The 42 reducts of the random ordered graph

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Outline

- **Part I:** The setting of The Answer
- **Part II:** The 42 reducts of the random ordered graph
- **Part III:** Discussion of The Answer
- **Part IV:** The question to The Answer

Part I: The setting of The Answer

Homogeneous structures

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- Random graph $(V; E)$
- Free Boolean algebra with $\aleph_0$ generators

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- Boolean algebras
- Lattices
- Universal Algebra
- Set theory
- Topology

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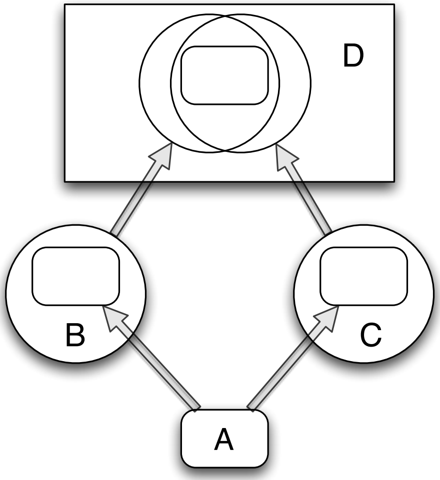
Then there exists a unique countable homogeneous structure Δ
whose age (=substructures up to iso) equals $\mathcal{C}$.

Amalgamation



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- Tournaments $\leftrightarrow$ random tournament
- Linearly ordered graphs $\leftrightarrow$ random ordered graph $(D; <, E)$

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Problem

Understand the reducts of homogeneous structures.

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  – Constraint Satisfaction Problems related to $\mathcal{C}$:
  Graph-SAT, Poset-SAT,...

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Question

How many inequivalent reducts?

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Conjecture (Thomas '91)

Homogeneous structures in finite relational language have finitely many reducts.

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Theorem (Corollary of Ryll-Nardzewski, Engeler, Svenonius)

Let Δ be homogeneous in a finite relational language.

Then the mapping

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is an anti-isomorphism
from the lattice of reducts
to the lattice of closed supergroups of $\mathrm{Aut}(\Delta)$.

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For $k \geq 1$, let $R^{(k)}$ consist of the k-tuples of distinct elements of V which induce an odd number of edges.

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- $\mathrm{Sym}(V)$

For $k \geq 1$, let $R^{(k)}$ consist of the k-tuples of distinct elements of V which induce an odd number of edges.

The random graph $(V; E)$

Let $-$ be any permutation of V which switches edges and non-edges.

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Part II: The 42 reducts of the random ordered graph

The random ordered graph

Definition

The random ordered graph $(D; <, E)$ is
the unique countable linearly ordered graph which

- contains all finite linearly ordered graphs
- is homogeneous.

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- $(D; <)$ is the order of the rationals
- $(D; E)$ is the random graph

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- $(D; <)$ is the order of the rationals
- $(D; E)$ is the random graph

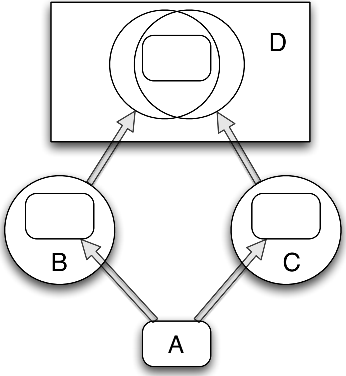
This is because the two structures are superposed freely, i.e.,
in all possible ways.

Strong amalgamation

Strong amalgamation

Definition

A class $\mathcal{C}$ has strong amalgamation $:\leftrightarrow$
it has amalgamation and
the amalgamation can be done without identifying elements outside A.



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- is a Fraïssé class and
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Lemma

The random ordered graph has at least 25 reducts.

Similarities between the order and the graph reducts

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Lemma

The random ordered graph has at least 27 reducts.

Hello random tournament!

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Definition

A tournament is a digraph with precisely one edge between any two vertices.

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Set $T(x, y)$ iff $x < y \wedge E(x, y)$ or $x > y \wedge N(x, y)$.

Then $(D; T)$ is the random tournament.

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The random ordered graph has at least 27+5-1=31 reducts.

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The random ordered graph has 41 reducts.

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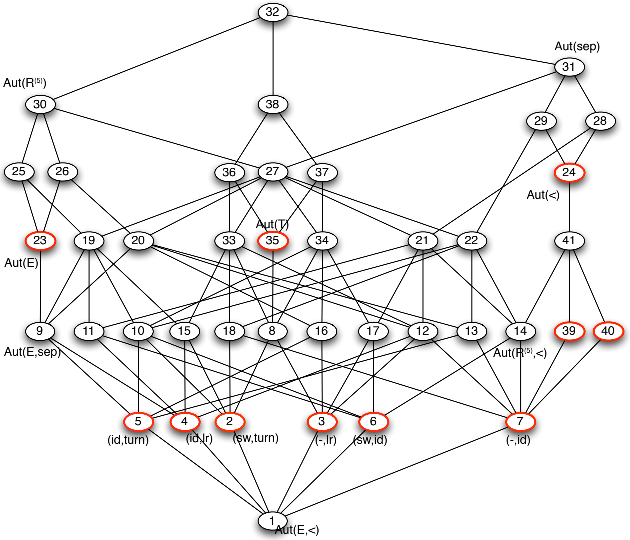
The random ordered graph has 41 reducts.



Part III: Discussion of The Answer

Discussion

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We have learnt from the result:

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- similarities between the symmetries of the order and the graph
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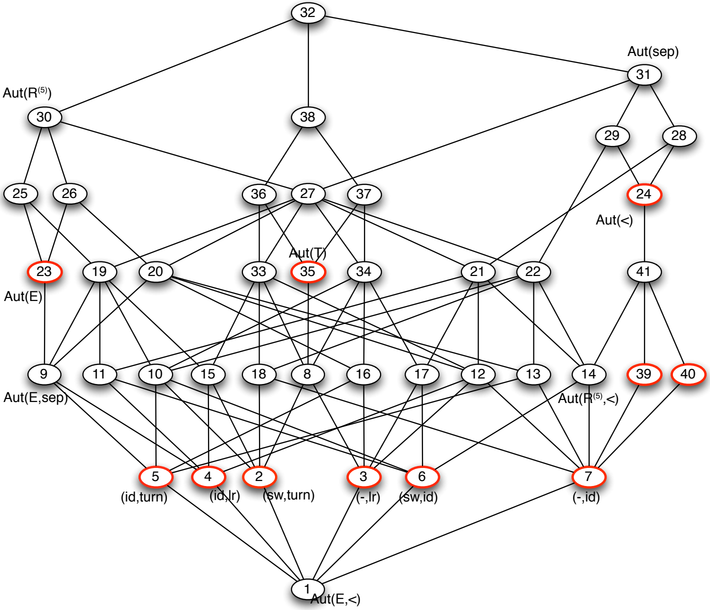
On a technical level:

- our Ramsey-theoretic method is quite efficient (first classification of free superposition)
- improved it to reduce work to the join irreducible elements
- our method is not sporadic (same for order, graph, tournament)



32
Aut(sep)
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Aut(T)
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Aut(E,sep)
Aut($R^{(5)}$,<)
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Ramsey structures

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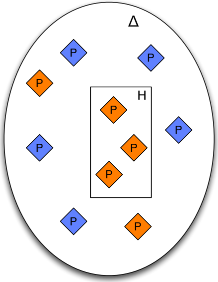

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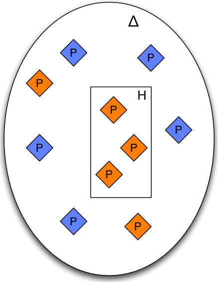

Theorem (Nešetřil-Rödl)

The random ordered graph is Ramsey.

Canonical functions

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Definition

Let Δ, Λ be structures.

$f : \Delta \to \Lambda$ is canonical iff

for all tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of the same type in Δ

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Canonizing functions on Ramsey structures

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Magical proposition (Bodirsky+MP+Tsankov '11)

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- Δ is ordered Ramsey homogeneous finite language
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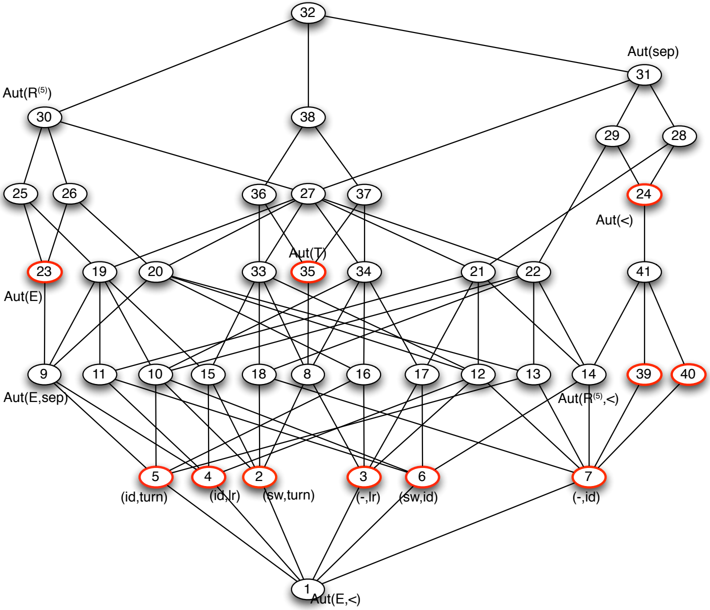
Note:

- only finitely many different behaviors of canonical functions.
- g, g' same behavior $\to$ generate one another (with $\mathrm{Aut}(\Delta)$).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3
6
7
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BLAST

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- Universal Algebra ✓
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Part IV: The Question to The Answer

The Question

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Does their free superposition have finitely many reducts?

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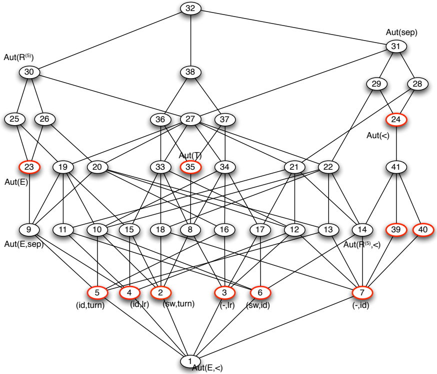
Problem

Suppose that Δ is homogeneous in a finite relational language.

Does it have a finite homogeneous extension which is Ramsey?

Thank you!



"The Answer to the Great Question. . .
Of Life, the Universe and Everything. . . Is. . . Forty-two,"
said Deep Thought, with infinite majesty and calm.

Douglas Adams