Nilpotence and dualizability

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JOHANNES KEPLER UNIVERSITY LINZ | JKU

FWF Der Wissenschaftsfonds.

What is a natural duality?

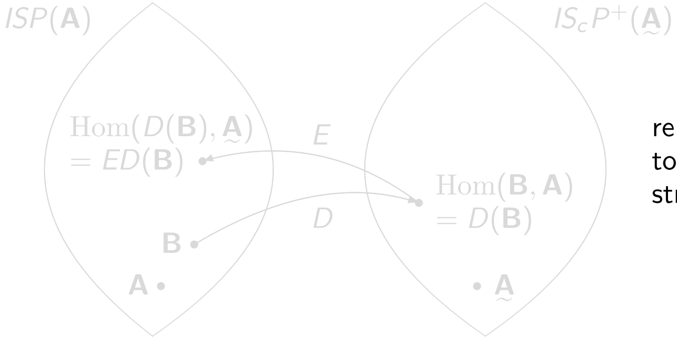
General idea (cf. Clark, Davey, 1998):

1. A duality is a correspondence between a category of algebras and a category of relational structures with topology.
2. **Representation:** Elements of the algebras are represented as continuous, structure preserving maps.
3. Classical example: **Stone duality** between Boolean algebras and Boolean spaces (totally disconnected, compact, Hausdorff)
4. Application, e.g., completions of lattices

For a finite algebra $\mathbf{A} = \langle A, F \rangle$, let $\underset{\sim}{\mathbf{A}} = \langle A, \mathcal{R}, \tau_d \rangle$ be an **alter ego**.

- $\mathcal{R} \subseteq \bigcup_{n \in \mathbb{N}} \{B \leq \mathbf{A}^n\} =: \mathrm{Inv}(\mathbf{A})$
- τ_d ... discrete topology on A

algebras



relational topological structures

A is **dualized** by $\underset{\sim}{\mathbf{A}}$ if $\forall \mathbf{B} \in ISP(\mathbf{A})$:

$$ED(\mathbf{B}) = \{e_b \colon \mathrm{Hom}(\mathbf{B}, \mathbf{A}) \to A, h \mapsto h(b) \mid b \in B\}$$

"Every morphism from $D(\mathbf{B})$ to $\underset{\sim}{\mathbf{A}}$ is an evaluation."

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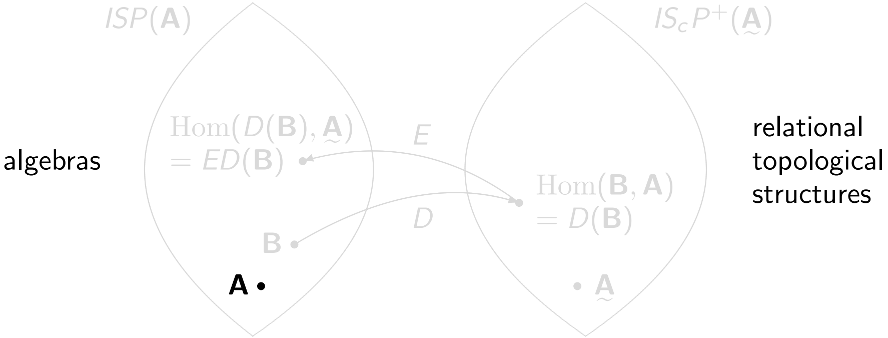

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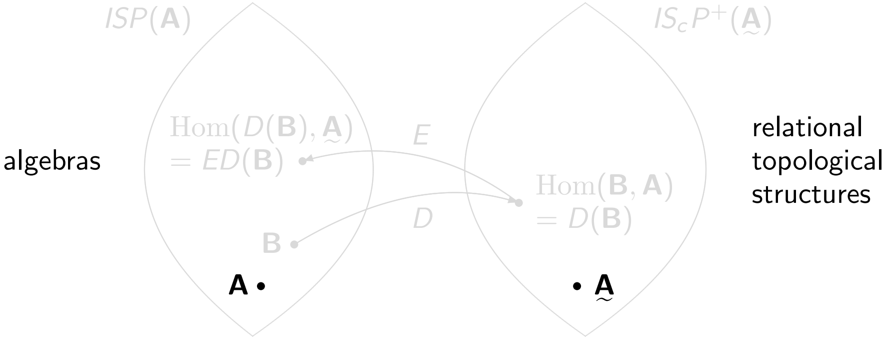

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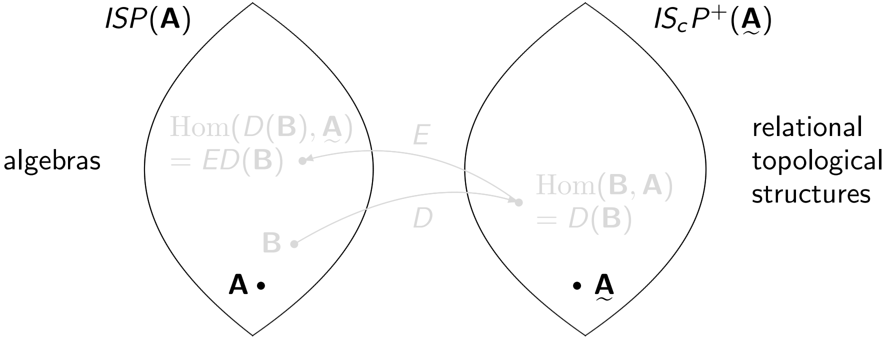

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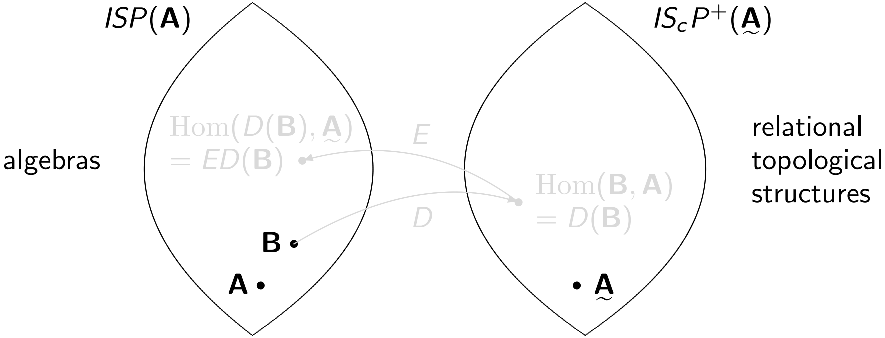

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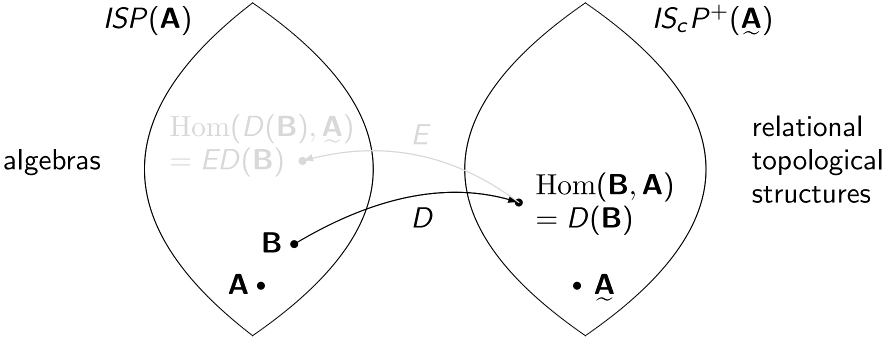

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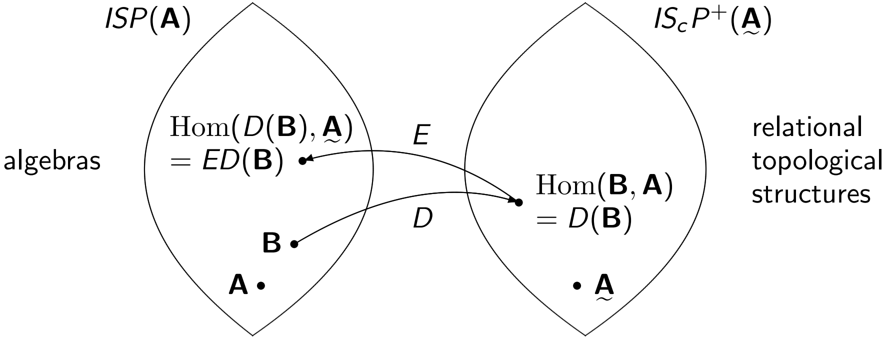

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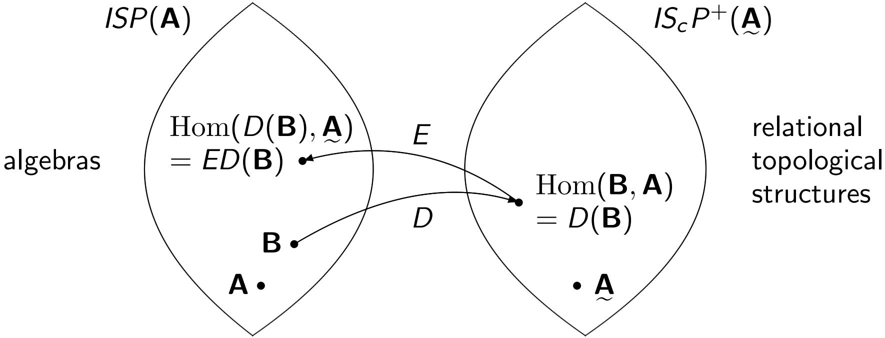

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When can $\mathbf{A}$ be dualized by some $\underset{\sim}{\mathbf{A}}$?

$\mathbf{A}$ is **not dualizable** iff $\exists \mathbf{B} \leq \mathbf{A}^S$ and a morphism α from $D(\mathbf{B}) \leq \underset{\sim}{\mathbf{A}}^B$ to $\underset{\sim}{\mathbf{A}} := \langle A, \mathrm{Inv}(\mathbf{A}), \tau_d \rangle$ that is not an evaluation.

Theorem (Davey, Heindorf, McKenzie, 1995)

Let $\mathbf{A}$, finite, in a CD variety. Then $\mathbf{A}$ is dualizable iff $\mathbf{A}$ has a NU-term.

Problem (Clark, Davey, 1998)

Characterize dualizable algebras in CP varieties (= **Mal'cev algebras**).

Theorem ($\Rightarrow$ Quackenbush, Szabó 2002, $\Leftarrow$ Nickodemus 2007)

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Nilpotence and beyond

There is a generalization of commutators, abelianess, nilpotence, ... from groups to algebras in CM varieties (Freese, McKenzie, 1987).
A Mal'cev algebra **A** is **supernilpotent** if $[1_A, \dots, 1_A] = 0_A$ for some higher commutator (Bulatov, 2001; Aichinger, Mudrinski, 2010).

Lemma (cf. Freese, McKenzie, 1987, Kearnes 1999)

For a finite nilpotent Mal'cev algebra **A** TFAE:

1. **A** is supernilpotent.
2. **A** is polynomially equivalent to a direct product of algebras of **prime power order and finite type**.
3. $\exists k \in \mathbb{N}$: every term operation on **A** is a "sum of at most k-ary commutator operations".

Examples of supernilpotent algebras

Finite nilpotent groups, nilpotent rings, $\langle \mathbb{Z}_4, +, 2x_1 \dots x_k \rangle \dots$

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Our main result

Theorem (Bentz, M, submitted 2012)

Finite non-abelian supernilpotent Mal'cev algebras are (inherently) non-dualizable.

Corollary

The following finite algebras are not dualizable:

1. groups with nonabelian Sylow subgroups (Quackenbush, Szabó, 2002)
2. rings with nilpotent subring S and $S^2 \neq 0$ (Szabó, 1999; Clark, Idziak, Sabourin, Szabó, Willard, 2001)
3. non-abelian loops with nilpotent multiplication group

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How to show that **A** is not dualizable

The ghost element method

Find $\mathbf{B} \leq \mathbf{A}^S$ and $\alpha\colon \mathrm{Hom}(\mathbf{B},\mathbf{A}) \to A$ such that

1. α is continuous,
   α depends only on a finite subset of indices of B,
2. $\mathrm{Inv}(\mathbf{A})$-preserving,
   on any finite set of homomorphisms, α is an evaluation at some $b \in B$
3. not an evaluation at any $b \in B$.
   the tuple $(\alpha(\pi_s))_{s\in S}$ is not in B.

Then **A** is not dualizable.

Proof idea for our Theorem

1. Supernilpotence of **A** yields a nice representation of term operations.
2. This allows to construct $\mathbf{B} \leq \mathbf{A}^{\mathbb{Z}}$ and $\alpha\colon \mathrm{Hom}(\mathbf{B},\mathbf{A}) \to A$ with properties 1,2,3.

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Nilpotence alone is not an obstacle

Theorem (Bentz, M, submitted 2012)

$\mathbf{A} := \langle \mathbb{Z}_4, +, 1, \{2x_1 \cdots x_k \mid k \in \mathbb{N}\} \rangle$ is nilpotent and dualized by
$\underset{\sim}{\mathbf{A}} := \langle \mathbb{Z}_4, \{R \leq \mathbf{A}^4\}, \tau_d \rangle$.

Fun fact

All reducts

$$\langle \mathbb{Z}_4, +, 2x_1x_2, \ldots, 2x_1 \cdots x_k \rangle \qquad (k \in \mathbb{N})$$

of finite type are supernilpotent, hence non-dualizable.

Duality via partial clones

Partial operations on "conjunct-atomic definable" domains

$\mathrm{Clo}(\mathbf{A})\ldots$ term operations on $\mathbf{A}$

$\mathrm{Clo}_{\mathrm{cad}}(\mathbf{A}) := \{f|_D \colon f \in \mathrm{Clo}(\mathbf{A}), D \text{ is } \underbrace{\text{solution set of term identities on } \mathbf{A}}_{\mathrm{cad}}\}$

For $D \subseteq A^k$, a partial op $f : D \to A$ **preserves** a relation $R \subseteq A^n$ if

$$\forall r_1, \ldots, r_k \in R : f(r_1, \ldots, r_k) \in R \text{ whenever defined.}$$

Lemma (Davey, Pitkethly, Willard, 2012)

Assume $\mathbf{A}$ and $\mathcal{R} \subseteq \mathrm{Inv}(\mathbf{A})$ are **finite** such that $\mathrm{Clo}_{\mathrm{cad}}(\mathbf{A})$ is the set of all $\mathcal{R}$-preserving operations with cad domains over $\mathbf{A}$.
Then $\mathbf{A}$ is dualized by $\underset{\sim}{\mathbf{A}} := \langle A, \mathcal{R}, \tau_d \rangle$.

Follows from Third Duality Theorem and Duality Compactness.

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$\mathbf{A} := \langle \mathbb{Z}_4, +, 1, \{2x_1 \cdots x_k \mid k \in \mathbb{N}\} \rangle$ is dualizable

Proof idea:

1. Solution sets $D \subseteq \mathbb{Z}_4^k$ of term identities can be described explicitly.
2. $\mathrm{Clo}_{\mathrm{cad}}(\mathbf{A})$ is determined by the unary term operations and the 4-ary commutator relations just like $\mathrm{Clo}(\mathbf{A})$.

Open

Problem

Is every finite abelian Mal'cev algebra dualizable?

Finite ring modules are dualizable (Kearnes, Szendrei, announced).

Problem

Let **A** be a finite Mal'cev algebra with a non-abelian supernilpotent congruence α, i.e., $[\alpha, \ldots, \alpha] = 0$. Is **A** non-dualizable?

Yes, if **A** is nilpotent (Bentz, M).

Supernilpotence is not the only obstacle for dualizability

$\langle S_3, \cdot, \text{all constants}\rangle$ is not dualizable (Idziak, unpublished) but all its (super)nilpotent congruences are abelian.

Wild guess

A finite nilpotent **A** is dualizable iff all supernilpotent algebras in $HSP(\mathbf{A})$ are abelian.

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