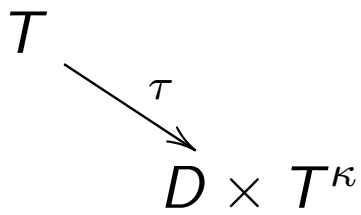

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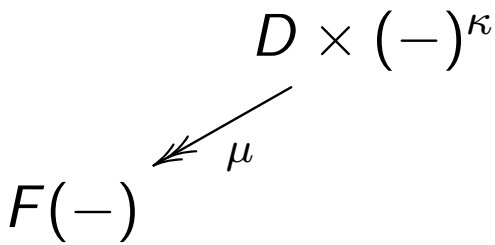

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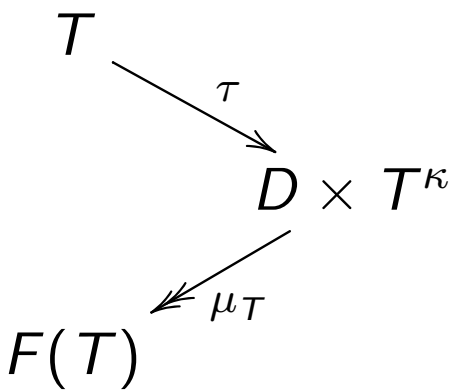

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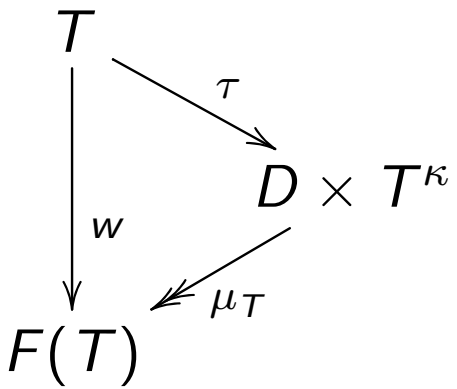

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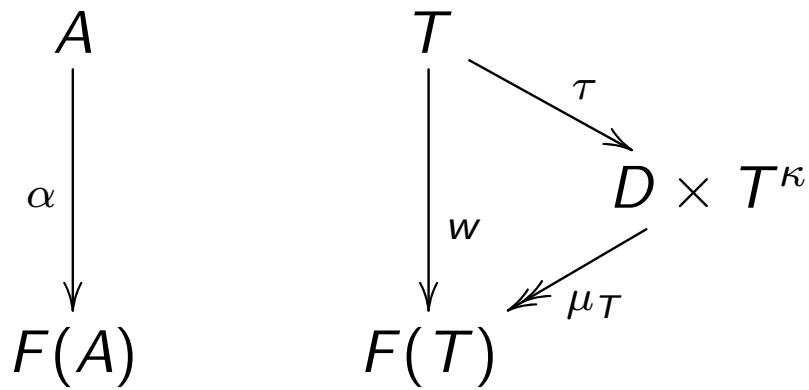

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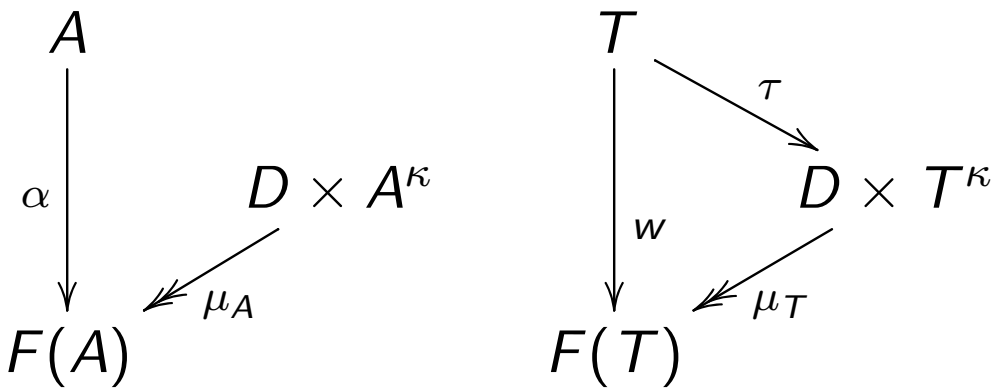

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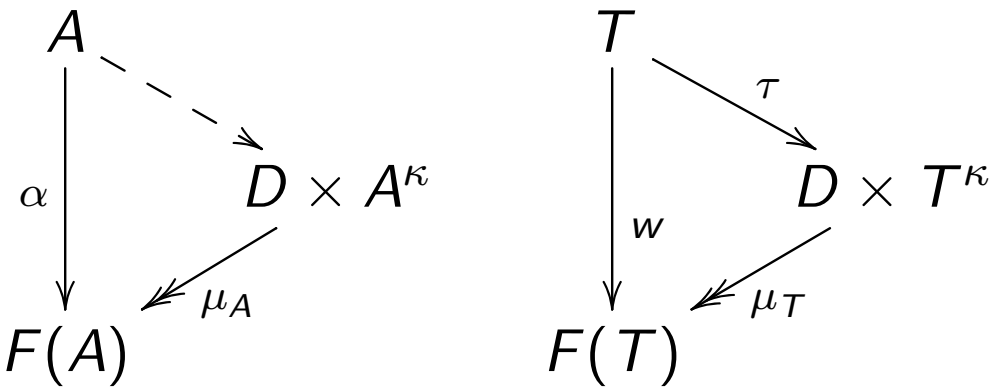

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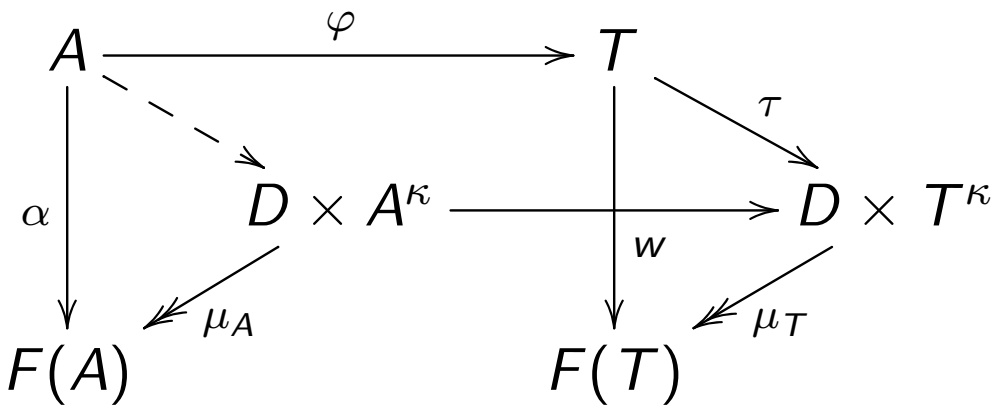

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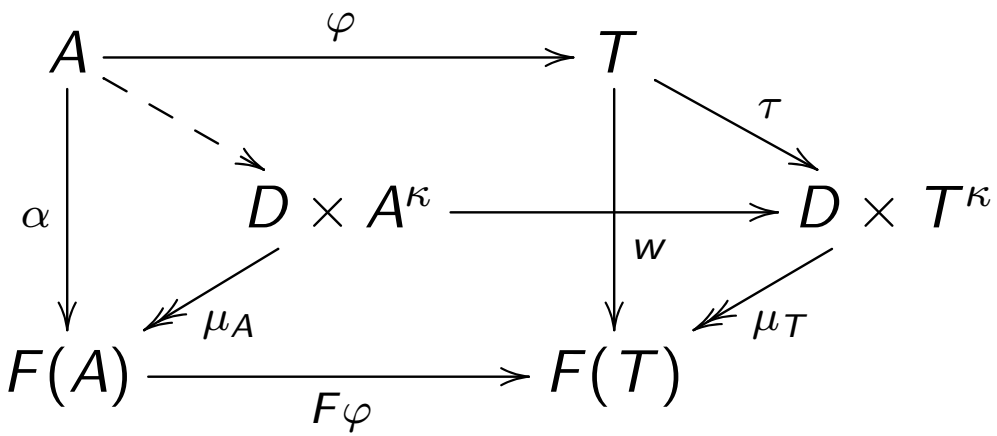

Existence of terminal algebras

Theorem

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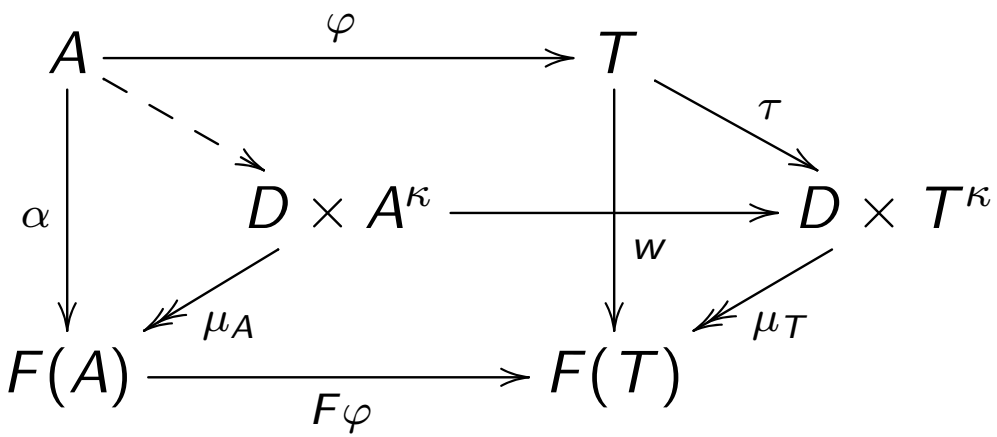

$\mathcal{T} = (T, w)$ is weakly terminal, so $\mathcal{T}/\nabla$ is terminal

Cofree coalgebras

- Let C be any set (of colors) ...
- $\mathcal{T}_C = (T_C, \alpha)$ with "coloring" map $\varepsilon_C : T_C \to C$ is called *cofree* if
  - for each coalgebra $\mathcal{A} = (A, \alpha)$ with coloring $c : A \to C$
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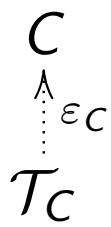

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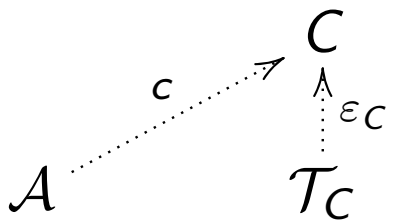

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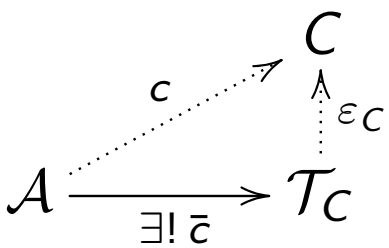

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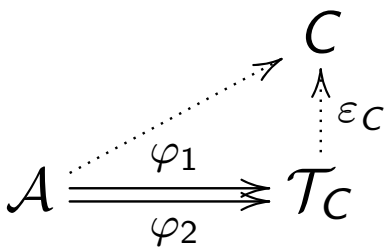

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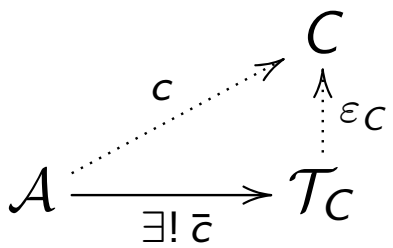

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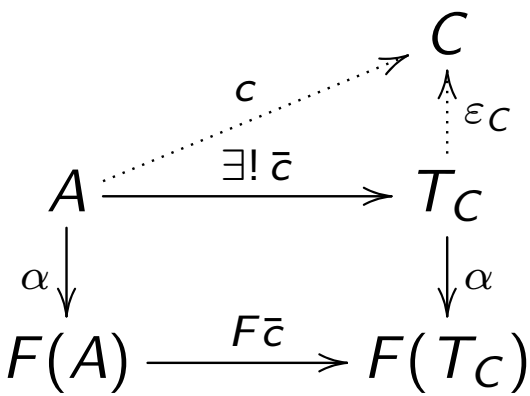

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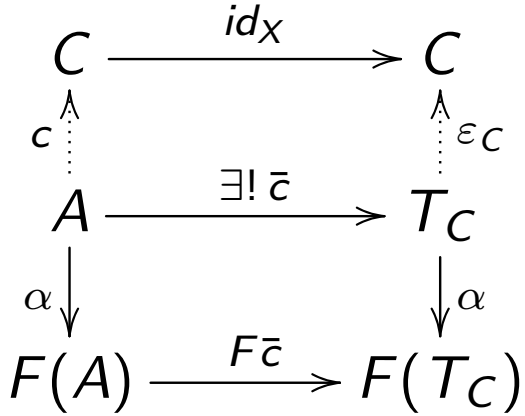

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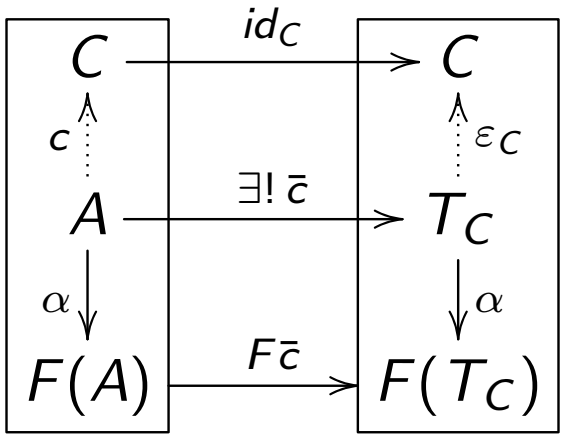

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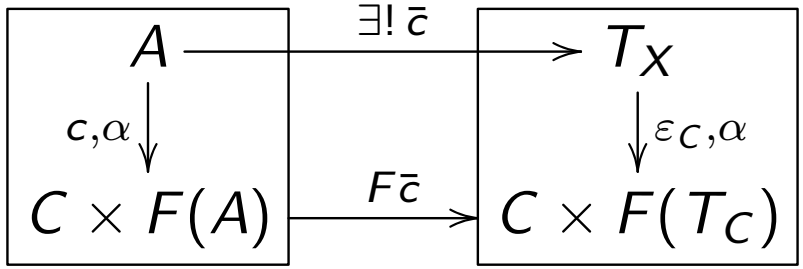

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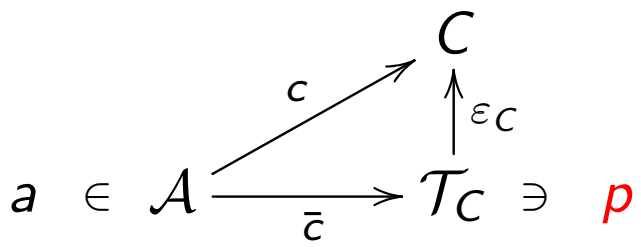

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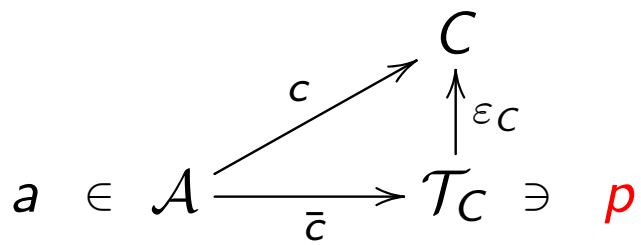

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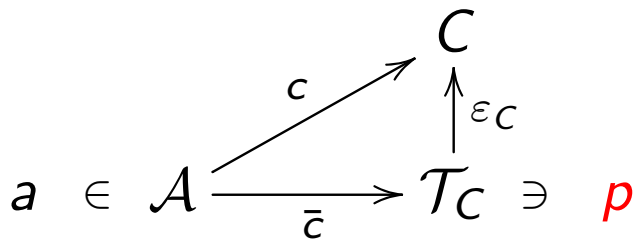

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If $\mathcal{A}, \mathcal{A}_i \models p$ then

- if $\mathcal{U} \leq \mathcal{A}$ then $\mathcal{U} \models p$.
- if $\varphi : \mathcal{A} \twoheadrightarrow \mathcal{B}$ then $\mathcal{B} \models p$
- $\Sigma \mathcal{A}_i \models p$

Covarieties

- $\mathfrak{K}$ a class of F-coalgebras.
  - $S(\mathfrak{K})$ all subcoalgebras
  - $H(\mathfrak{K})$ all homomorphic images
  - $\Sigma(\mathfrak{K})$ all sums
- $\mathfrak{K}$ a *is a covariety if* $\mathfrak{K}$ *is closed under* S, H, Σ
- if and only if $\mathfrak{K} = SH\Sigma(\mathfrak{K})$
- E a set of coequations, $Mod(E) := \{\mathcal{A} \mid \forall e \in P.\mathcal{A} \models p\}$ is covariety
- $\mathfrak{K}$ a class of coalgebras, $Eq_C(\mathfrak{K}) := \{p \in \mathcal{T}_C \mid \forall \mathcal{A} \in \mathfrak{K}.\mathcal{A} \models p\}$

Co-Birkhoff

Theorem

F bounded (by κ). Covarieties are the model classes for coequations

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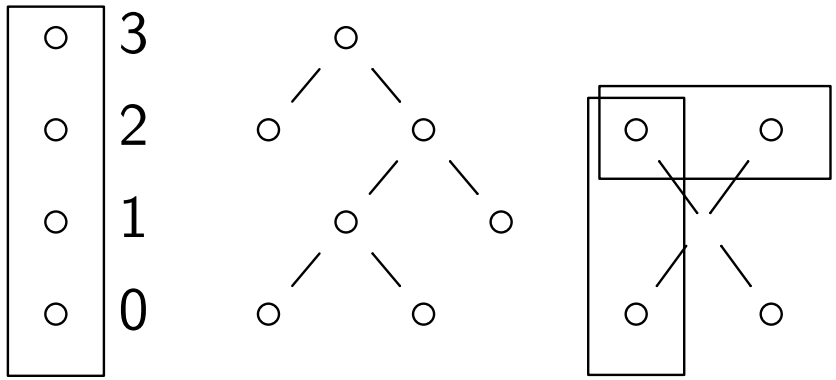
Proof.

- Clearly: $\mathfrak{K} \subseteq Mod(Eq_C(\mathfrak{K}))$ for $|C| \geq \kappa$.
- Conversely: Let $\mathcal{A} \in Mod(Eq_C(\mathfrak{K}))$:
- First reduce to case: $|\mathcal{A}| \leq \kappa$:
  - $\mathcal{A}$ is subdirect sum of $\mathcal{B}_i \leq \mathcal{A}$ with $|\mathcal{B}_i| \leq \kappa$.
  - each $\mathcal{B}_i \in Mod(Eq_C(\mathfrak{K}))$
  - if $\mathcal{B}_i \in \mathfrak{K}$, then $\mathcal{A} \in \mathfrak{K}$
- Now may assume $|\mathcal{A}| \leq \kappa$.
  - choose injective coloring $c : A \to C$, then $\mathcal{A} \cong \bar{c}(A) \leq \mathcal{T}_C$.
  - for each $u \in \bar{c}(A)$ there is $\mathcal{A}_u \in \mathfrak{K}$ and coloring c_u with $u \in \bar{c_u}(\mathcal{A}_u) \leq \mathcal{T}_C$
  - $\mathcal{A} \in SH\Sigma(\{\mathcal{A}_u \mid u \in \bar{c}(A)\}) \subseteq \mathfrak{K}$

Useful Intuition for functors

- containers
  - $F(1)$: shapes
  - $F(2)$: 0-1-patterns
  - ...
  - $F(X)$: X -pattern.
- $F\sigma$ is substitution with σ
  - $\sigma : x_0, x_2 \mapsto \bullet, x_1 \mapsto \circ :$



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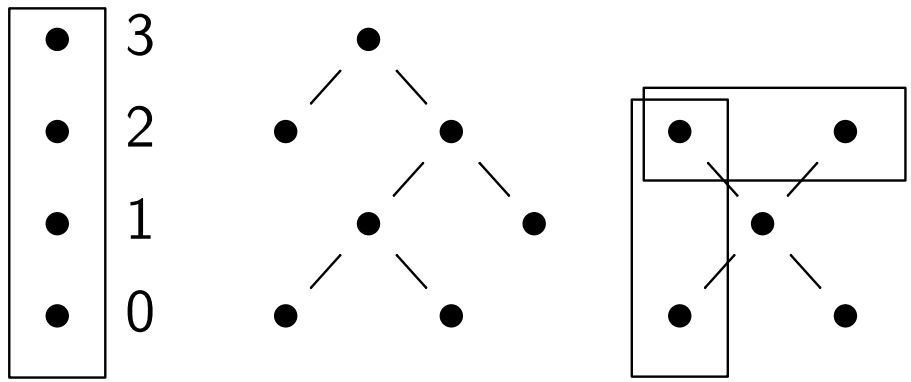

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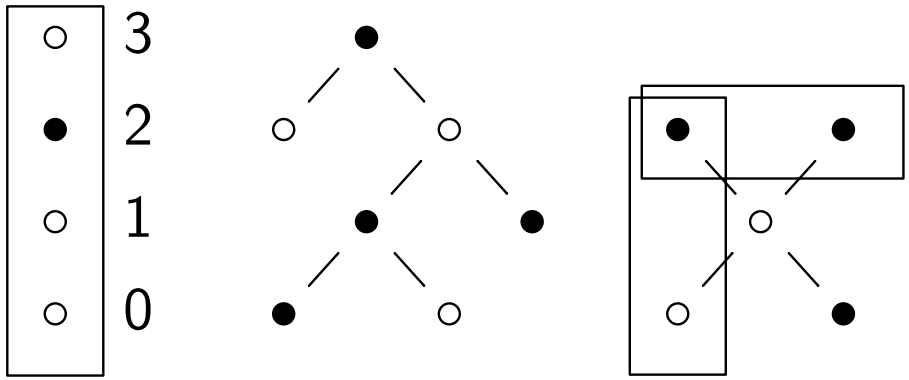

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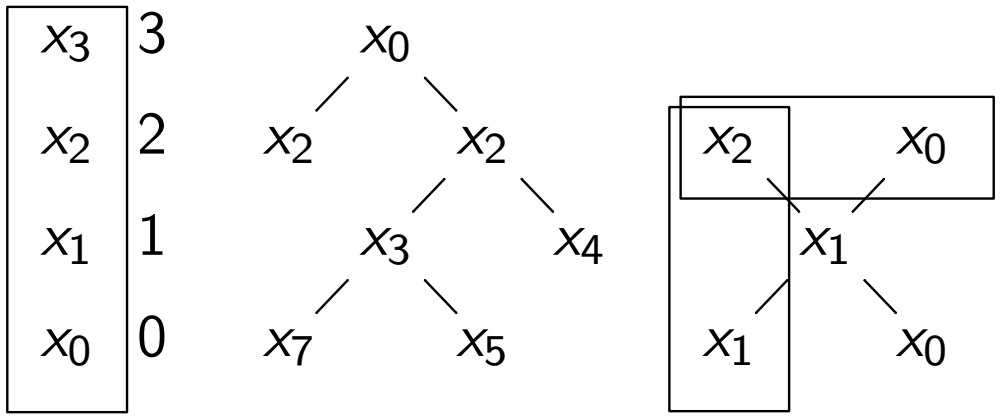

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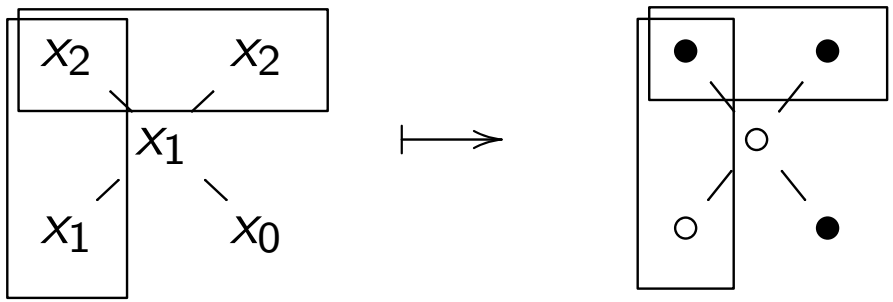

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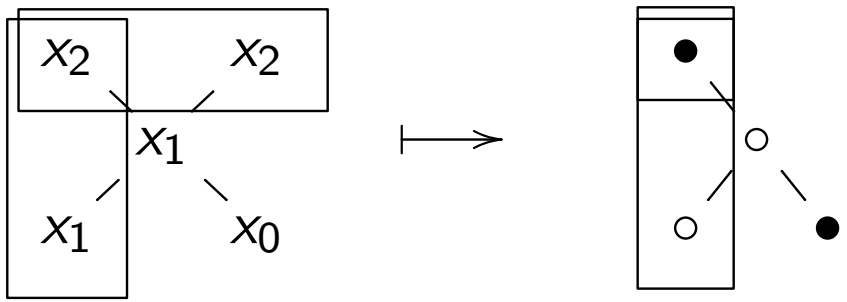

Patterns and substitutions

$X = \{x_1, ..., x_\kappa\}$ a set of variables

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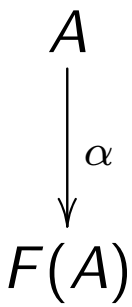
Modalities

- Use formulae ϕ as test: $[\![\phi]\!] : A \to 2$
- F $[\![\phi]\!] : F(A) \to F(2)$ produces $0-1-$patterns
- judge $a \in A$ by the $0-1-$pattern of $\alpha(a)$
- For $\Box \subseteq F(2)$:



$a_0, a_1 \models \phi$
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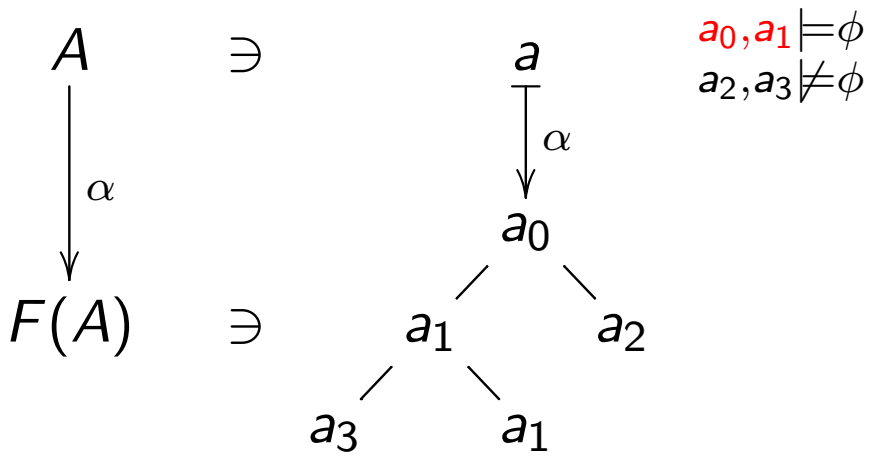

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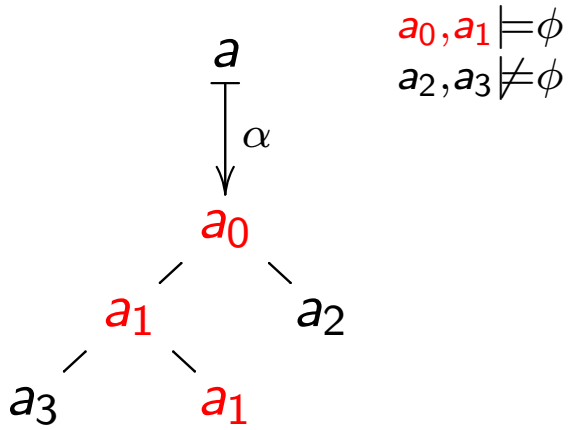

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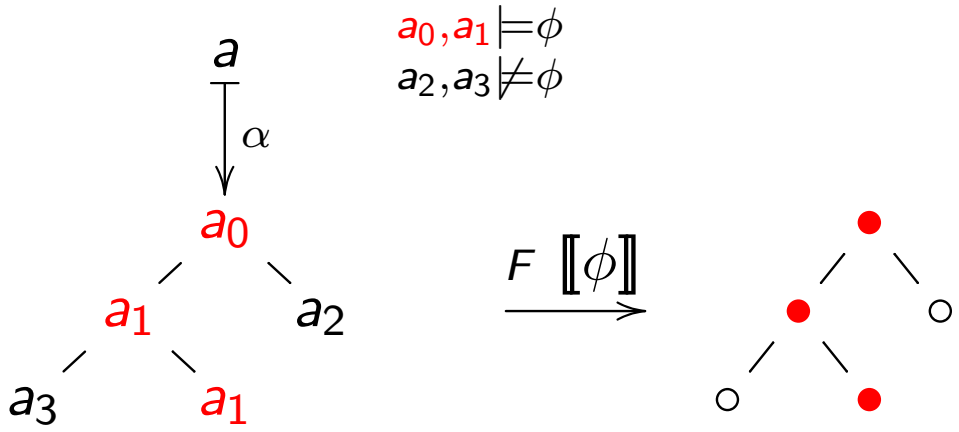

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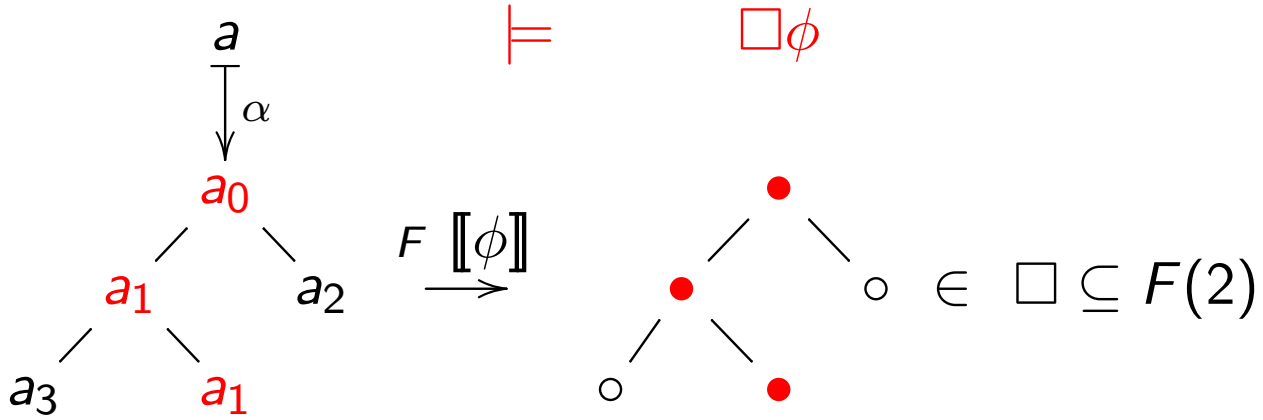

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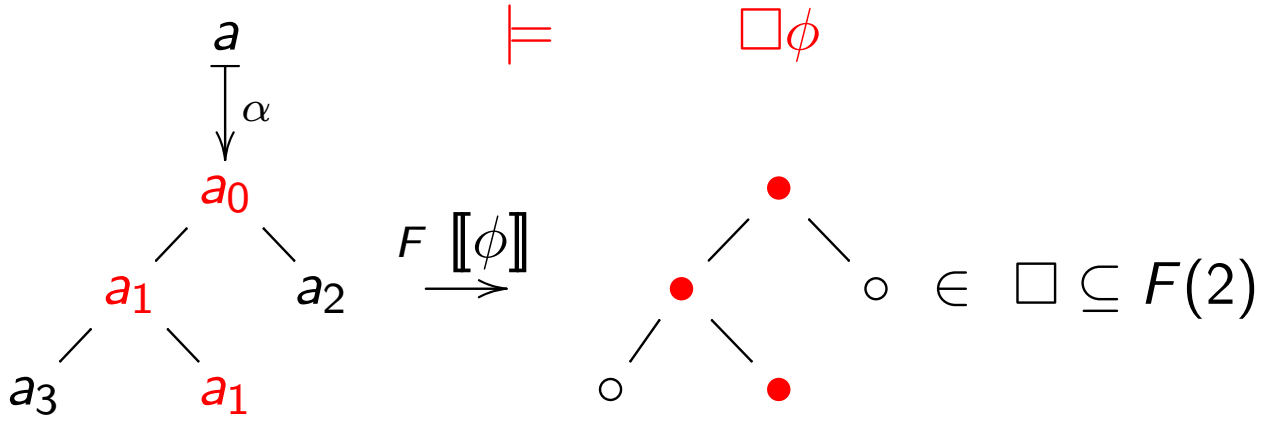

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- Formulae
  - pick $\Box_i \subseteq F(2)$

$$\begin{array}{rcl} \phi & :: & \mathit{true} \mid \neg\phi \mid \bigwedge_{i\in I} \phi_i \\ & \mid & \Box\phi \quad \text{for } \Box \subseteq F(2) \end{array}$$

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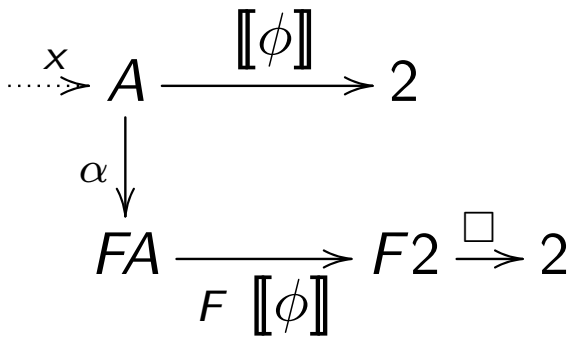
Modal Logic

- Formulae
  - pick $\Box_i \subseteq F(2)$

$$\begin{array}{rcl} \phi & :: & \mathit{true} \mid \neg\phi \mid \bigwedge_{i\in I} \phi_i \\ & \mid & \Box\phi \quad \text{for } \Box \subseteq F(2) \end{array}$$



- Semantics
  - $x \models \bigwedge_{i\in I} \phi_i \iff \forall i \in I. x \models \phi_i$
  - $x \models \Box\phi \iff F\,[\![\phi]\!](\alpha(x)) \in \Box$

Stability: $\nabla \subseteq \approx$

- $\models$ is homomorphism stable

$$\varphi : \mathcal{A} \to \mathcal{B} \implies (\forall a \in A.a \models \phi \iff \varphi(a) \models \phi)$$

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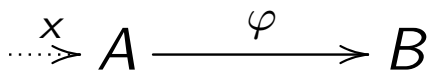

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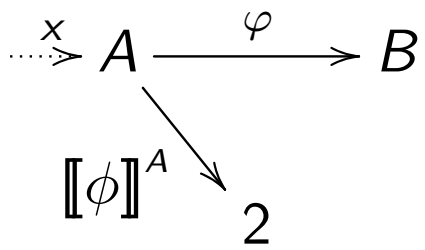

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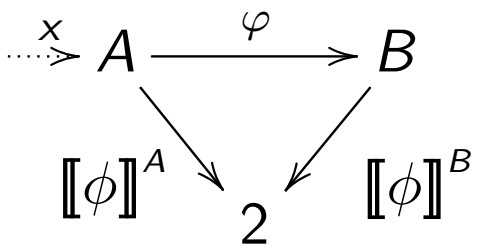

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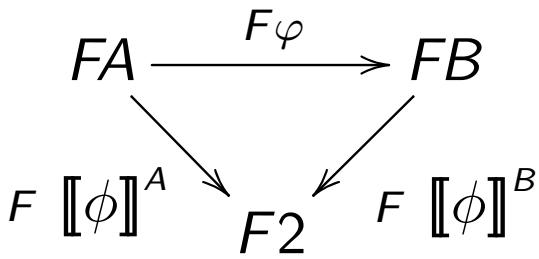

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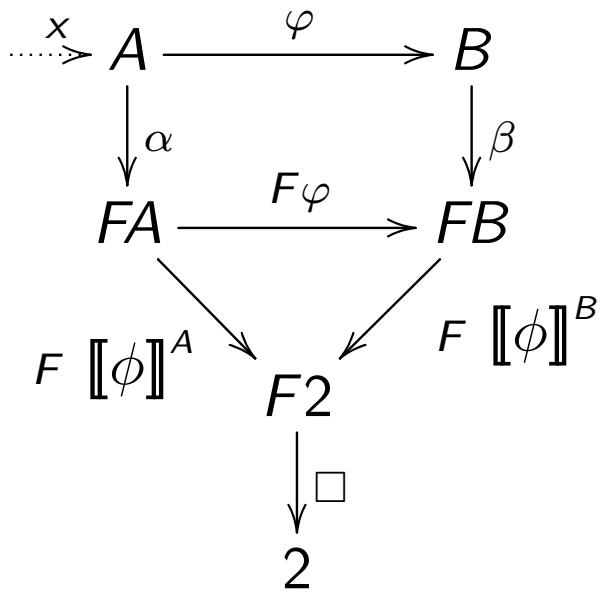

Completeness: $\approx \subseteq \nabla$

- F is $2-$*separable* if $\{Ff \mid f : X \to 2\}$ separates points

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3. For $f : A/\approx \to 2$
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$$X \xrightarrow{f} 2$$

$$\overset{a}{\underset{b}{\rightrightarrows}} F(X) \xrightarrow{Ff} F(2)$$

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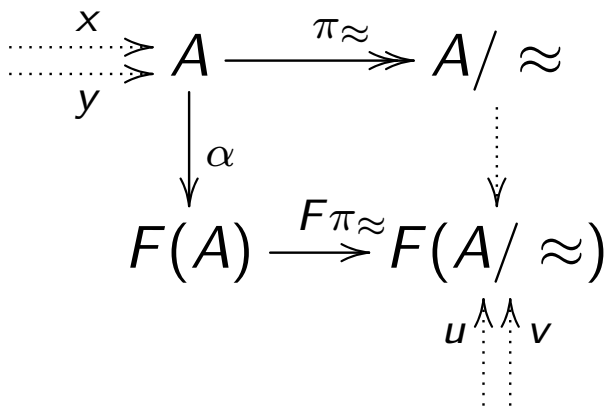

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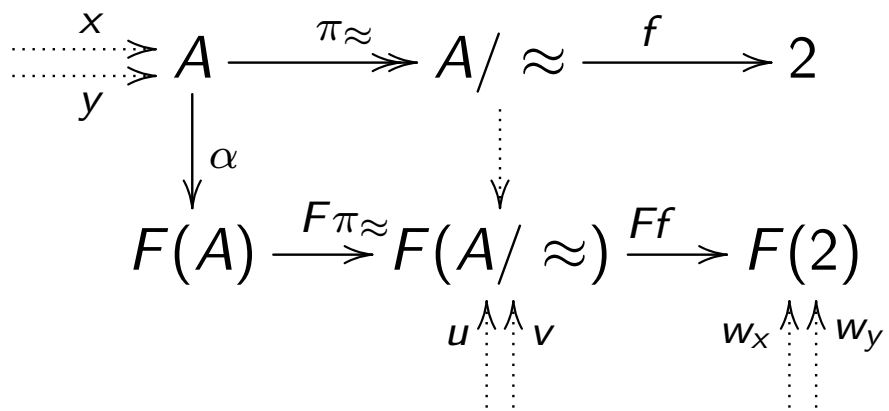

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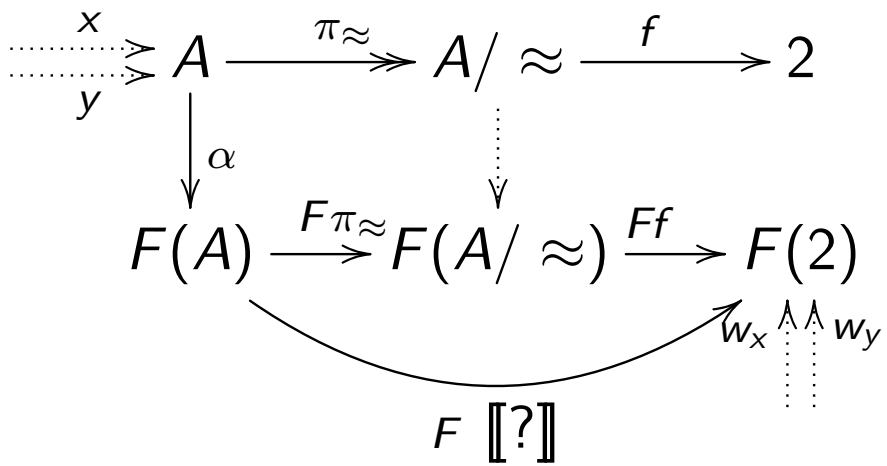

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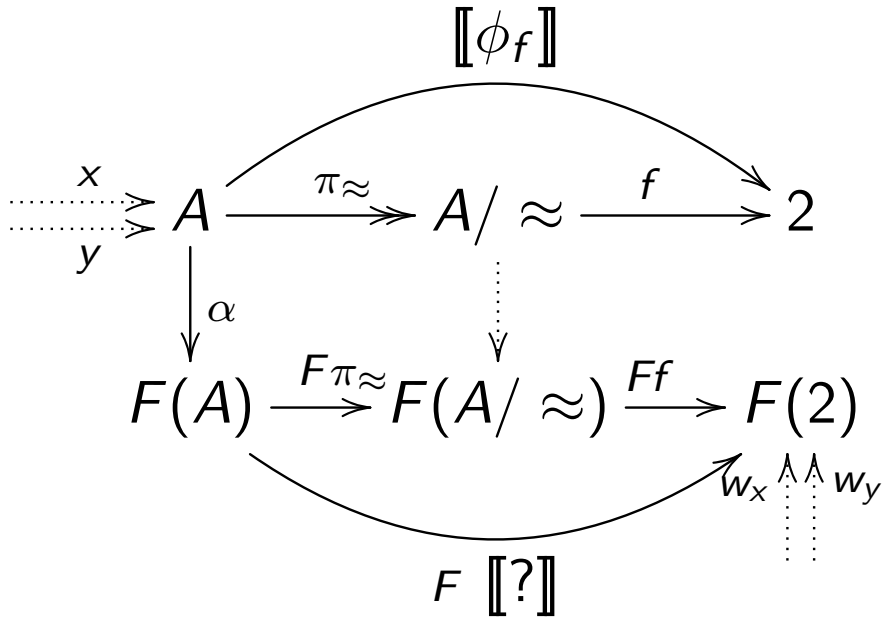

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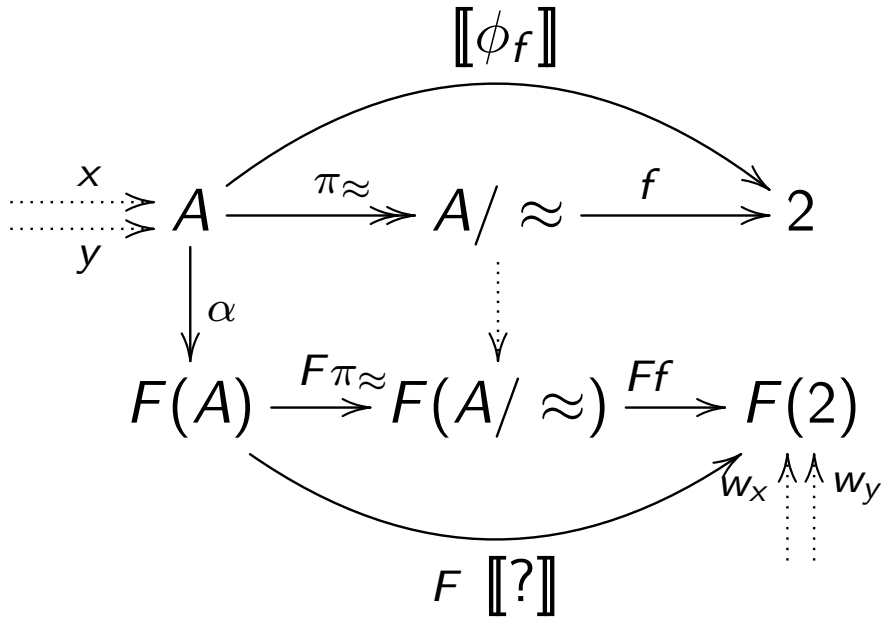

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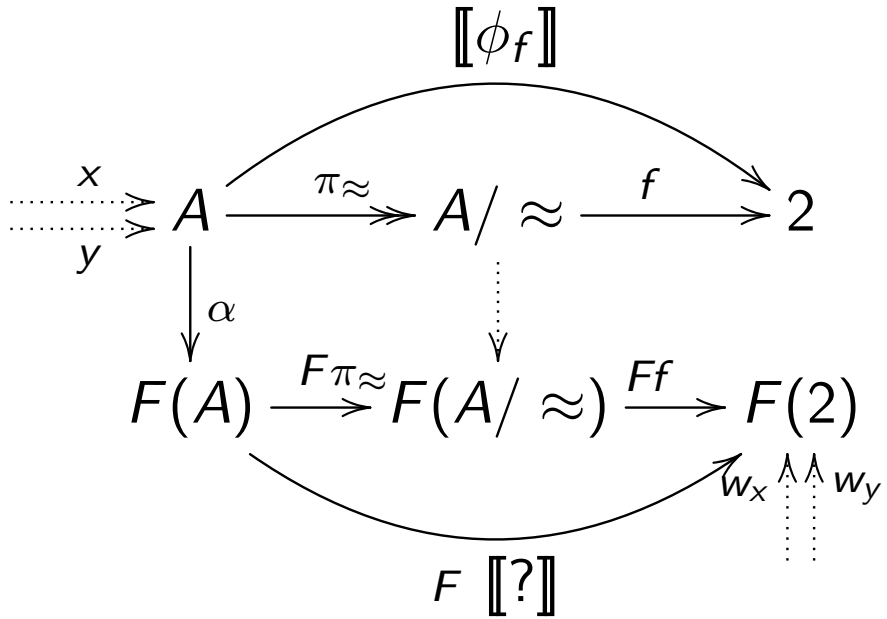

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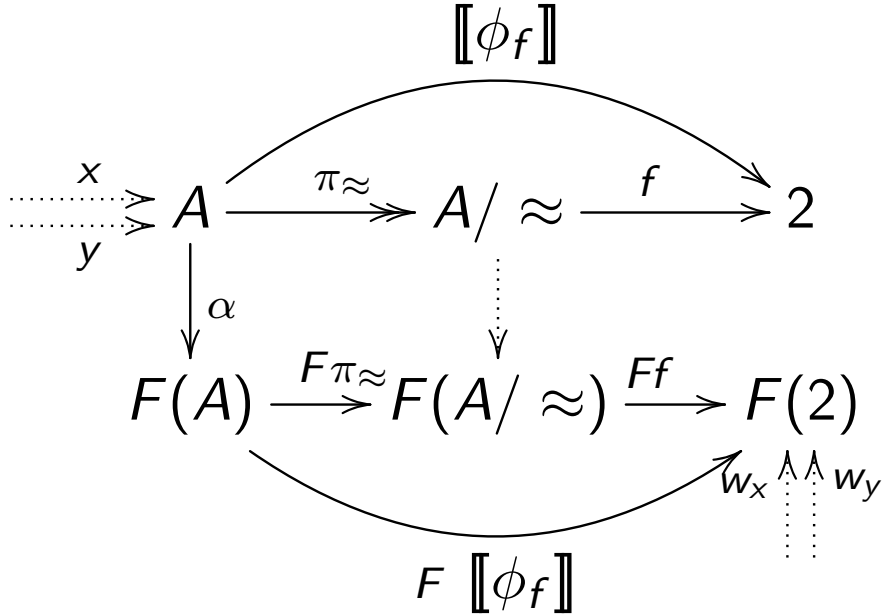

Example: Probabilistic systems

- $\mathcal{D}_\omega(X) =$ finitary prob. distr. on X
- Assumption satisfied:
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- Formulae

$$\begin{array}{rcl} \phi & :: & [\bar{w}]\phi \text{ where } \bar{w} = [w, 1] \\ & \mid & \phi \wedge \phi \mid \neg\phi \mid \mathit{true} \end{array}$$

- Semantics
  - $x \models [\bar{w}]\phi \iff \sum\{p \mid x \xrightarrow{p} y \models \phi\} \geq w$

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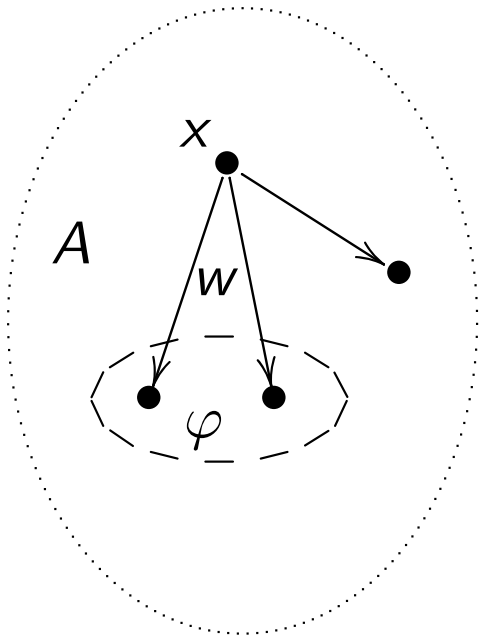

Towards Finitary Logic

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- Finite conjunctions

Towards Finitary Logic

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Finitary completeness

- Assume F is $\omega - bounded$
  - $F(X) = \bigcup\{F[U]X \mid U \subseteq_{fin} X\}$
  - Choose $U \subseteq X$ with $\alpha(x), \alpha(y) \in F(U)$
  - $\phi_a := \bigvee\{\phi_{a,b} \mid a \not\approx b \in U\}$ defines a relative to U
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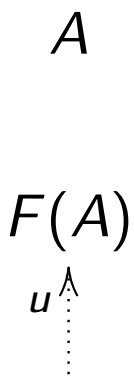
Wrap up



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$$\begin{array}{ccc} U & \xrightarrow{\quad\iota\quad} & A \\ & & \\ F(U) & \xrightarrow{\quad F\iota\quad} & F(A) \\ r \uparrow & & u \uparrow \end{array}$$

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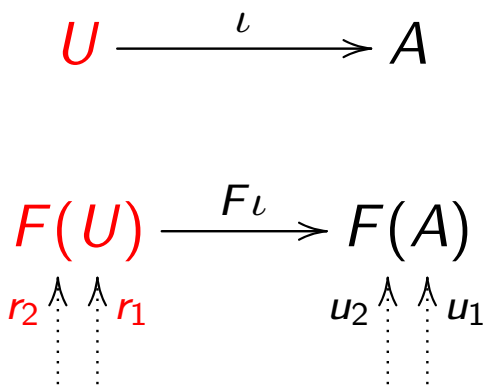

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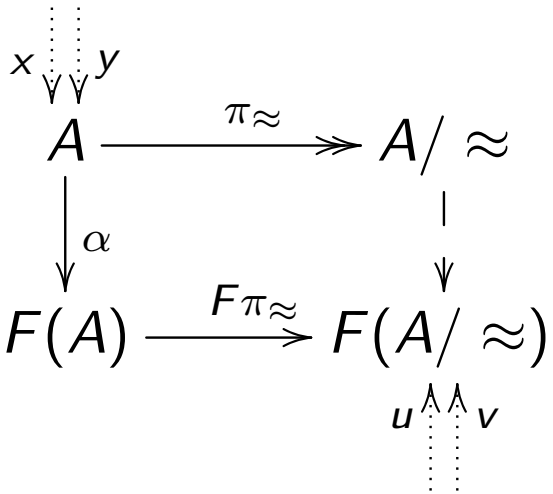

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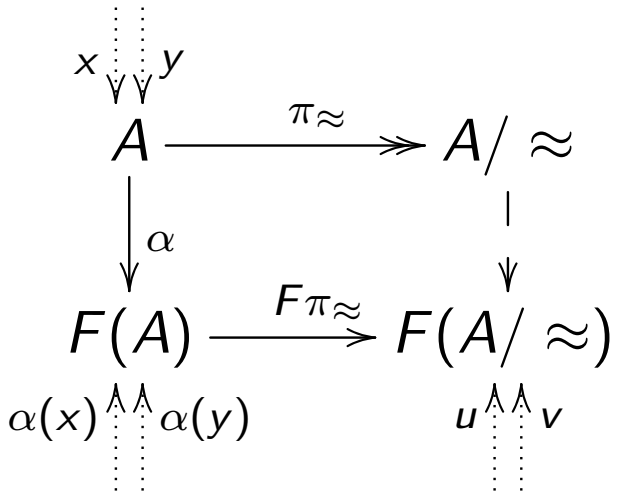

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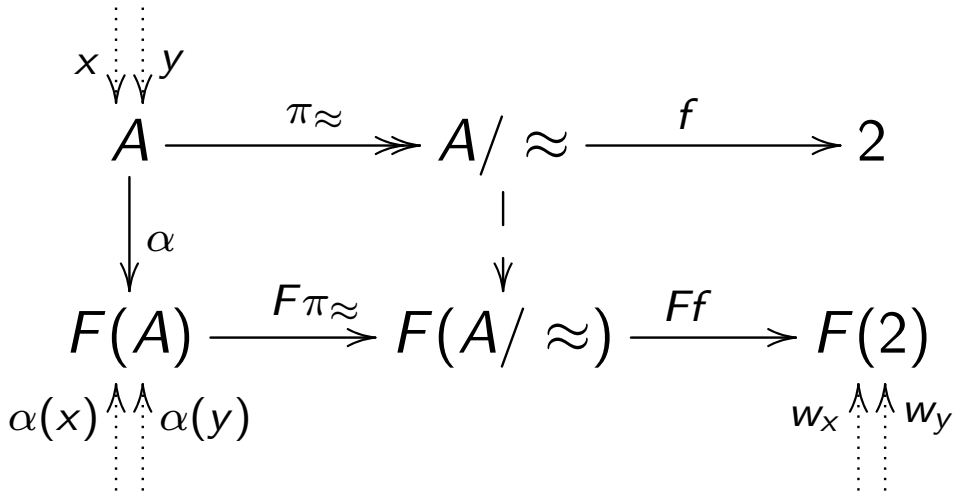

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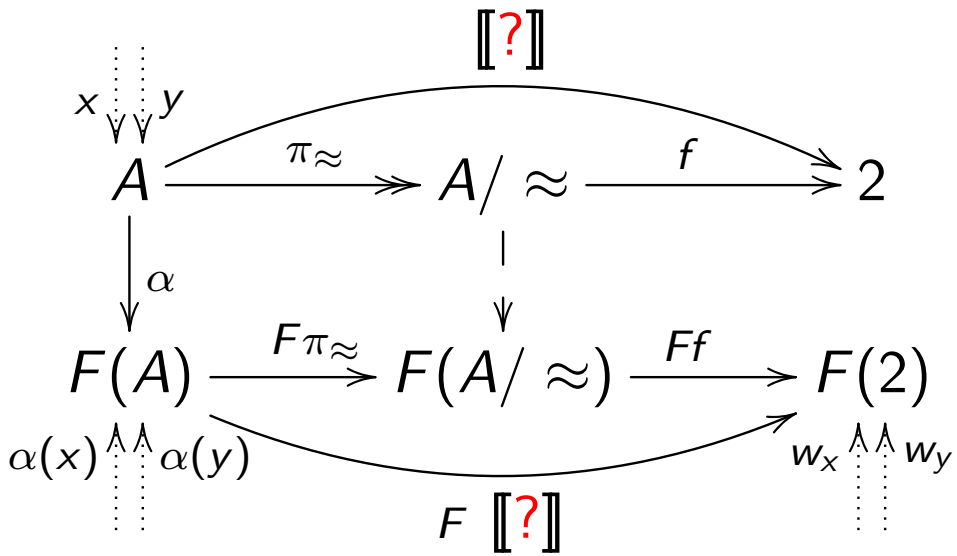

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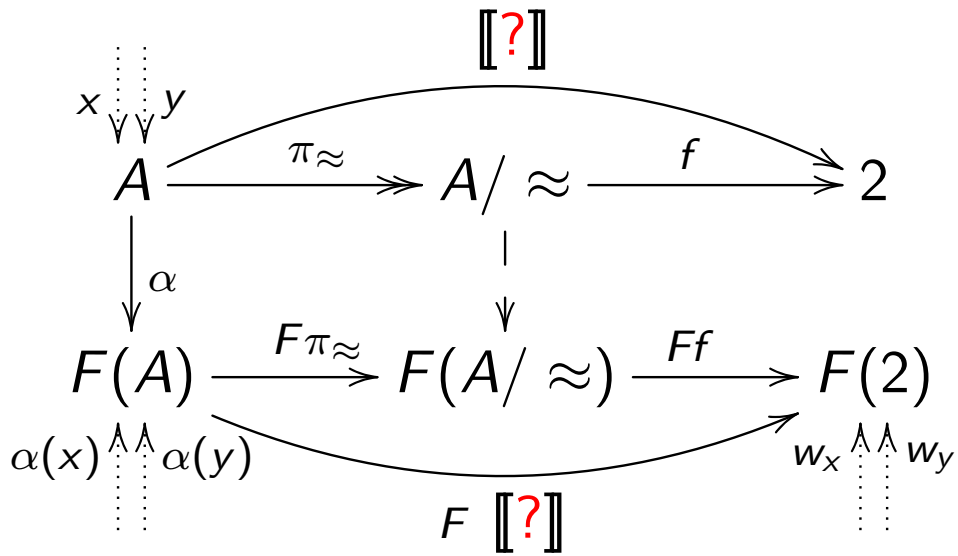

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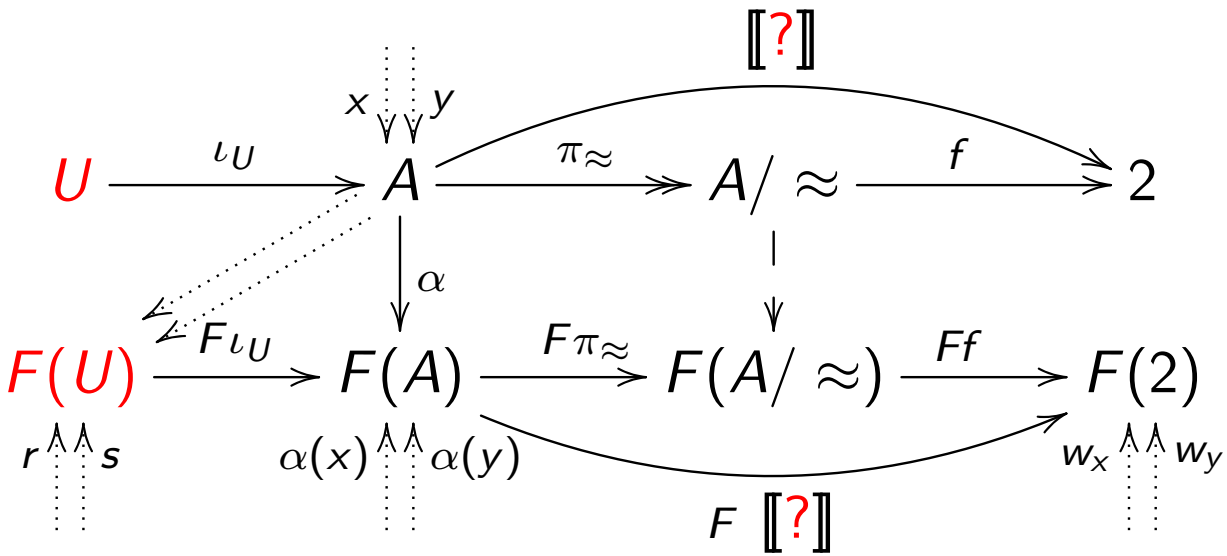

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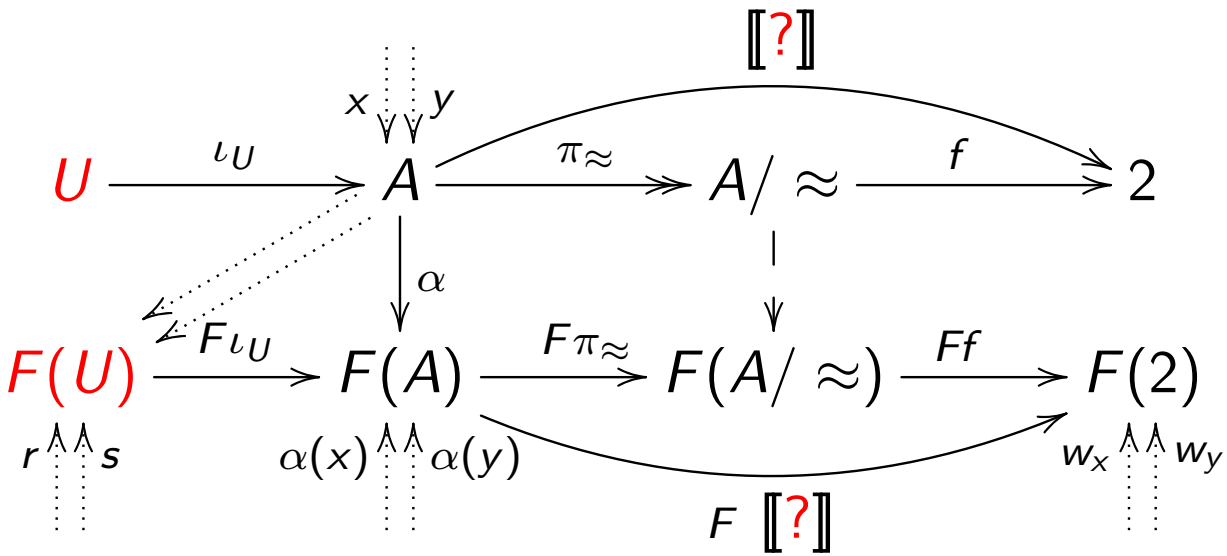

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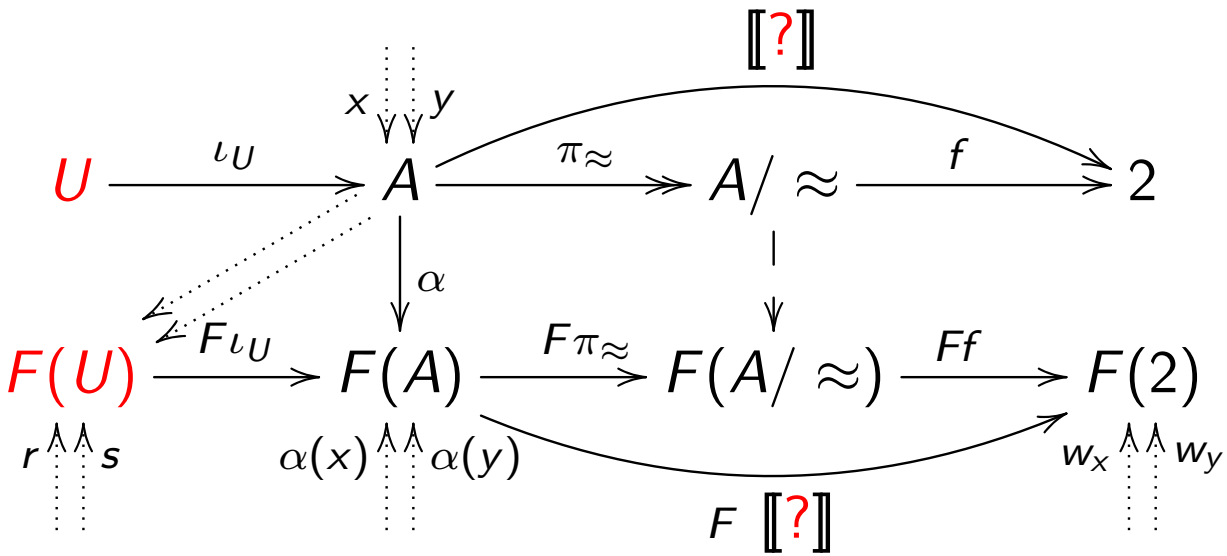

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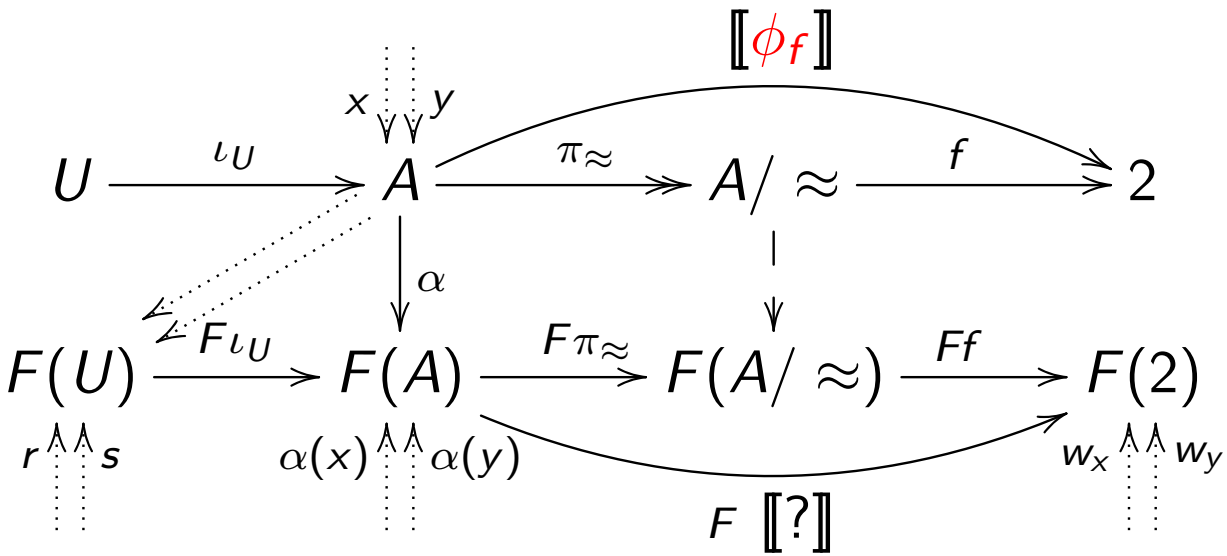

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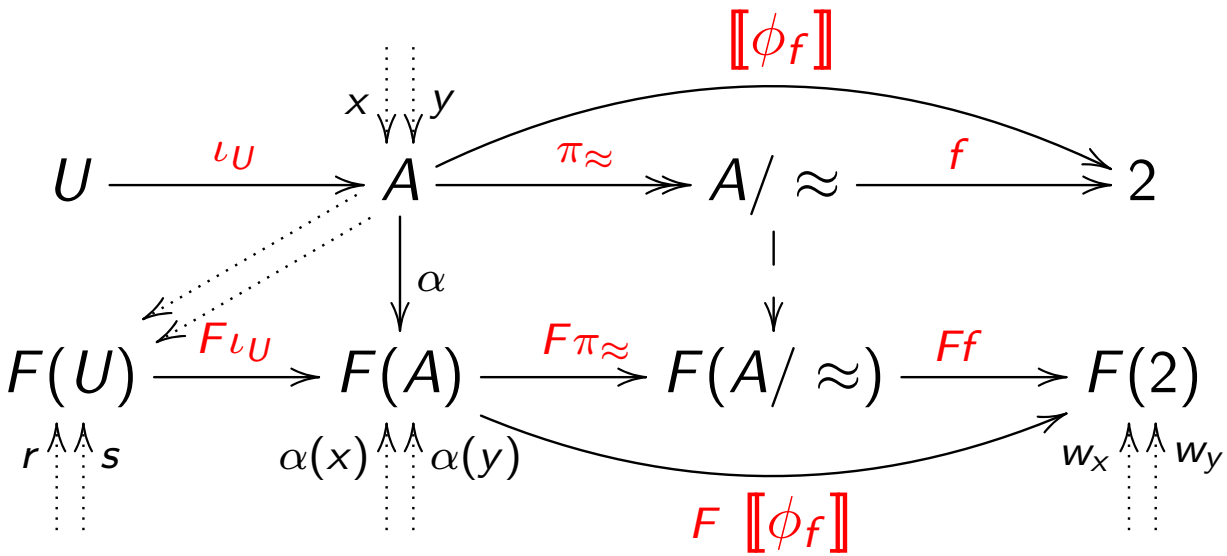
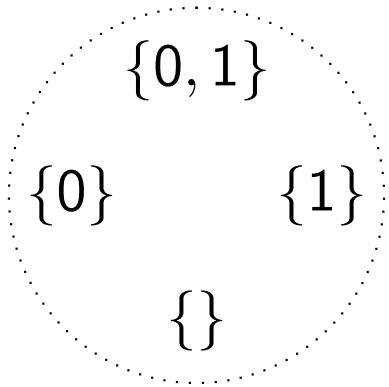


Modalities for $\mathbb{P}(-)$

$$x \models \Box\phi : \Longleftrightarrow \; x \models [\{\}]\phi \;\vee\; x \models [\{1\}]\phi$$



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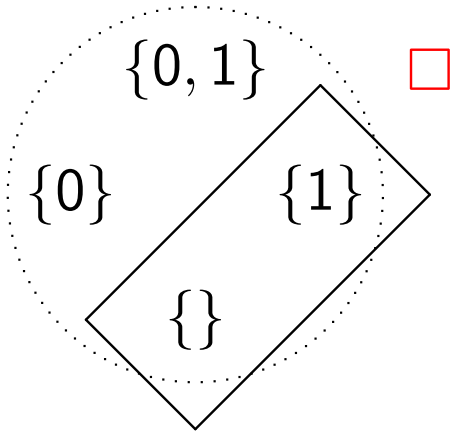

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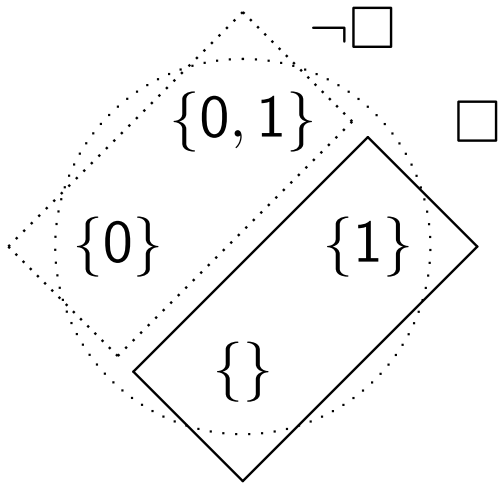

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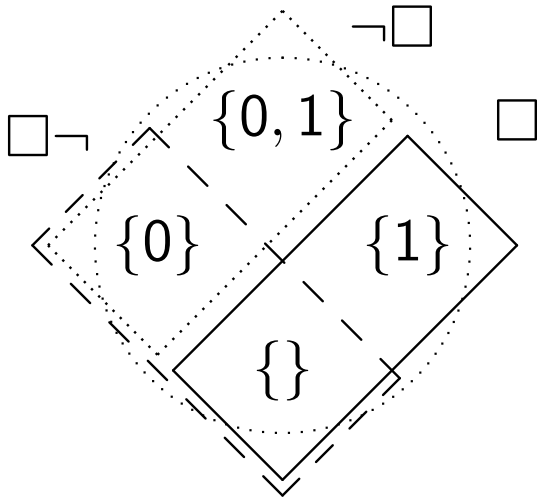

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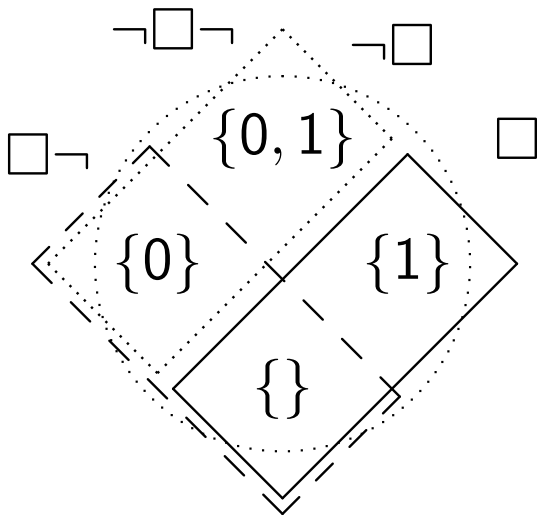

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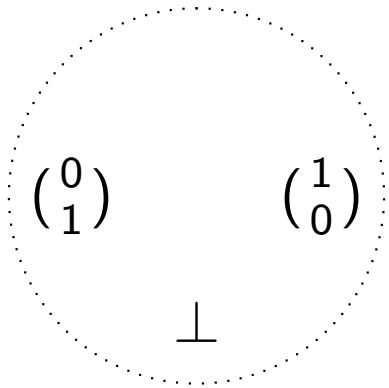

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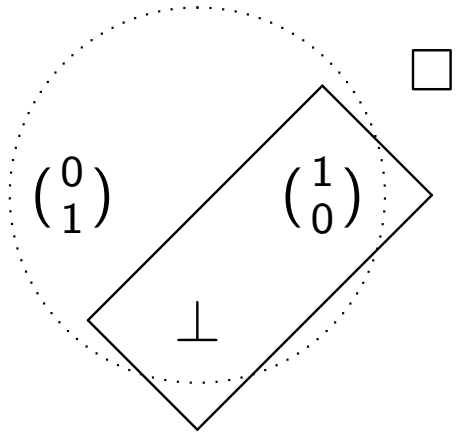

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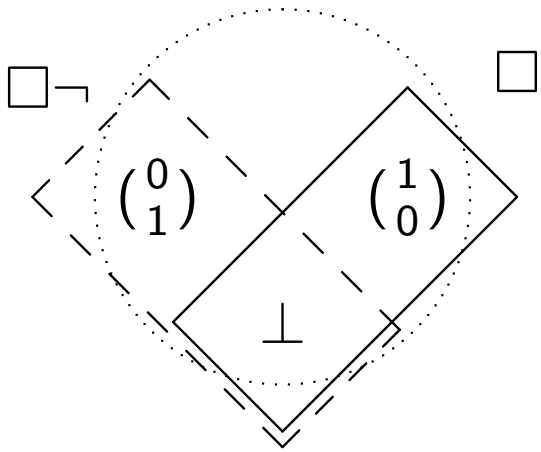

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$$\begin{array}{rcl} & & \Box T \\ \Box v_1 \wedge \Box v_2 & \Longrightarrow & \Box(v_1 \wedge v_2) \\ (v_1 \Longrightarrow v_2) & \Longrightarrow & (\Box v_1 \Longrightarrow \Box v_2) \end{array}$$

Validity depends on choice of $\Box \subseteq F(2)$.

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- $\models \Box T$ **iff** $\frac{}{p(1) \in \Box}$
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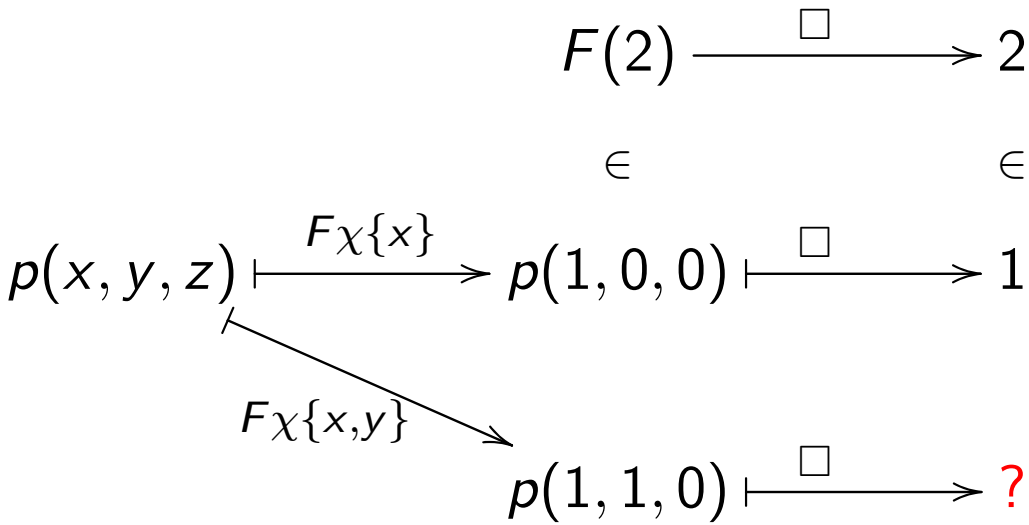

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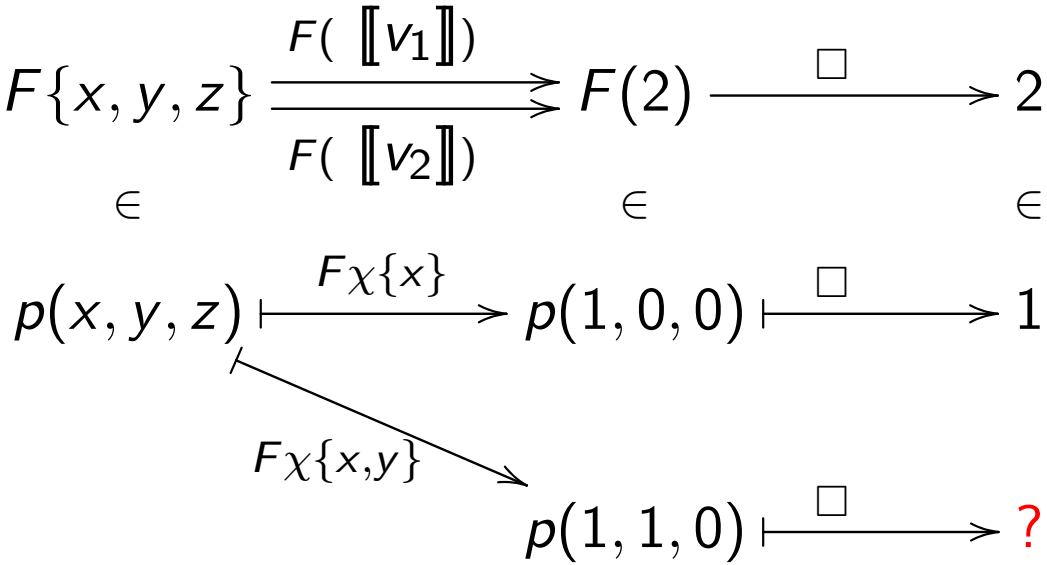

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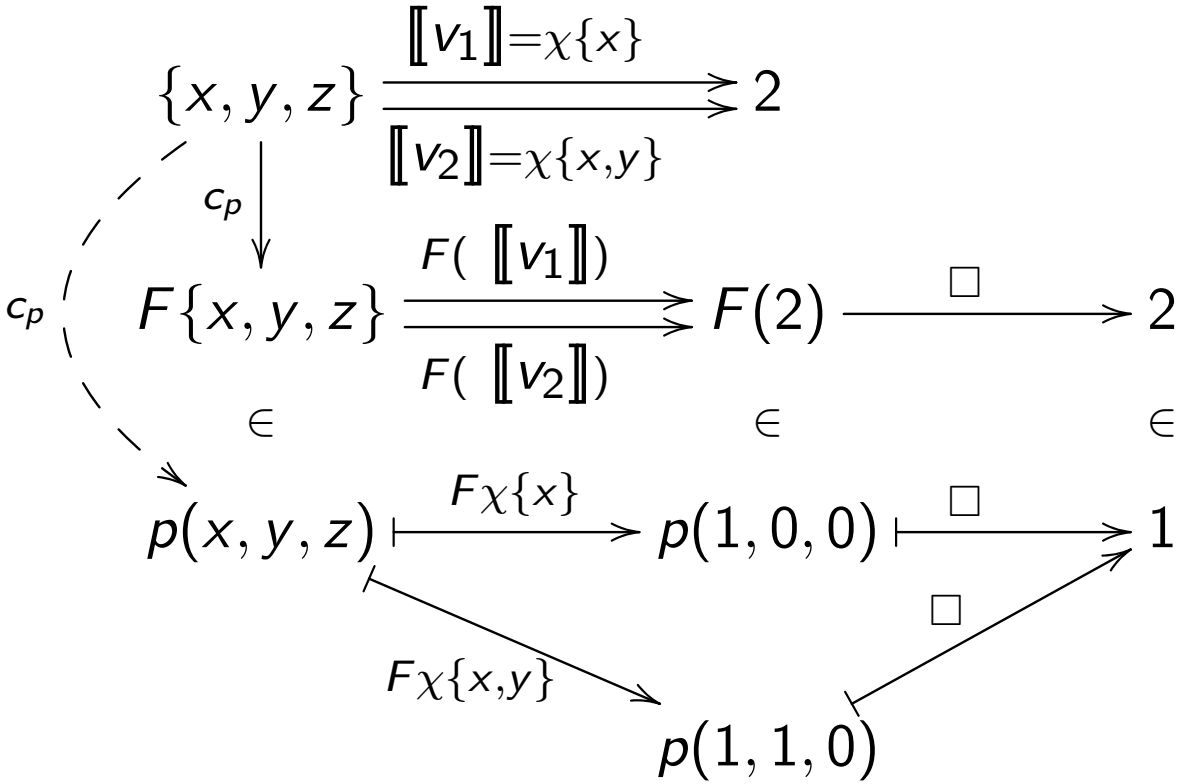

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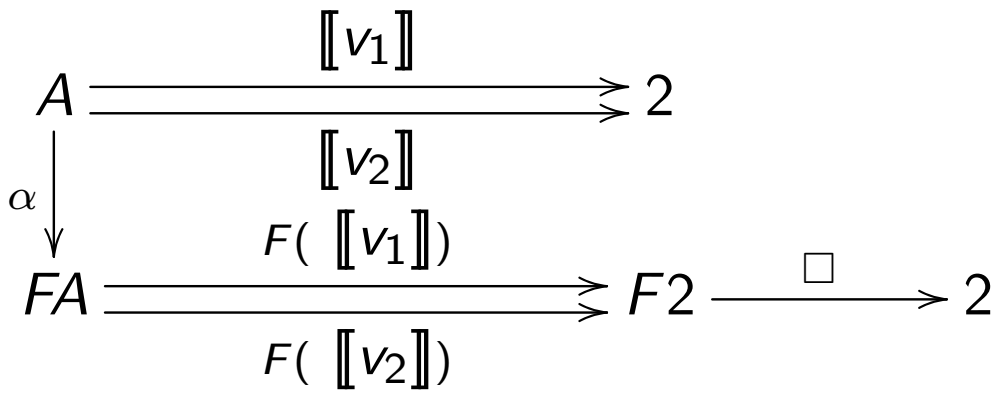

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 & & \alpha \downarrow \\
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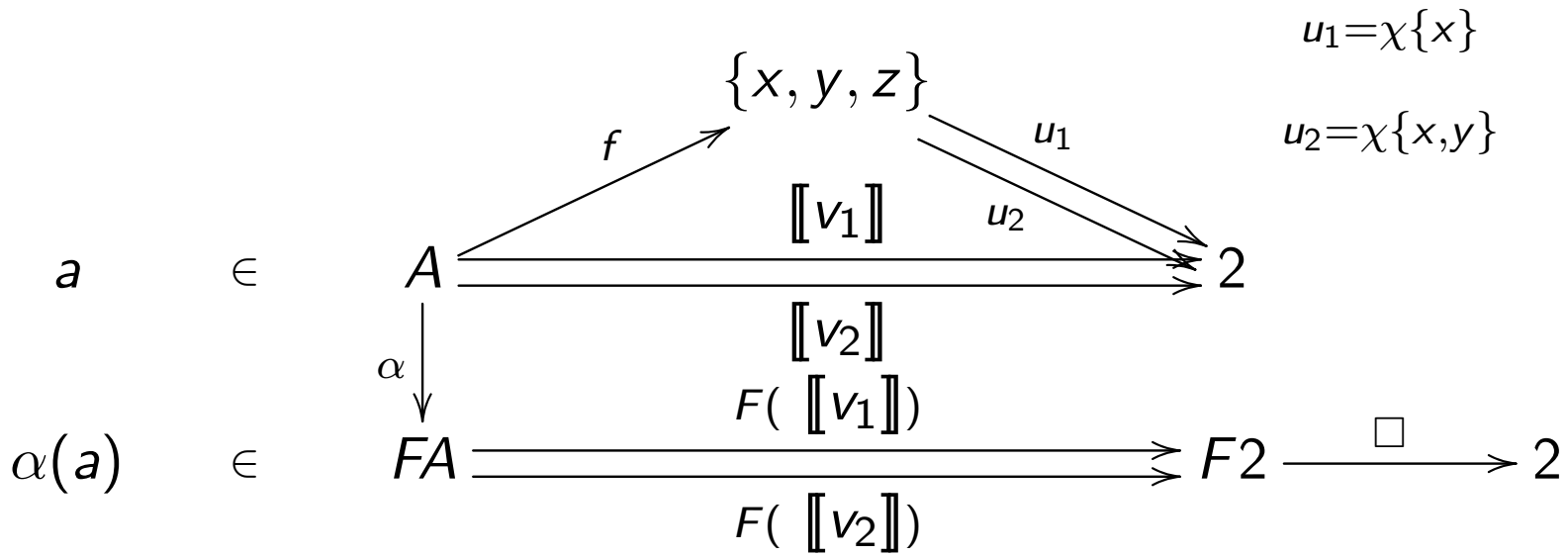

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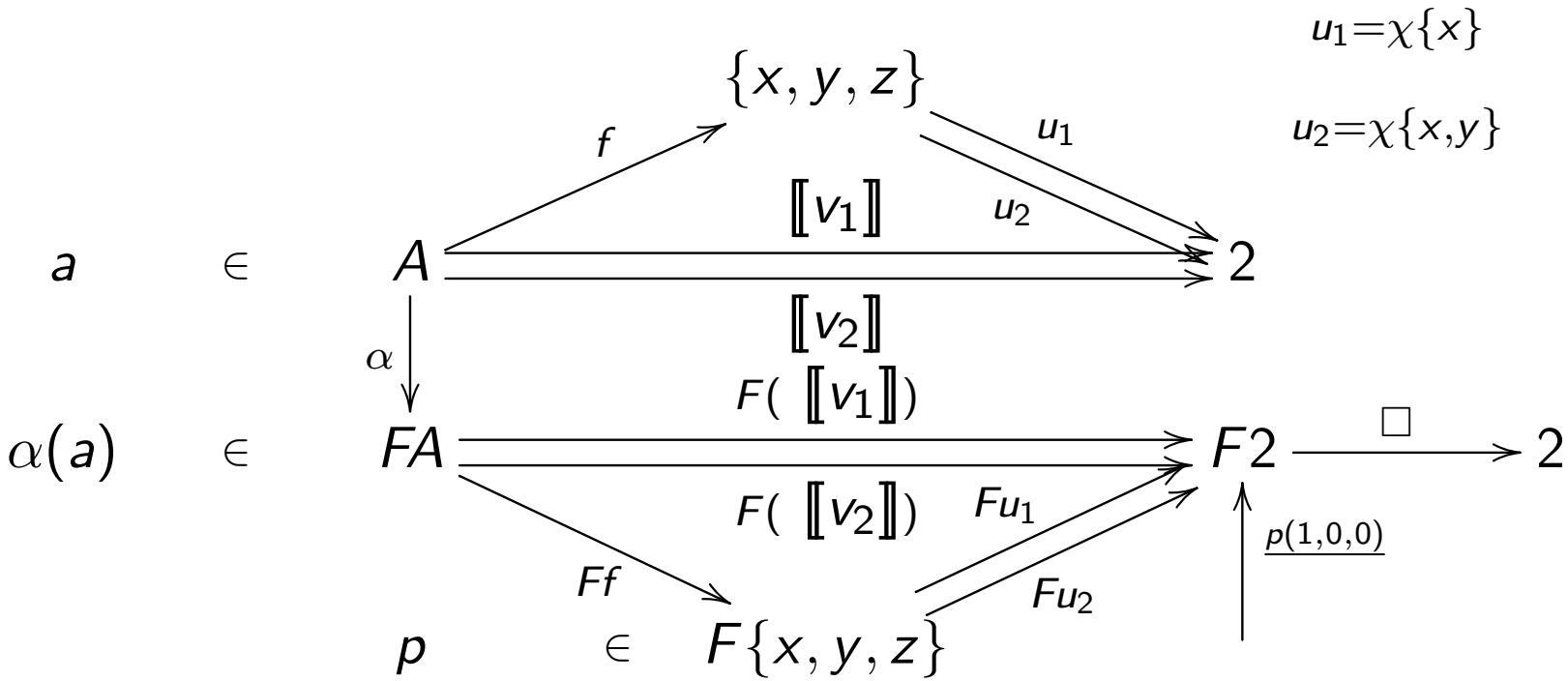

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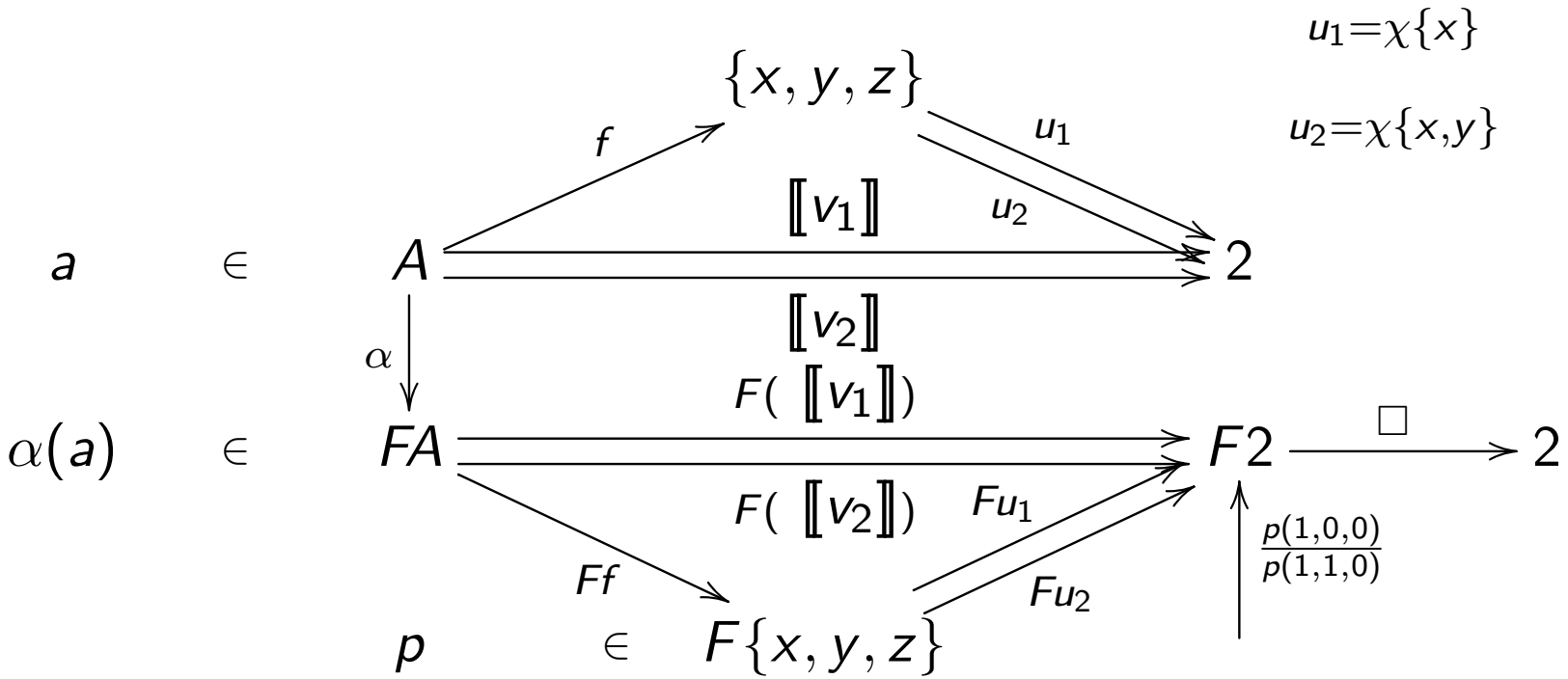

Topological Spaces

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$$T = (X, \tau)$$

- Neighbourhoods

$$\mathfrak{N}(x) := \{U \subseteq X \mid \exists \mathcal{O} \in \tau . a \in \mathcal{O} \subseteq U\}$$

- Topologies as Coalgebras

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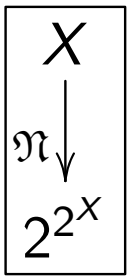

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On objects:

$$2^{2^X} = \mathbb{PP}(X)$$

On maps:

$$f : X \longrightarrow Y \qquad \text{yields} \qquad 2^{2^f} : 2^{2^X} \longrightarrow 2^{2^Y}$$

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2^{2^-}-coalgebras: Neighbourhood systems

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- Notation:

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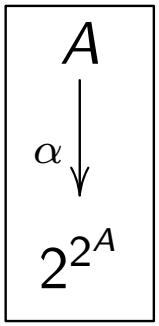

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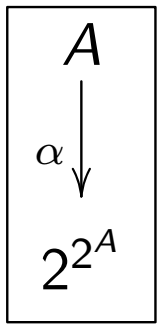

2^{2^-}-coalgebras: Neighbourhood systems

- 2^{2^-}-coalgebra:
- Notation:
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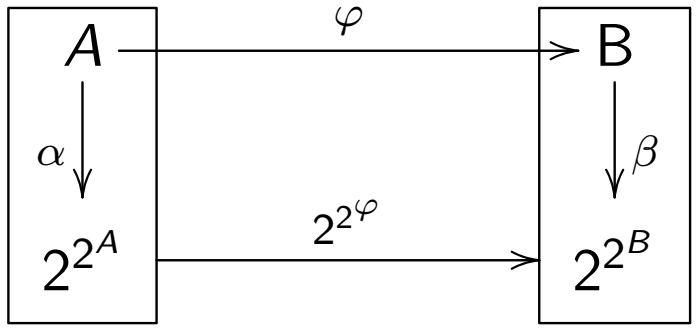

2^{2^-}-coalgebras: Neighbourhood systems

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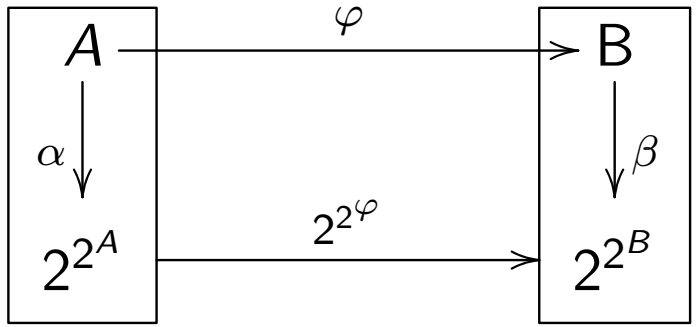

$$a \longrightarrow\!\!\!\!\!\!\triangleright\ \varphi^{-1}[V] \iff \varphi(a) \longrightarrow\!\!\!\!\!\!\triangleright\ V$$

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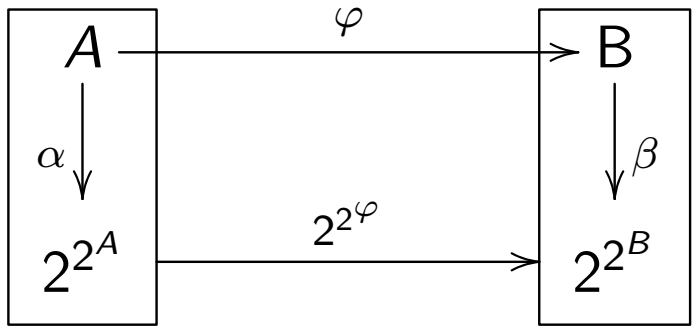

Topological spaces with open maps form a subcovariety of neighbourhood systems.

UnCurry-Curry

Curry:

$$X \times Y \to Z \cong X \to Z^Y$$

UnCurry-Curry

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Functor action:

$$A \xrightarrow{f} 2 \qquad \mapsto \qquad F(A) \xrightarrow{Ff} F(2)$$

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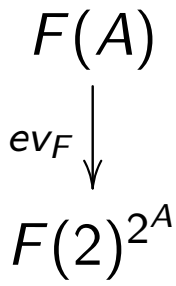
Curried:

$$F(A) \xrightarrow{\quad ev_F \quad} F(2)^{2^A}$$

Naturality

It's a natural transformation ...



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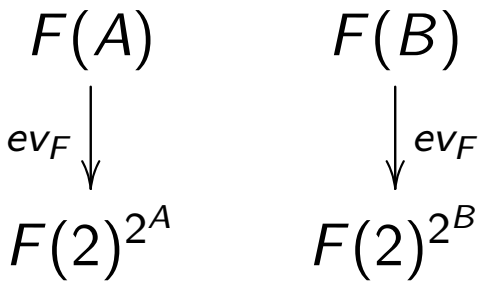

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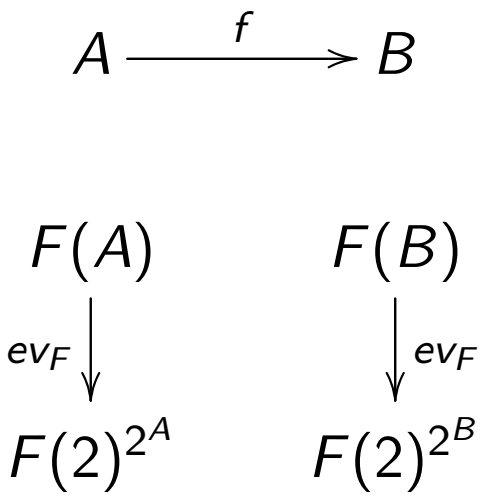

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$$A \xrightarrow{f} B$$

$$\begin{array}{ccc} F(A) & \xrightarrow{Ff} & F(B) \\ {\scriptstyle ev_F}\downarrow & & \downarrow{\scriptstyle ev_F} \\ F(2)^{2^A} & \xrightarrow{2^{2^f}} & F(2)^{2^B} \end{array}$$

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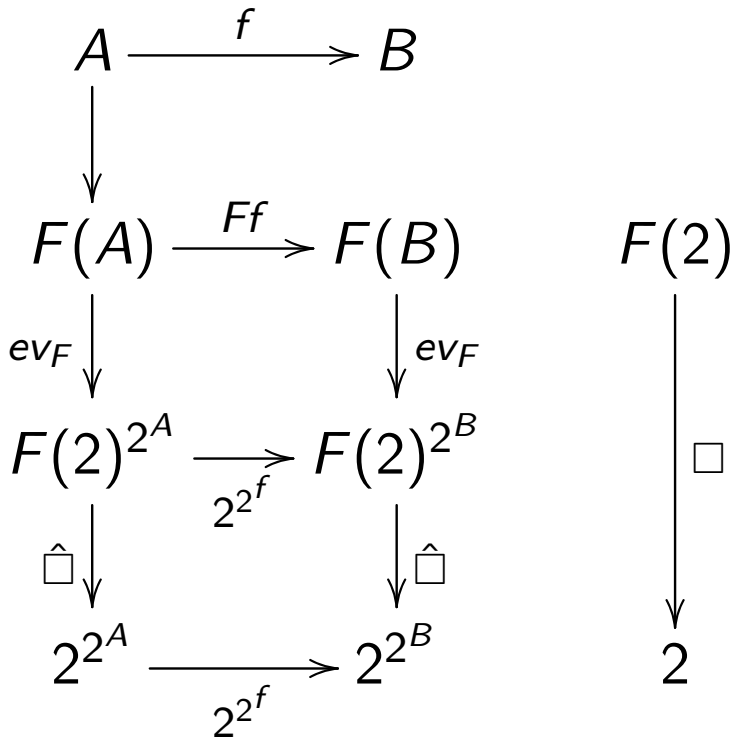

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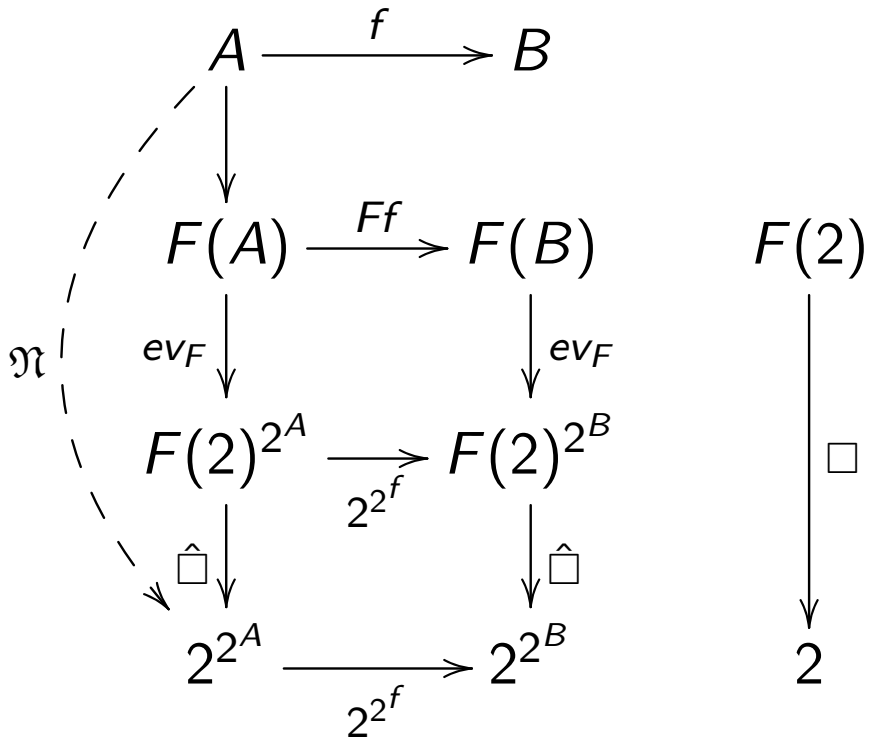

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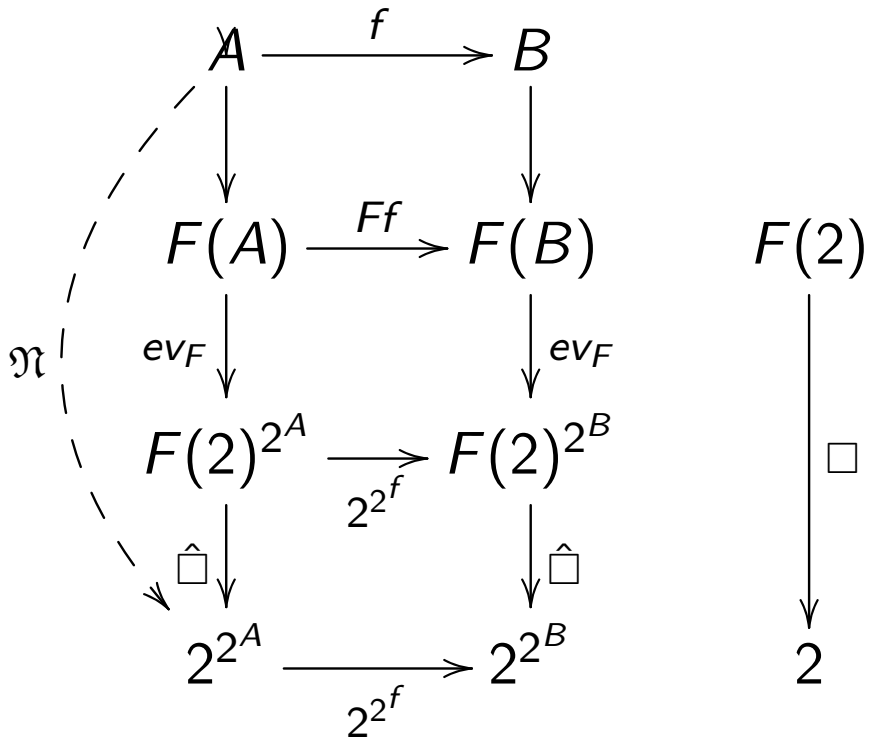

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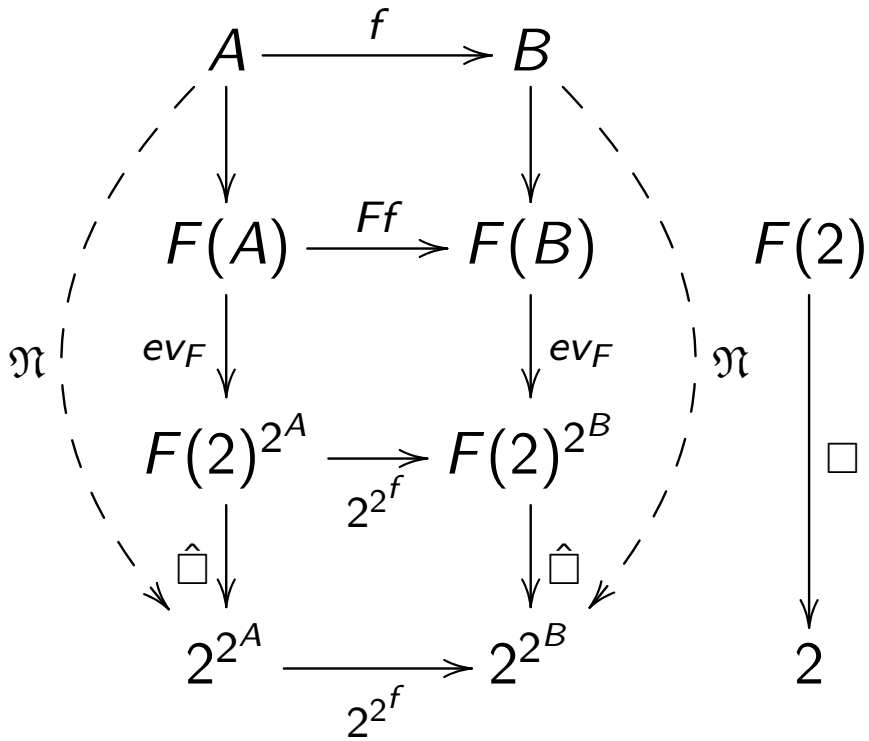


A full embedding

- Each $\Box_i : F(2) \to 2$ yields a functor

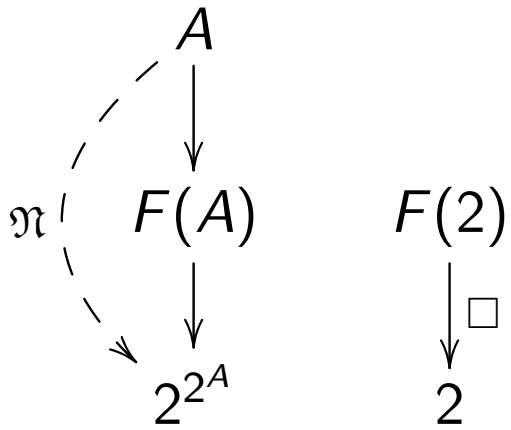
$$\mathcal{S}et_F \to \mathcal{S}et_{2^{2^-}}$$

- each bit-encoding $\Box : F(2) \hookrightarrow 2^\kappa$ where $\Box_i = \pi_i \circ \Box$ yields a full embedding:

$$\mathcal{S}et_F \hookrightarrow (\mathcal{S}et_{2^{2^-}})^\kappa$$

- Relation to modal logic:

$$a \models \Box_i \phi \qquad \Longleftrightarrow \qquad a \xrightarrow{\quad}_i \triangleright [\![\phi]\!]$$



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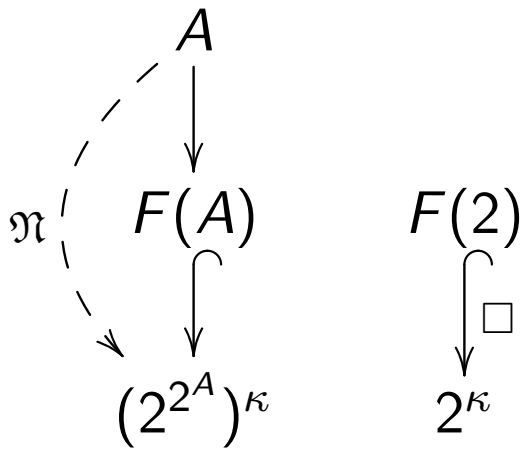

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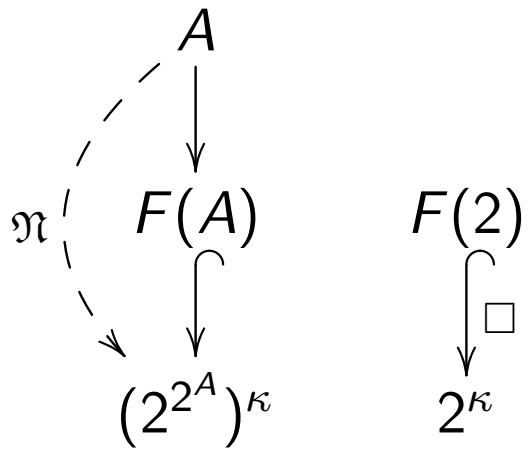

Conclusion

- Coalgebras model state based systems
  - elegant algebraic theory
  - covers all familiar systems
    - minimality
    - observational equivalence
    - coequational (global) logic
- Modal logic
  - framework for arbitrary coalgebra
  - adequate and expressive
  - finitary for finitary type functor
- A lot more
  - Algebra-coalgebra maps
  - Recursive coalgebras
    - ... but that's another story ...

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Thanks

$$T^h @_n x$$