Graph Theory and Modal Logic

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Contents of this Talk

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1. Graphs = Kripke frames.

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2. Completeness for the basic hybrid logic **H**.

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3. The hybrid logic **G** for all graphs.

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2. Completeness for the basic hybrid logic **H**.

3. The hybrid logic **G** for all graphs.

4. Hybrid formulas characterizing some properties of graphs .

Why symmetric frames?

= My research history =

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Quantum Logic = a logic of quantum mechanics

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$\Downarrow$

Orthologic /orthomodular logic

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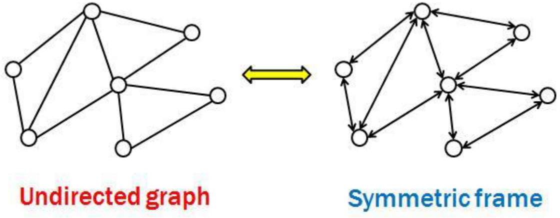
Modal logic **KTB** and its extension

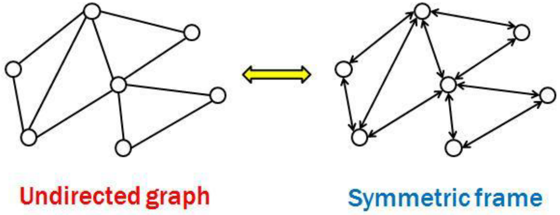
$\cdots$ complete for reflexive and symmetric frames.

Undirected Graphs = Symmetric Kripke frames



Kripke frames and graphs

Undirected Graphs = Symmetric Kripke frames



Every point (node) in an undirected graph must be treated as an **irreflexive point**!

To characterize irreflexivity

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Employ a kind of **hybrid language** (NOMINALS)

A Hybrid Language

- ▷ 2 sorts of variables:
  - $\Phi := \{p, q, r, \ldots\} \cdots$ the set of prop. variables
  - $\Omega := \{i, j, k, \ldots\} \cdots$ the set of **nominals**

  where $\Phi \cap \Omega = \emptyset$.

  Nominals are used to distinguish points (states) in a frame from one another.

- ▷ Our language $\mathcal{L}$ (the set of formulas) consists of
  $A ::= \ p \mid i \mid \bot \mid \neg A \mid A \wedge B \mid \Box A$

  $\cdots$ No satisfaction operator ($@_i$)

Normal hybrid logic(1)

A normal hybrid logic **L** over $\mathcal{L}$ is a set of formulas in $\mathcal{L}$ that contains:

(1) All classical tautologies

(2) $\Box(p \to q) \to (\Box p \to \Box q)$

(3) $(i \wedge p) \to \Box^n(i \to p)$ for all $n \in \omega$: **(nominality axiom)**

and closed under the following rules:

(4) Modus Ponens

$$\frac{A, \ A \to B}{B}$$

(5) Necessitation

$$\frac{A}{\Box A}$$

(6) Sorted substitution

$$\frac{A}{A[B/p]}\ ,\ \frac{A}{A[j/i]}$$

p : prop. variable, i, j: nominals

(7) Naming

$$\frac{i \to A}{A}$$

i: not occurring in A

(8) Pasting

$$\frac{(i \wedge \Diamond(j \wedge A)) \to B}{(i \wedge \Diamond A) \to B}$$

$j \not\equiv i$, j:not occurring in A or B.

Normal hybrid logic(3)

H: the smallest normal hybrid logic over $\mathcal{L}$

For $\Gamma \subseteq \mathcal{L}$,

$\mathbf{H} \oplus \Gamma$: the smallest normal hybrid extension containing Γ

Semantics

$\mathcal{F} := \langle W, R \rangle$: a (Kripke) frame

$\mathfrak{M} := \langle \mathcal{F}, V \rangle$: a model,
where, $V : \Phi \cup \Omega \to 2^W$ such that:

For $p \in \Phi$, $V(p)$: a subset of W,
for $i \in \Omega$, $V(i)$: a singleton of W.

Interpretation of a nominal:

$$(\mathfrak{M}, a) \models i \text{ if and only if } V(i) = \{a\}$$

In this sense, i is a name for the point a in this model $\mathfrak{M}$!

Soundness for H

For a frame $\mathcal{F}$,

$$\mathcal{F} \models A \iff_{\text{def}} \forall V \text{ on } \mathcal{F}, \forall a \in W, \big((\langle \mathcal{F}, V \rangle, a) \models A\big)$$

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Theorem (Soundness for the logic H)

For $A \in \mathcal{L}$, $A \in \mathbf{H}$ implies $\mathcal{F} \models A$ for any frame $\mathcal{F}$.

Completeness for $\mathbf{H}$

For $\Gamma \subseteq \mathcal{L}$, $A \in \mathcal{L}$,

$\mathbf{H} : \Gamma \vdash A$
$\Leftarrow \text{def} \Rightarrow \exists B_1, B_2, \ldots, B_n \in \Gamma \big(\mathbf{H} \vdash (B_1 \wedge B_2 \wedge \cdots \wedge B_n) \to A\big)$

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Theorem (Strong completeness for the logic $\mathbf{H}$)

For $\Gamma \subseteq \mathcal{L}$, $A \in \mathcal{L}$, suppose that $\mathbf{H} : \Gamma \nvdash A$.
Then there exists a model $\mathfrak{M}$ and a point a such that:

(1) $(\mathfrak{M}, a) \models B$ *for all* $B \in \Gamma$,

(2) $(\mathfrak{M}, a) \not\models A$

FMP and Decidability for $\mathbf{H}$

Theorem

(1) $\mathbf{H}$ *admits filtration, and so, it has the finite model property.*

(2) $\mathbf{H}$ *is decidable.*

Axiom for Irreflexivity

Proposition

For any frame $\mathcal{F} = \langle W, R \rangle$,
$\mathcal{F} \models i \to \Box\neg i$ *if and only if* $\mathcal{F} \models \forall x \in W \big(Not(xRx)\big)$.

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Proof.

($\Rightarrow$:) Suppose that there is a point $a \in W$ s.t. aRa. Define a valuation V as: $V(i) := \{a\}$. Then $a \not\models i \to \Box\neg i$
($\Leftarrow$:) Suppose $\mathcal{F} \not\models i \to \Box\neg i$. Then, ther exists $a \in W$, s.t. $a \models i$, but $a \not\models \Box\neg i$, which is equivalent to $a \models \Diamond i$. The latter means that there is $b \in W$ s.t. aRb and $b \models i$. Then, $V(i) = \{a\} = \{b\}$. Thus $a = b$ and that aRa $\Box$

The logic G for undirected graphs

$$\mathbf{G} := \mathbf{H} \oplus (p \to \Box\Diamond p) \oplus (i \to \Box\neg i)$$

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Lemma

(1) *For any frame $\mathcal{F}$, $\mathcal{F} \models (p \to \Box\Diamond p) \wedge (i \to \Box\neg i)$ if and only if $\mathcal{F}$ is an undirected graph.*

(2) *The canonical frame for $\mathbf{G}$ is also irreflexive and symmetric.*

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The logic $\mathbf{G}$ is strong complete for the class of all undirected graphs.

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Question: Does $\mathbf{G}$ admit filtration?

Formulas charactering some graph properties

$\mathcal{F}$: a graph (irreflexive and symmetric frame)

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(1) Degree of a graph
Every point in $\mathcal{F}$ has at most n points that connects to it iff $\mathcal{F} \models \mathbf{Alt_n}$

$$\mathbf{Alt_n} := \Box p_1 \vee \Box(p_1 \to p_2) \vee \cdots \vee \Box(p_1 \wedge \cdots \wedge p_n \to p_{n+1})$$

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(2) Diameter of a graph
The diameter of $\mathcal{F}$ is less than n iff $\mathcal{F} \models \neg\varphi_n$.

$$\begin{cases} \varphi_1 := p_1. \\ \varphi_{n+1} := p_{n+1} \wedge \neg p_n \wedge \cdots \wedge \neg p_1 \wedge \Diamond\neg\varphi_n. \end{cases}$$

(3) Hamilton cycles
$\mathcal{F}$: a graph that has n points.
$\mathcal{F}$ has a Hamilton cycle iff $\mathcal{F}$ **sat** ψ_n, so
$\mathcal{F}$ does NOT have a Hamilton cycle iff $\mathcal{F} \models \neg\psi_n$.

$$\psi_n := \sigma_1 \wedge \Diamond(\sigma_2 \wedge \Diamond(\cdots \Diamond(\sigma_n \wedge \Diamond\sigma_1)\cdots)), \text{ where}$$
$$\sigma_k := \neg i_1 \wedge \neg i_2 \wedge \cdots \wedge i_k \wedge \cdots \wedge \neg i_n$$

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(**Q**:) How to characterize having Euler cycles?

(4) Coloring

$\mathcal{F}$: a graph whose diameter is at most n.

$\mathcal{F}$ is k-colorable iff $\mathcal{F}$ **sat** $\text{color}(k)$, so
$\mathcal{F}$ is NOT k-colorable iff $\mathcal{F} \models \neg\text{color}(k)$

$$\text{color}(k) := \Box^{(n)}\Big(\bigvee_{\ell=1}^{k} c_\ell \wedge \bigwedge_{\ell=1}^{k}(c_\ell \to \Box\neg c_\ell)\Big),$$

each c_ℓ is a prop. variable representing a color.

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(**Q**:) How to characterize being planar?

Future Study

(1) What kind of graph properies are definable over the logic **G**?

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(2) Can we prove theorems from graph theory by constructing a fromal proof?