Non-symmetric generalized nearness and subfitness

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— joint work with Aleš Pultr (Charles Univ., Prague, Czech Republic)

LOWER SEPARATION AXIOMS POINT-FREELY

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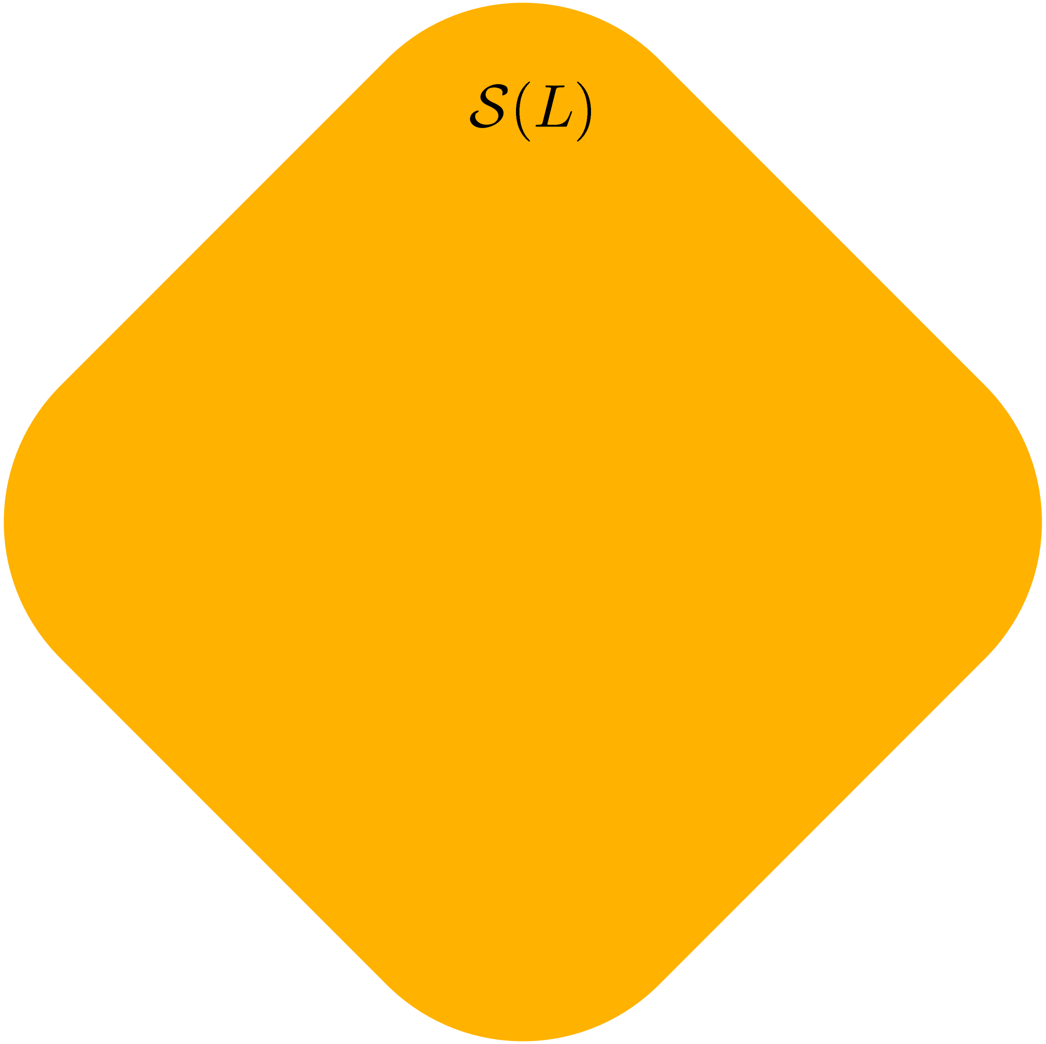
There are two point-free properties loosely related to T_1:

FITNESS, SUBFITNESS [J. Isbell (1972)]

Background: THE SUBLOCALE LATTICE $\mathcal{S}(L)$

It is a co-frame



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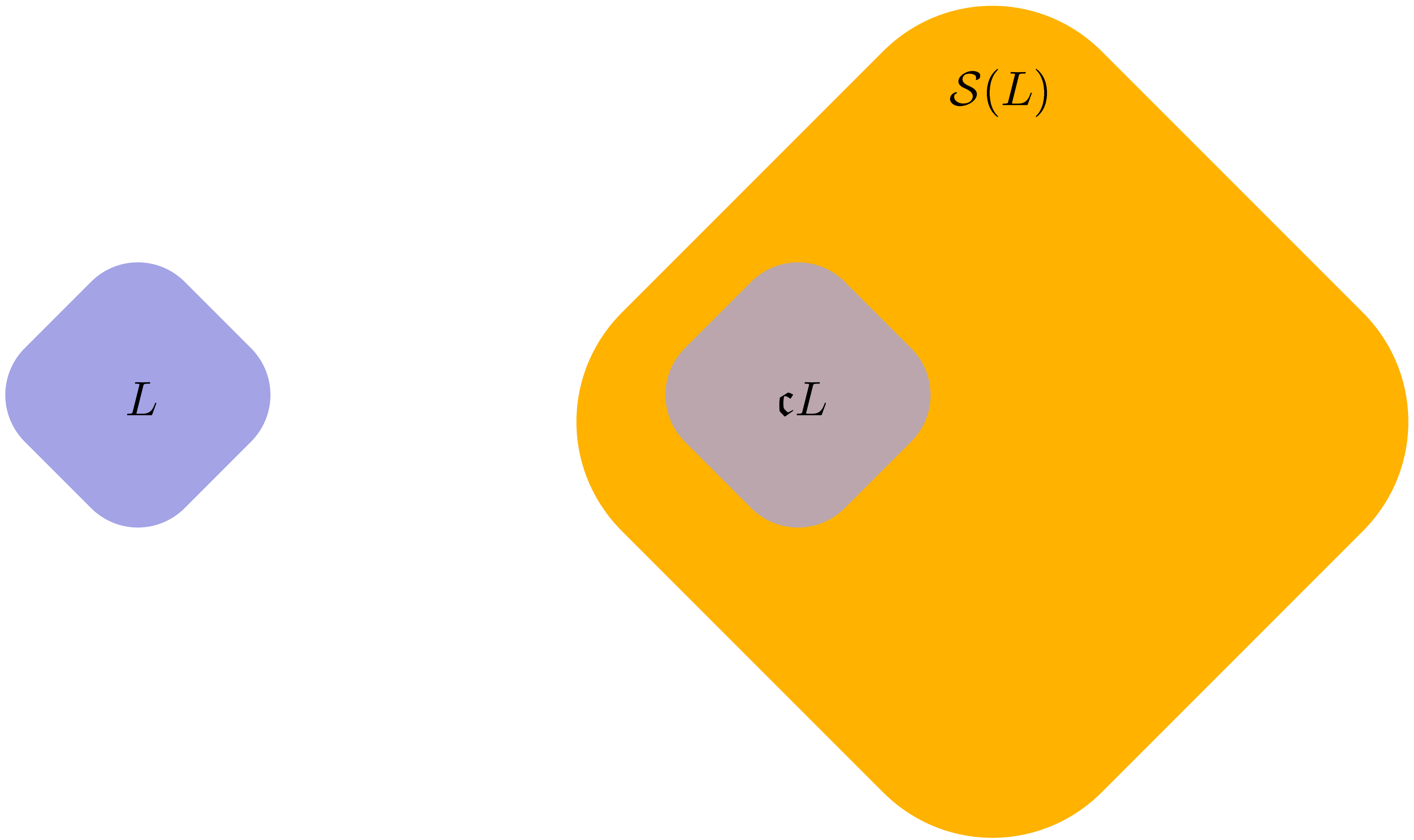

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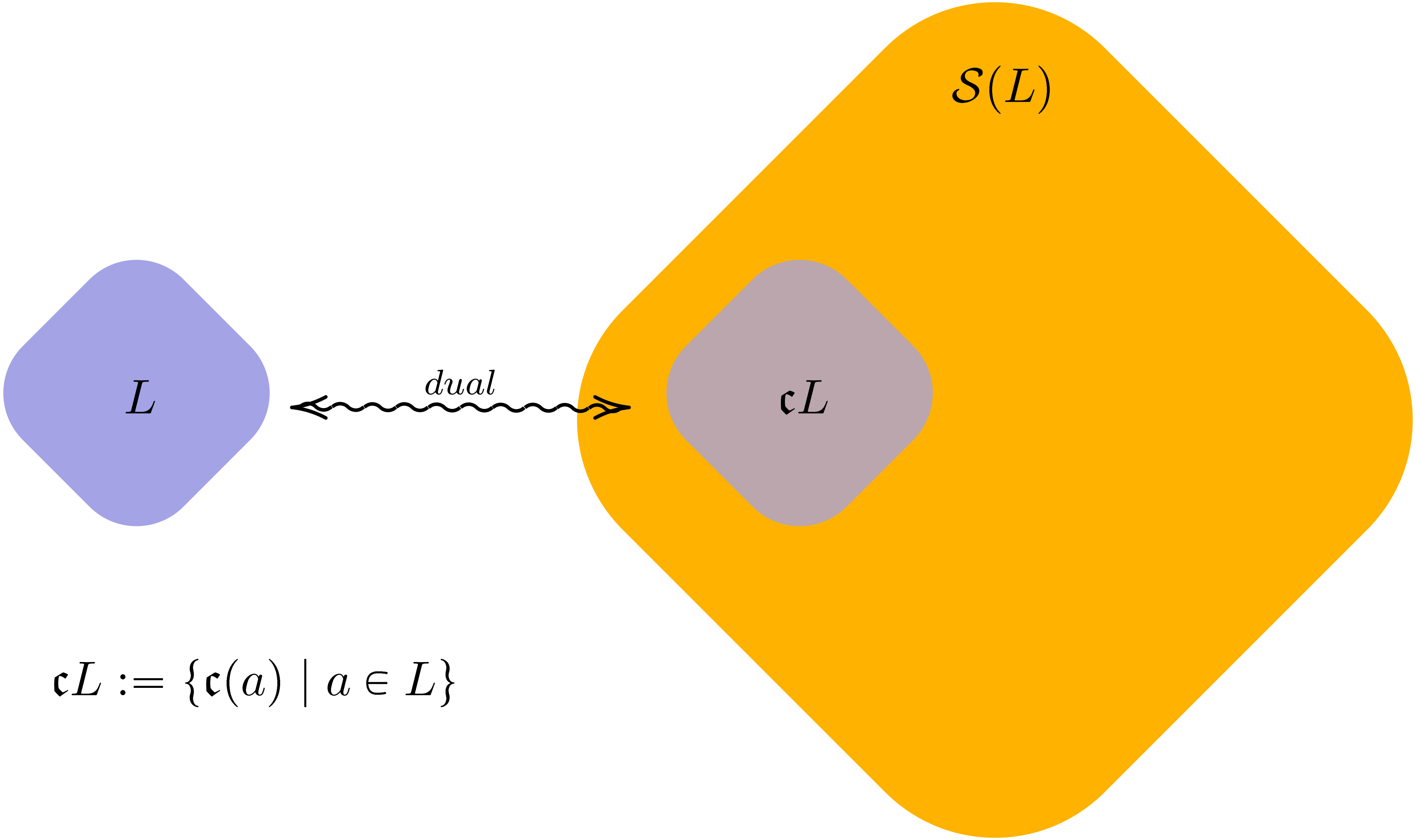

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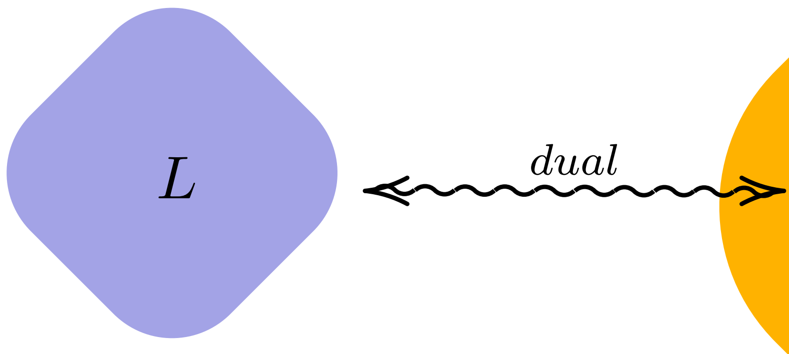

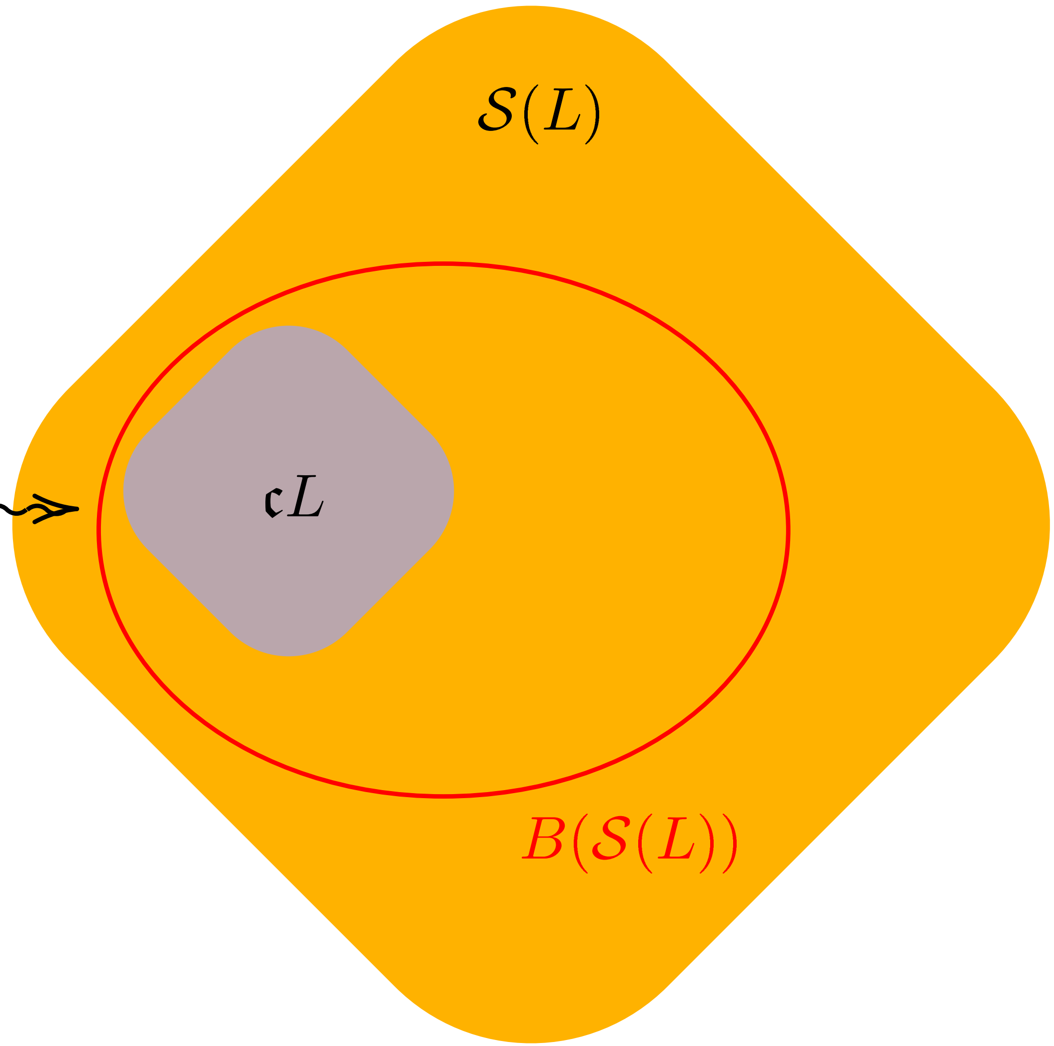

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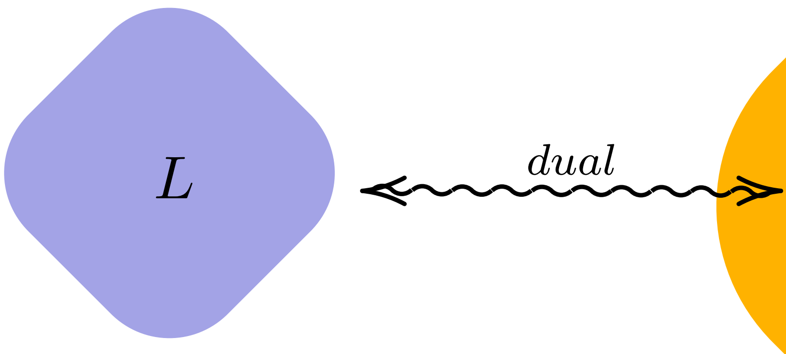

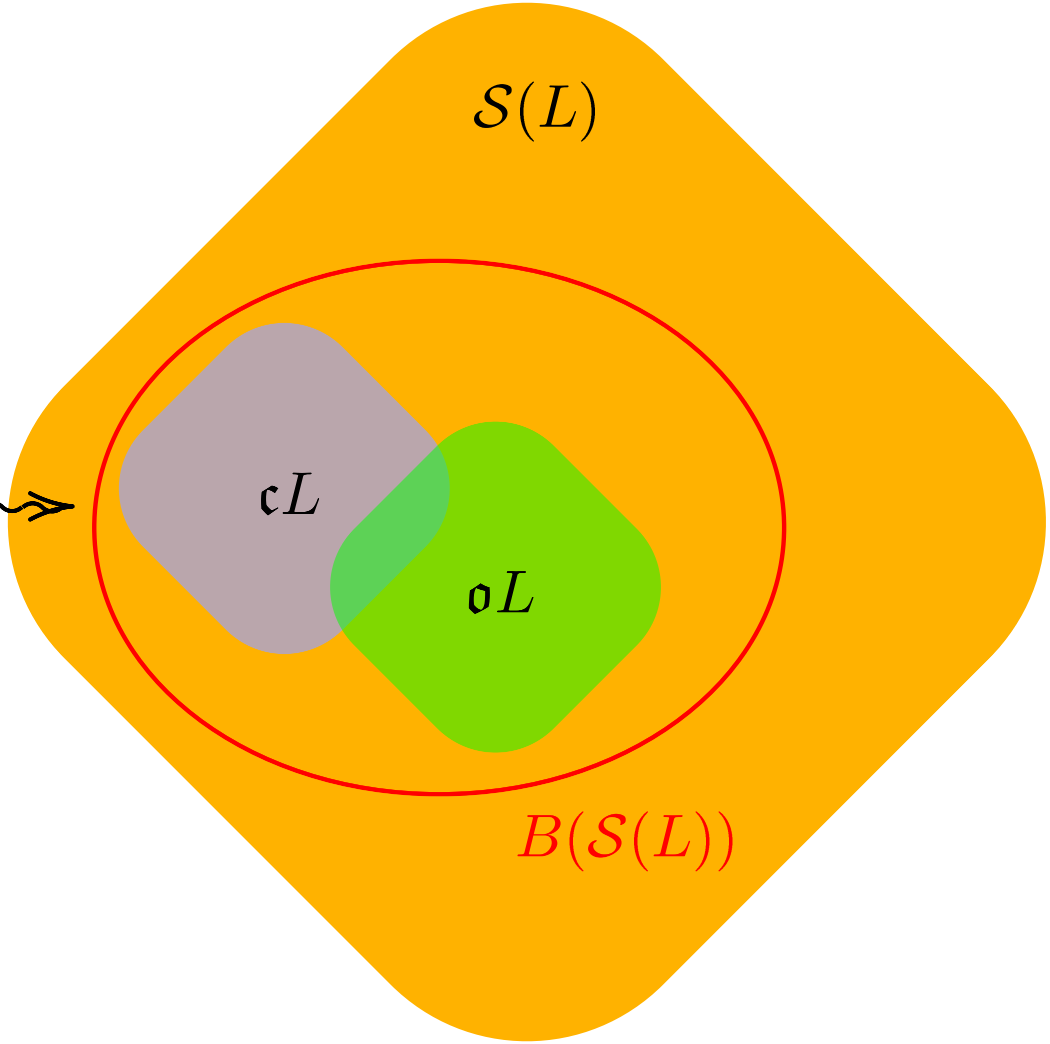

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[J. Isbell (1972)]

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X is fit iff ???

AIMS

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- monotone normality and stratifiable frames [J. Gutiérrez García, J. P. & M. A. de Prada Vicente (2013)]

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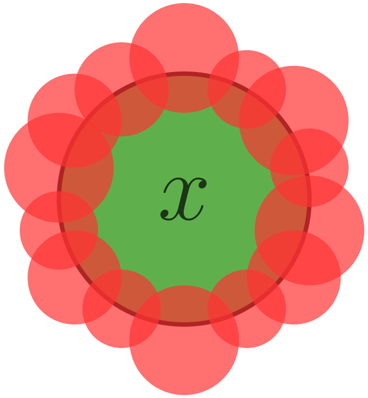

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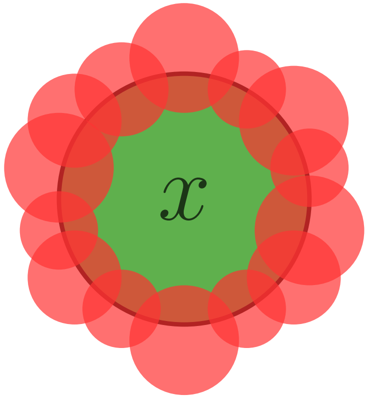

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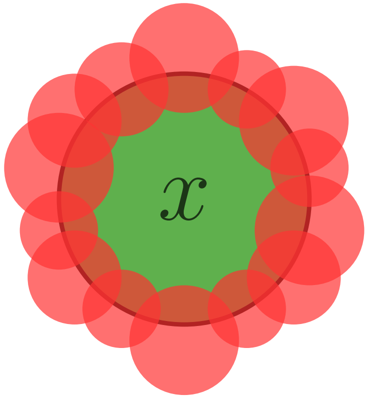

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[B. Banaschewski & A. Pultr (1996)]

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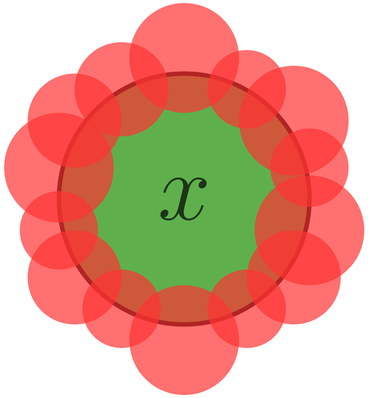

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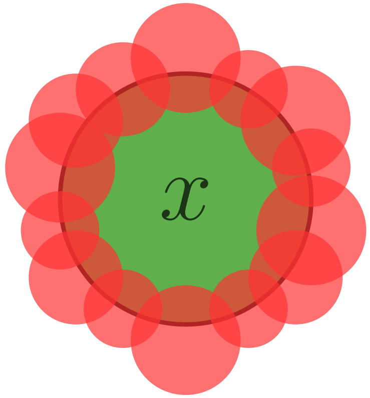

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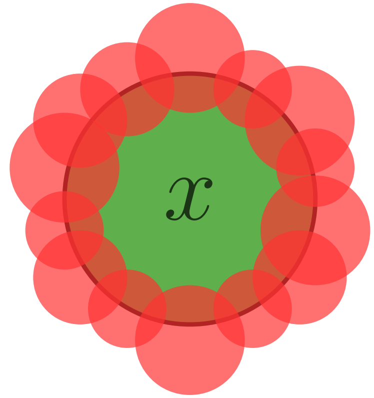

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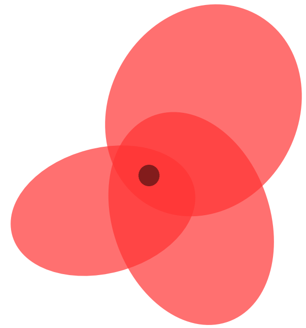

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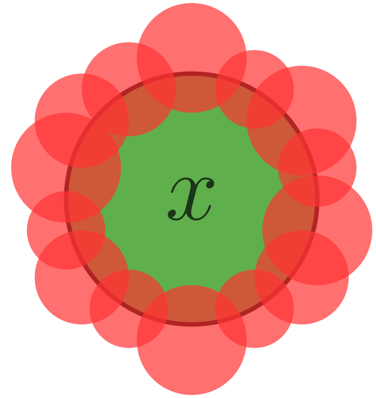

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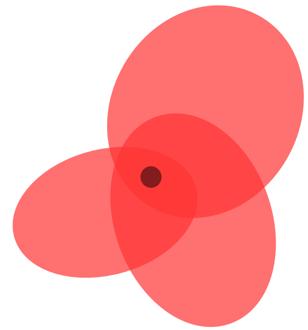

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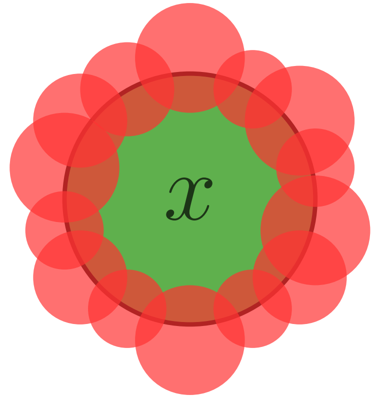

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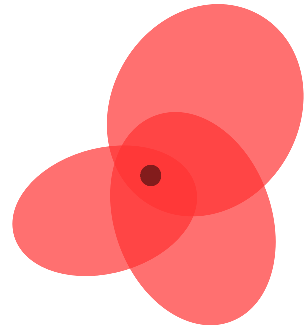

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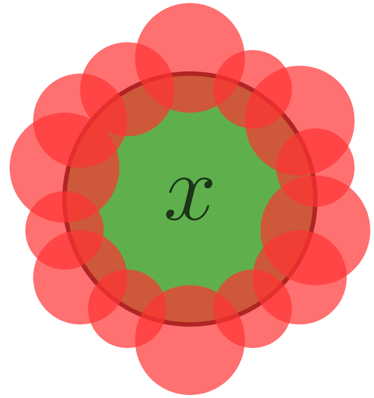

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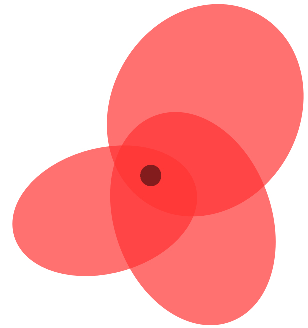

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- But L admits a nearness iff it is regular.

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Weakly subfit: $a \nleqslant 0 \ \Rightarrow \ \exists c \neq 1,\ a \vee c = 1$. (wsfit)

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THEOREM. L is fit $\Leftrightarrow$ every semiclosed sublocale is closed

$\Leftrightarrow$ every semiclosed sublocale is subfit.

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$F_\alpha(L)$ decrease $\Rightarrow \exists\gamma(L)\colon F_1(F_{\gamma(L)}(L)) = F_{\gamma(L)}(L)$.

$$F(L) = F_{\gamma(L)}(L)$$

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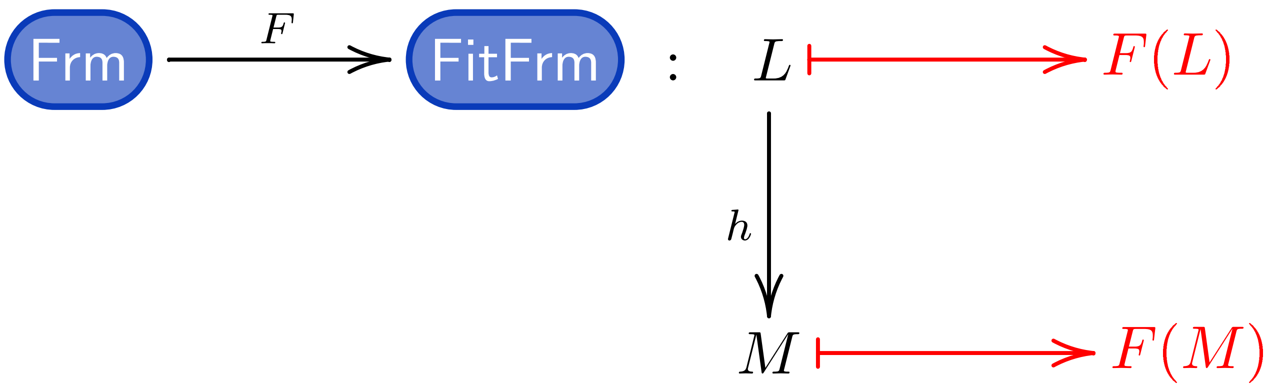

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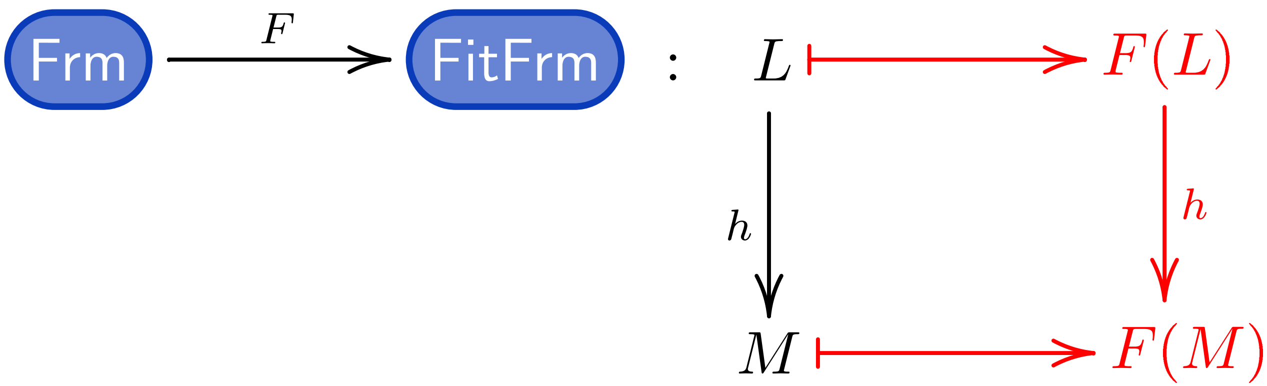

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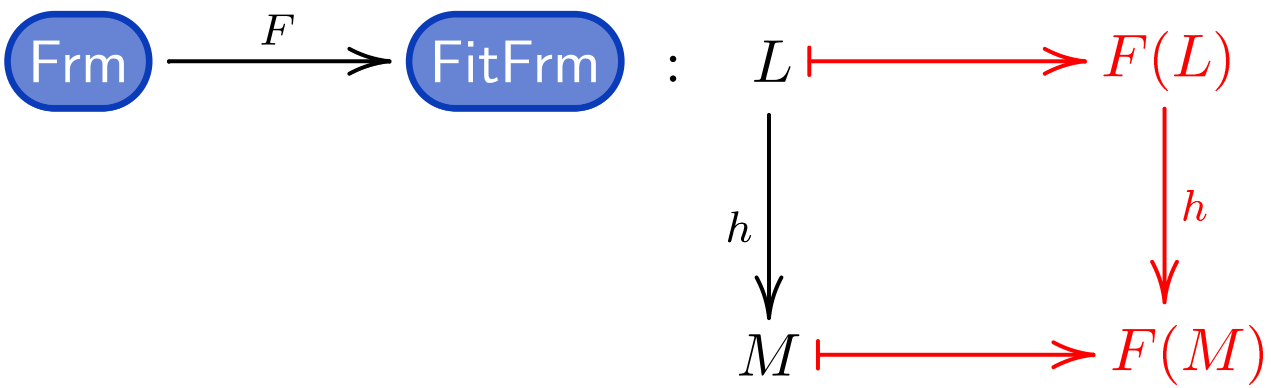

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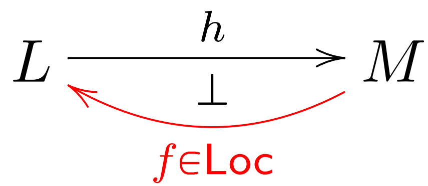

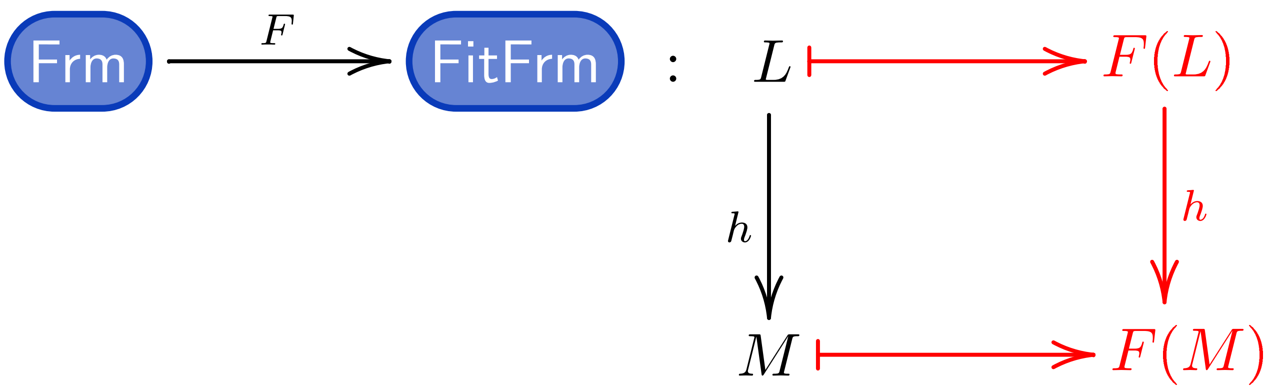

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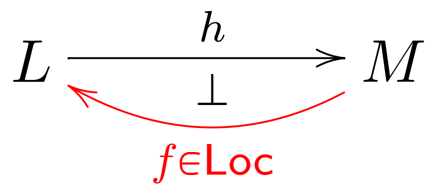

f^{-1} ↓

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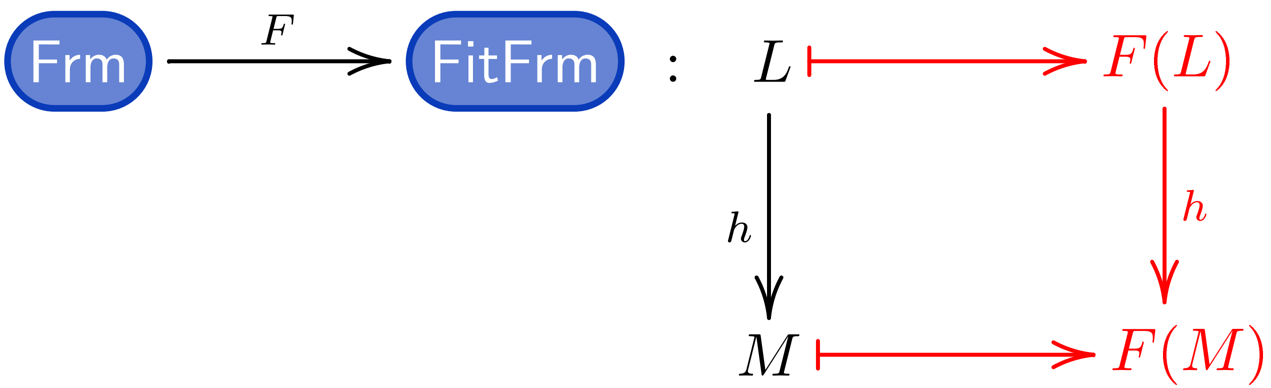

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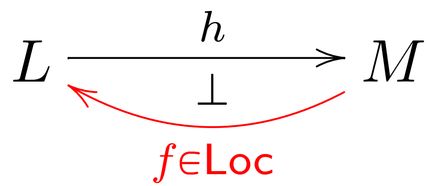

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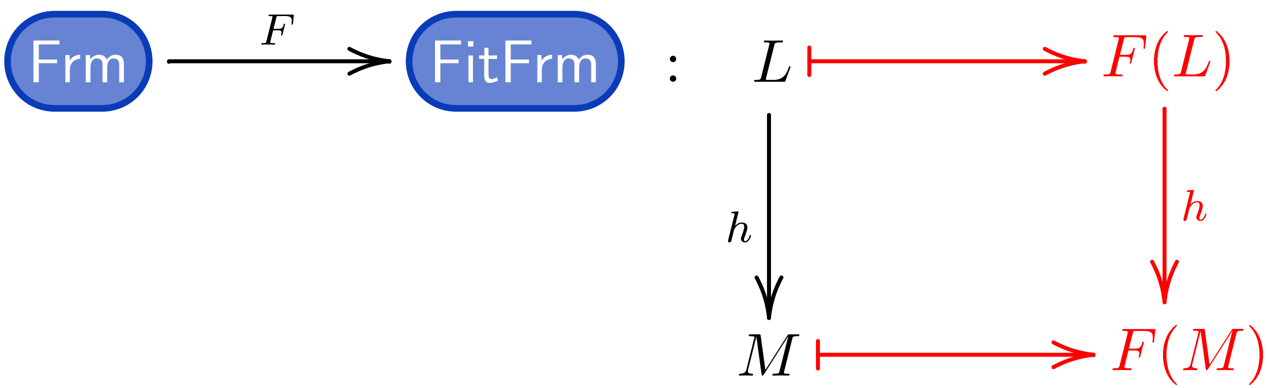

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$L \xrightarrow{h} M$, $\perp$, $f \in \mathsf{Loc}$ ($f: M \to L$)

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Prefit: $\forall a \neq 0 : \; \exists b \neq 0, \; b \prec a$. (pfit)

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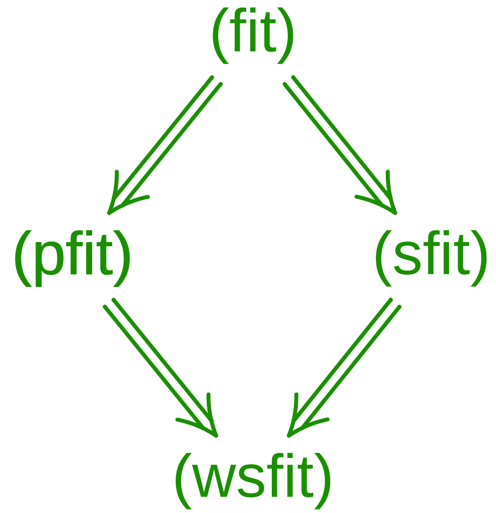

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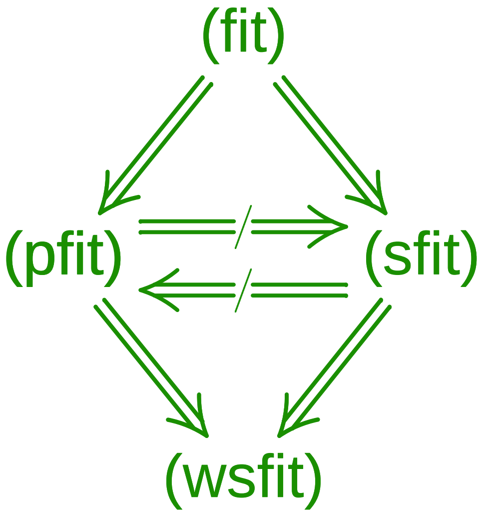

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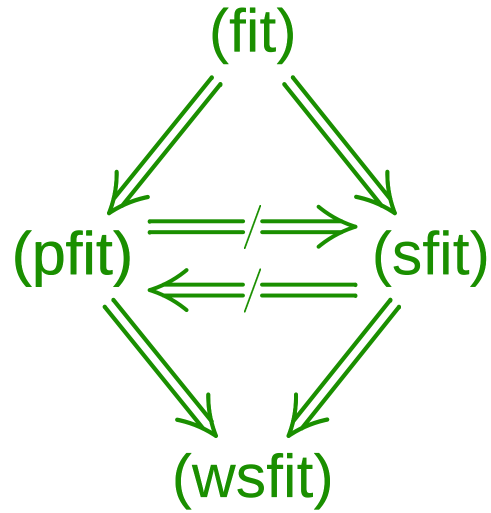

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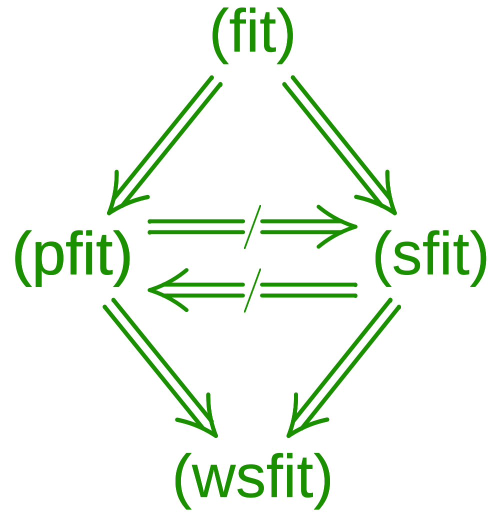

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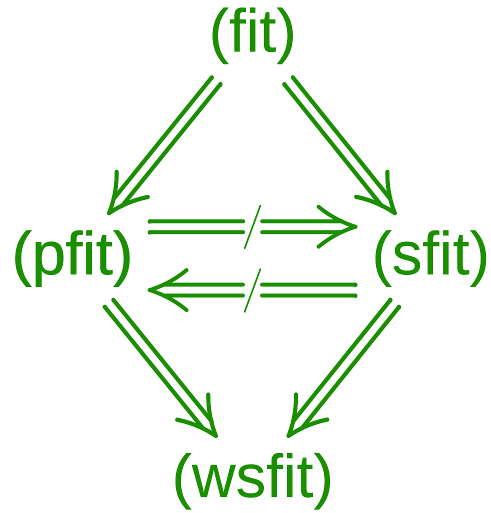

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COROLLARY. X is fit iff

any closed F, any open U, $U \cap F \neq \emptyset \; \Rightarrow$

$\Rightarrow \; \exists$ open $V \; : \; V \cap F \neq \emptyset, \; \overline{V} \cap F \subseteq U \cap F$.

NON-SYMMETRIC GENERALIZED NEARNESS on a biframe (L, L_1, L_2)

BACKGROUND: THE CATEGORY OF BIFRAMES

[Banaschewski, Brümmer & Hardie (1983)]

bitopological space

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biframe maps
- $h : L \to M$ frame homomorphism
- $h(L_i) \subseteq M_i \quad (i = 1, 2)$.

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PROPOSITION. A biframe admits a quasi-nearness iff it is subfit.

BIFRAME VARIANTS OF FITNESS AND SUBFITNESS

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[subbilocales, heredity, ...]