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The assembly, Smyth's stable compactifications and the patch frame

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What's what

A frame

...has the algebraic structure of a topology. Use frames (locales) as substitutes for spaces.

The assembly of a frame

- ...categorically, is the analogue of the powerset (object of subobjects).
- ...topologically, is the analogue of declaring open sets to be closed.

The patch topology

...declares all compact (saturated) sets to be closed.

Stable compactification

...is the T_0 analogue of Hausdorff compactification.

The assembly of a frame as a pushout

$$
\begin{array}{ccc}
\mathrm{Idl}\, L & \overset{\bigsqcup}{\twoheadrightarrow} & L \\
\big\downarrow & & \big\downarrow \\
\mathrm{Patch}\,\mathrm{Idl}\, L & \twoheadrightarrow & \mathrm{N}\, L
\end{array}
$$

Figure: Idl L = largest stable compactification (ideal completion), Patch Idl L = compact regular reflection, N L = assembly.

How to construct the patch of a stably continuous frame (Jung, Moshier)

Start with a stably continuous frame M (e.g. Idl L). Its *Lawson dual* $M^\wedge$ (Scott open filters, ordered by inclusion) is another stably continuous frame. Construct a frame by generators and relations:

Generators

- One generator $\ulcorner a \urcorner^+$ for every element of M,
- One generator $\ulcorner \phi \urcorner^-$ for every Scott open filter $\phi \in M^\wedge$.

Relations enforcing that

- $\ulcorner - \urcorner^+$ and $\ulcorner - \urcorner^-$ are frame homomorphisms.
- If a is a lower bound of ϕ then $\ulcorner a \urcorner^+ \sqcap \ulcorner \phi \urcorner^- = 0$
- If ϕ contains a then $\ulcorner \phi \urcorner^- \sqcup \ulcorner a \urcorner^+ = 1$

How to construct the pushout

Start with a frame L. The Lawson dual of the ideal completion is the frame $\mathrm{Filt}\, L$ of all filters, ordered by inclusion. Construct the frame $\mathrm{N}\, L$ by generators and relations:

Generators

- One generator $\ulcorner a \urcorner^{+}$ for every element of L,
- One generator $\ulcorner \phi \urcorner^{-}$ for every filter $\phi \in \mathrm{Filt}\, L$.

Relations enforcing that

- $\ulcorner - \urcorner^{+}$ and $\ulcorner - \urcorner^{-}$ are frame homomorphisms.
- If a is a lower bound of ϕ then $\ulcorner a \urcorner^{+} \sqcap \ulcorner \phi \urcorner^{-} = 0$
- If ϕ contains a then $\ulcorner \phi \urcorner^{-} \sqcup \ulcorner a \urcorner^{+} = 1$

The patch of a continuous frame

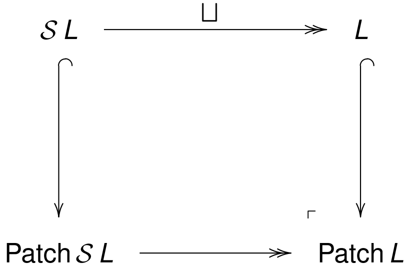

Figure: $\mathcal{S}\,L =$ smallest stable compactification.

How to construct the pushout

Start with a continuous frame L. The Lawson dual of L is a continuous preframe $L^\wedge$. Construct the frame Patch L by generators and relations:

Generators

- One generator $\ulcorner a \urcorner^+$ for every element of L,
- One generator $\ulcorner \phi \urcorner^-$ for every Scott open filter $\phi \in L^\wedge$.

Relations enforcing that

- $\ulcorner - \urcorner^+$ and $\ulcorner - \urcorner^-$ preserve all existing joins and finite meets.
- If a is a lower bound of ϕ then $\ulcorner a \urcorner^+ \sqcap \ulcorner \phi \urcorner^- = 0$
- If ϕ contains a then $\ulcorner \phi \urcorner^- \sqcup \ulcorner a \urcorner^+ = 1$

The patch of a locally compact space is a pullback

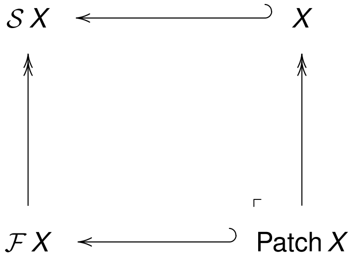

Figure: $\mathcal{S}\,X =$ Smyth's smallest stable compactification, $\mathcal{F}\,X =$ Fell compactification.

Perfect frame homomorphisms

A frame homomorphism is *perfect* if its right adjoint is Scott continuous. Lawson duality is a contravariant endofunctor on preframes. Our patch construction is functorial on perfect frame homomorphisms.

$$L \underset{f_*}{\overset{f}{\rightleftarrows}} M \quad (\bot)$$

$$L^{\wedge} \xrightarrow[(f_*)^{\wedge}]{} M^{\wedge}$$

Summary (in terms of locale theory)

- The general construction universally solves the problem of transforming an auxiliary relation into the well-inside relation.
- New, easy construction of the assembly as an ordered locale. Frame of filters serves as lower opens w.r.t. the specialisation order of the original locale
- Extended the patch construction from stably locally compact locales to locally compact locales. Previous patches are sublocales of ours.
- Retain the universal property of the patch, retain functoriality, but lose the coreflection.

Details to appear in *Algebra Universalis*:

A presentation of the assembly of a frame by generators and relations exhibits its bitopological structure.

Yet another patch construction for continuous frames, and connections to the Fell compactification.