Forcing, Equivalence Relations and Marker Structures

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Basic objects of study are Borel equivalence relations E on Polish spaces X. We frequently regard X as a standard Borel space.

The notion of complexity is provided be the concept of reduction.

Definition

- We say E is reducible to F, $E \leq F$, if there is a Borel function $f \colon X \to Y$ such that $xE\,y \Leftrightarrow f(x)F\,f(y)$.
- We say E is bi-reducible with F, $E \sim F$, if $E \leq F$ and $F \leq E$.
- We say E is emdeddable into F, $E \sqsubseteq F$, if in addition f is one-to-one.

Note that a reduction gives a definable injection from X/E to Y/F so reduction can be viewed as a notion of definable cardinality for these quotient spaces.

We say E is a countable (Borel) equivalence relation if all classes of E are countable.

If G is a Polish group and G acts on X, then the orbit equivalence relation E_G is defined by

$$xE_G\, y \Leftrightarrow \exists g \in G\ (g \cdot x = y).$$

The Feldman-Moore theorem says that every countable Borel equivalence relation is given by the Borel action of a countable group G. The case $G = \mathbb{Z}$ is the classical case of discrete-time dynamics.

So, we can study the equivalence relations E_G group by group.

The simplest equivalence relations are the smooth or tame ones.

Definition

E is smooth if there is a Borel reduction of E to equality relation on a Polish space.

So, for a smooth E, X/E can be regarded as a subset of a standard Borel space.

For countable Borel E, smooth is the same as saying there is a Borel selector for E.

Definition

E_0 is the equivalence relation on 2^ω given by

$$xE_0\, y \Leftrightarrow \exists n\, \forall m \geq n\, (x(m) = y(m)).$$

The Harrington-Kechris-Louveau theorem says that if E is a Borel equivalence relation then either E is smooth or $E_0 \sqsubseteq E$.

So, there is no complexity class of equivalence relation strictly between the smooth relation $E_=$ and E_0.

If G is a Polish group, G acts of $F(G)$ by the shift action

$$g \cdot F = \{gf \colon f \in F\}$$

We can view this action as being on 2^G by

$$g \cdot x(h) = x(g^{-1}h)$$

We call this the Bernoulli (left) shift action of G on 2^G. When G is countable, 2^G is a compact Polish space in the natural product topology.

Countable Equivalence Relations

We let $E(2^G)$ denote the shift action of G on 2^G, and $F(2^G)$ denote the free part of 2^G with the shift action.

Theorem (Dougherty-J-Kechris)

The shift action of F_2 on 2^{F_2} is a universal countable Borel equivalence relation, that is, $E \leq E(2^{F_2})$ for any countable Borel E.

In general, the shift action is more or less universal for actions of G:

Fact

Let E the the orbit equivalence relation for a Borel action of the countable group G on a Polish space X. Then

$$E \leq E((2^\omega)^G) \leq E(2^{G \times \mathbb{Z}}).$$

Definition

A countable Borel equivalence relation E is hyperfinite if E is the increasing union of relations E_n with finite classes.

Theorem (Slaman-Steel)

The following are equivalent:

- *E is hyperfinite.*
- *$E = E_G$ where $G = \mathbb{Z}$.*
- *The classes of E can be uniformly Borel ordered in type $\mathbb{Z}$ (or are finite).*

Markers

Definition

Let E be a Borel equivalence relation. A marker set M is a Borel set $M \subseteq X$ such that $M \cap [x] \neq \emptyset$, $M^c \cap [x] \neq \emptyset$ for every $x \in X$.

Usually we require some additional properties on M, related to the structure of G.

Many argument in dynamics/ergodic theory and descriptive dynamics use markers sets with certain properties (e.g., Rochlin's lemma, Ornstein's theorem, Slaman-Steel theorem).

Hyperfiniteness proofs also typically use marker arguments.

Theorem (Weiss)

Every Borel action by $\mathbb{Z}^n$ is hyperfinite.

Theorem (Gao-J)

Every Borel action by a countable abelian group is hyperfinite.

Weiss' proof (and several other proofs of this result) use a basic marker lemma:

Lemma

For each m, there is a relatively clopen $M_m \subseteq F(2^{\mathbb{Z}^n})$ such that

1. $\forall x \neq y \in M_m\ [\rho(x, y) > m]$
2. $\forall x \in F(2^{\mathbb{Z}^n})\ \exists y \in M_m\ [\rho(x, y) \leq m]$

For the abelian result, we need markers with more regularity.

By a set of marker regions we mean a Borel equivalence relation $\mathcal{R} \subseteq E$ with $\mathrm{dom}(\mathcal{R})$ a complete section and all classes of $\mathcal{R}$ finite.

We say $\mathcal{R}$ is clopen if for each $g \in G$ the set $\{x \in X : x \mathcal{R}\, g \cdot x\}$ is relatively clopen in $\mathrm{dom}(E)$.

We say the marker regions from a tiling if $\mathrm{dom}(\mathcal{R}) = \mathrm{dom}(E)$.

Lemma

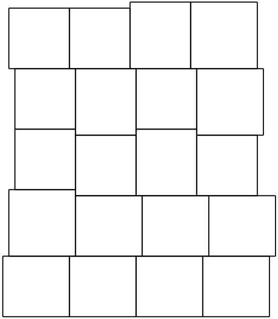
For each n, there is a clopen set of markers $\mathcal{R}_n$ for $F(2^{\mathbb{Z}^m})$ which form a tiling and such that each $\mathcal{R}$ class is a rectangle with each side length in $\{n, n+1\}$.

We call this a clopen, almost square tiling.

The following question arises in several problems.

Question

Can we get a (Borel or clopen) rectangular tiling of $F(2^{\mathbb{Z}^m})$ which is "almost lined-up"?



Note that a (Borel or clopen) almost lined-up tiling would have the following consequences:

- There would be a (Borel or clopen) "lining" of $F(2^{\mathbb{Z}\times\mathbb{Z}})$.
- There would be a (Borel or continuous) proper action of $\mathbb{Z}\times\mathbb{Z}$ on each class of $F(2^{\mathbb{Z}\times\mathbb{Z}})$.

The existence of a lining seems to be related to the (Borel, continuous) chromatic number problem for $F(2^{\mathbb{Z}^m})$.

Theorem (Kechris-Soleci-Todorcevic)

$3 \leq \chi_b(m) \leq m+1.$

Theorem (Gao-J)

$3 \leq \chi_b(m) \leq \chi_c(m) \leq 4.$

2-colorings and minimality

Definition

A 2-coloring of a group G is an $x \colon G \to \{0,1\}$ satisfying the following: for every $s \neq 1_G$, there is a finite $T = T(s) \subseteq G$ such that:

$$\forall g \in G \; \exists t \in T \; (x(gt) \neq x(gst)).$$

The notion of a 2-coloring was formulated independently by Pestov, and Glassner-Uspensky independently showed many groups admit 2-colorings.

Fact

$x \in 2^G$ is a 2-coloring iff $\overline{[x]} \subseteq F(2^G)$.

Definition

$x \in 2^G$ is minimal if $\overline{[x]}$ is a minimal closed invariant set (subflow), that is, $\forall y \in \overline{[x]}\ (\overline{[y]} = \overline{[x]})$.

Being minimal has a combinatorial reformulation.

Fact

$x \in 2^G$ is minimal iff for every $A \in G^{<\omega}$ there is a $T \in G^{<\omega}$ such that

$$\forall g \in G\ \exists t \in T\ \forall a \in A\ (x(gta) = x(a)).$$

Remark

Minimal x exist in any subflow of any 2^G (don't need AC in fact).

Theorem (Gao-J-Seward)

Every countable group G has a 2-coloring.

So, there is a compact invariant set $\overline{[x]} \subseteq F(2^G)$.

An early consequence of this was the following. Recall (Slaman-Steel) that for any countable equivalence relation there are Borel complete sections B_n such that $\bigcap_n B_n = \emptyset$.

Corollary

Let $B_n \subseteq F(2^G)$ be relatively clopen complete sections. Then $\bigcap_n B_n \neq \emptyset$.

Theorem (GJS; minimal 2-coloring forcing)

For any countable group Γ *there is separative forcing notion* $\mathbb{P}_{mc}$ *on which* Γ *acts by automorphisms and such that*

$$\emptyset \Vdash (x_G \text{ is a minimal 2-coloring of } \Gamma).$$

The forcing can be described directly, or an instance of orbit-forcing.

Definition

Let $x \in F(2^\Gamma)$. $\mathbb{P}_x$ is the forcing notion

$$\mathbb{P}_x = \{p \in 2^{<\Gamma} \colon \exists g \in \Gamma \ (p = g \cdot x \restriction \mathrm{dom}(p))\}$$

A generic G for $\mathbb{P}_x$ produces an $x_G \in \overline{[x]}$.

If x is a minimal 2-coloring, then x_G will also be a minimal 2-coloring.

- Varying x can produce different forcing effects.
- The forcings can also be described directly by (usually) finitary $\hat{p} \in 2^{<G}$ with extra side-conditions.

To illustrate the give the direct definition of $\mathbb{P}_{mc}$ for the case $\Gamma = \mathbb{Z} \times \mathbb{Z}$.

$\mathbb{P}_{mc}$ consists of conditions

$$p = (\hat{p}; s_0, \dots, s_n; T_0, \dots, T_n; A_0, \dots, A_m; U_0, \dots, U_m)$$

satisfying the following:

1. $\hat{p} \in 2^R$ where $R = [a, b] \times [c, d] \subseteq \mathbb{Z} \times \mathbb{Z}$.
2. $T_0, \dots, T_n,\ U_0, \dots, U_m \in 2^{<(\mathbb{Z}\times\mathbb{Z})}$.
3. $A_i \in 2^{<(\mathbb{Z}\times\mathbb{Z})}$ and $\exists h\, [\hat{p} \restriction (h \cdot (\mathrm{dom}(A_i))) = A_i$.
4. $\forall g \in \mathrm{dom}(\hat{p})\ \forall i \leq n\ \exists t \in T_i\ [gt, gst \in \mathrm{dom}(\hat{p}) \wedge \hat{p}(gt) \neq \hat{p}(gst)]$
5. $\forall g \in \mathrm{dom}(\hat{p})\ \forall i \leq m\ \exists t \in U_i\ [\hat{p} \restriction (gt \cdot (\mathrm{dom}(A_i))) = A_i]$ and
   $\forall g \in \mathrm{dom}(\hat{p})\ \forall i \leq m\ \exists t \in U_i\ [\hat{p} \restriction (gt \cdot (\mathrm{dom}(A_i))) = 1 - A_i]$

We have the following facts about $\mathbb{P}_{mc}$.

Lemma

For any $g \in \mathbb{Z} \times \mathbb{Z}$, $D_g = \{p \colon g \in dom(\hat{p})$ *is dense.*

Lemma

For each $s \neq (0,0)$ *in* $\mathbb{Z} \times \mathbb{Z}$, $D_s = \{p \colon \exists i \ (s = s_i)\}$ *is dense.*

Lemma

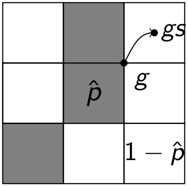
$\forall p \in \mathbb{P}_{mc} \ \forall A \subseteq \hat{p}$

$$D_{p,A} = \{q \colon \exists i \leq m_q \ A \subseteq A_i(q)\} \text{is dense below } p].$$

Let G be a generic for $\mathbb{P}_{mc}$, and let $x_G = \cup\{\hat{p} \colon p \in G\}$. So, $x_G \in 2^{\leq(\mathbb{Z}\times\mathbb{Z})}$.

The first lemma shows that $x_G = 2^{\mathbb{Z}\times\mathbb{Z}}$, the second lemma shows that x_G is a 2-coloring, and the third lemma shows that x_G is minimal.

For example, to show second lemma, copy the domain R of $\hat{p}$ to a larger rectangular domain using copies of $\hat{p}$ and $1-\hat{p}$ in such a way that we block the shift s.



Two theorems for general groups

The following two theorems are proved using $\mathbb{P}_{mc}$.

Theorem (GJS)

Let G be a countable group and E_G the equivalence relation generated by the shift action of G on $F(2^G)$. Let $B_n \subseteq X$ be Borel complete sections, and let $f : \omega \to \omega$ with $\limsup f = \infty$. There there an $x \in F(2^G)$ such that $\exists^\infty n\ \rho(x, B_n) < f(n)$.

Remark

The Slaman-Steel markers are Borel complete sections $B_n \subseteq F(2^{\mathbb{Z}})$ with $\bigcap_n B_n = \emptyset$.

Remark

There does exists a sequence $B_n \subseteq F(2^{\mathbb{Z}^n})$ of relatively clopen complete sections such that for all $x \in F(2^{\mathbb{Z}^n})$ we have $\rho(x, B_n) \to \infty$.

Theorem (GJS)

Let G be a countable group and E_G the equivalence relation generated by the shift action of G on $F(2^G)$. Let $f\colon (F(2^G), E_G) \to (Y, F)$ be a Borel invariant map (i.e., F is a factor of E_G). Then F has a recurrent point.

By a recurrent point $y \in Y$ we mean that for every non-empty open set $U \subseteq Y$ there is a $A \in G^{<\omega}$ such that $\forall z \in [y]\ \exists g \in A\ g \cdot y \in U$.

In fact, for any non-empty Borel set $B \subseteq Y$, there is a $y \in Y$ which is recurrent for B.

Special Groups

We specialize to the groups $G = \mathbb{Z}^n$.

Some of these results are related to the coloring problem for $\mathbb{Z}^n$.

Question (Kechris-Solecki-Todorcevic)

What the Borel/clopen chromatic number of $F(2^{\mathbb{Z}^n})$?

It is known (Gao-Jackson) that

$$3 \leq \chi_b(F(2^{\mathbb{Z}^n})) \leq \chi_c(F(2^{\mathbb{Z}^n})) \leq 4$$

Theorem

There does not exists a Borel coloring $c\colon F(2^{\mathbb{Z}^n}) \to k$ *such that for every* $x \in F(2^{\mathbb{Z}^n})$ *there are arbitrarily large regions in* $[x]$ *which are 2-colored by* c*.*

To prove this we need a variation of the minimal 2-coloring forcing which we call the odd minimal 2-coloring forcing.

Conditions in this forcing $\mathbb{P}_o$ are just like those of $\mathbb{P}$ (the minimal 2-coloring forcing) except we require that the domain of $\hat{p}$ have odd side lengths.

Previous density lemmas go through just as before.

Suppose $c\colon F(2^{\mathbb{Z}^n}) \to k$ is Borel. Let $x = x_G$ where G is generic for $\mathbb{P}_o$.

Suppose $p = (\hat{p}; \cdots) \in G$ and $p \Vdash c(x_G) = 0$, say.

Let $q \leq p$, $q \in G$, be such that $\hat{p} \subseteq A_i$ for some $A_i \in \vec{A}(q)$.

Let $r \leq q$, $r \in G$ be such that there are copies of $\hat{q}$ an odd distance apart in $\hat{r}$ (such sets are dense).

Let $g \in \mathbb{Z}^n$ be such that $g \cdot x \restriction R$ is 2-colored by c, where R is sufficiently large (say twice the size of R).

For some $h \in \mathbb{Z}^n$ we have $hg \cdot x \restriction \mathrm{dom}(r) = \hat{r}$ and $hg(\mathrm{dom}(r)) \subseteq R$. This is a contradiction as $gh \cdot x$ is still generic.

Theorem

Let $B \subseteq F(2^{\mathbb{Z}^n})$ be Borel. Then there is an $x \in F(2^{\mathbb{Z}^n})$ and a rectangular lattice $L \subseteq [x]$ such that either $L \subseteq B$ or $L \subseteq B^c$. If B is a complete section, then we have $L \subseteq B$.

We use another variation of the minimal 2-coloring forcing. We use a forcing which builds a minimal 2-coloring but all conditions have a periodicity requirement.

Conditions of the form

$$p = (R, \Delta, \{a, b\}, c, \Lambda)$$

- $R \subseteq \mathbb{Z} \times \mathbb{Z}$ is a rectangle.
- Δ is a translate of a rectangular lattice L and $\mathbb{Z}^2$ is the disjoint union of δR for $\delta \in \Delta$.
- $\{a, b\} \subseteq R$
- $c \colon (\cup_{\delta \in \Delta} \delta(R - \{a, b\}) \to \{0, 1\}$.
- $\Lambda \subseteq L$ is a rectangular lattice and c has period Λ.
- (local recognizability) If $x \in \Delta$, $y \notin \Delta$, then there is a $g \in R$ such that $c(gx) \neq c(gy)$ and both are defined.

Remark

The local recognizability condition is not necessary as it will hold generically.

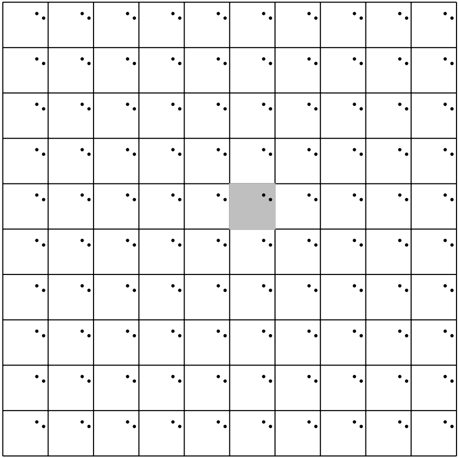


Figure: a condition in the forcing

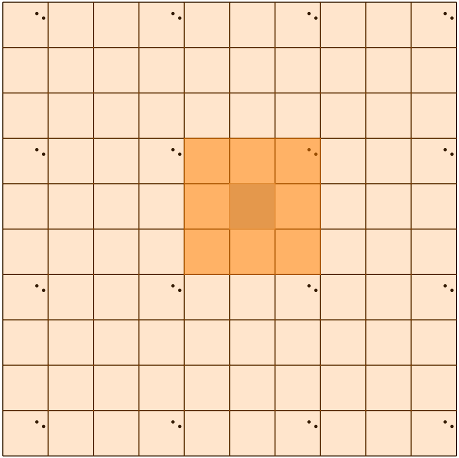


Figure: the extension relation

Recent Results

Using variations of minimal 2-colorings we have the following.

Theorem

There is no continuous "lining" of $F(2^{\mathbb{Z}\times\mathbb{Z}})$.

Corollary

This is no clopen, almost lined up rectangular marker regions for $F(2^{\mathbb{Z}\times\mathbb{Z}})$.

Extending (and simplifying) these arguments Ed Krohne has shown:

Theorem

There is no continuous 3-coloring of $F(2^{\mathbb{Z}\times\mathbb{Z}})$.

So we have:

$$\chi_c(F(2^{\mathbb{Z}^n})) = \begin{cases} 3 & \text{if } n = 1 \\ 4 & \text{if } n \geq 2 \end{cases}$$

For $n \geq 2$ we still don't know $\chi_b(F(2^{\mathbb{Z}^n}))$.