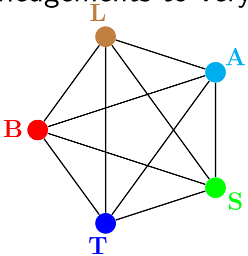
DUALITY THEORY AND B L A S T : Selected Themes Part II: Canonical Extensions, Proginite Completion and Notural Dualities

Hilary Priestley

Mathematical Institute, University of Oxford

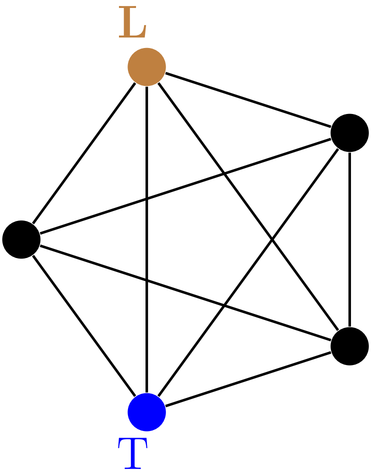
With acknowledgements to very many people



Outline: Part II

- Canonical extensions of unital semilattices and bounded lattices: a fast-track approach
- Canonical extensions of bounded lattices and choice principles

- The natural extension construction: Bohr compactfications of discrete structures
- Natural dualities via Pro- and Ind-completion?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The historical development of the theory of canonical extensions

Motivation came from study of ordered algebras. But shall focus here on the underlying ordered structures.

$\mathfrak{B}$ Boolean algebras Jónsson and Tarski 1951

$\mathfrak{D}$ Bounded distributive lattices Gehrke and Jónsson 1995

Bounded lattices Gehrke and Harding 2001

Posets Dunn, Gehrke and Palmigiano 2005

But what about (unital) semilattices?

Canonical extensions of semilattices and lattices: a fast-track approach

(Gouveia and Priestley, 2012, with acknowledgements to Cabrer and to Jipsen & Moshier)

Let $\mathcal{S} \in \mathcal{S}_\wedge$—meet semilattices with 1. Then

$\mathrm{Filt}(S) =$	filters of $\mathcal{S}$	non-empty up-sets closed under $\wedge$,
$\mathrm{Idl}(\mathcal{S}) =$	ideals of $\mathcal{S}$	directed down-sets

We have order-reversing principal filter embeddings of $\mathcal{S}$ into $\mathrm{Filt}(\mathcal{S})$ denoted by $\uparrow$ and of $\mathrm{Filt}(\mathcal{S})$ into $\mathrm{Filt}^2(\mathcal{S}) = \mathrm{Filt}(\mathrm{Filt}(^2(\mathcal{S}))$, denoted by $\Uparrow$.
We can embed $\mathcal{S}$ in $\mathrm{Filt}^2(\mathcal{S})$ (right way up) via $e\colon a \mapsto \Uparrow(\uparrow a)$.
Note $\mathrm{Filt}^2(\mathcal{S})$ is an algebraic closure system: complete lattice in which meet is given by intersection and and directed join by union.
$(\mathrm{Filt}^(\mathcal{S})$ concretely models the free join completion of the free meet completion of $\mathcal{S}$.)

Properties of the completion $(e, \mathrm{Filt}^2(\mathcal{S}))$

Let $\mathcal{S} \in \mathcal{S}_\wedge$.
Operations in $\mathrm{Filt}^2(\mathcal{S})$:

$$\begin{aligned} \textstyle\bigsqcap e(F) &= \Uparrow F && \text{if } F \in \mathrm{Filt}(\mathcal{S}), \\ \textstyle\bigsqcup e(J) &= \textstyle\bigcup e(J) && \text{if } J \text{ is directed.} \end{aligned}$$

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Theorem

2/3-canonicity property of $\mathrm{Filt}^(\mathcal{S})$ *The completion* $(e, \mathrm{Filt}^2(\mathcal{S}))$ *of* $\mathcal{S}$ *is*

- *compact : for* $F \in \mathrm{Filt}(\mathcal{S})$ *and* $J \in \mathrm{Idl}(\mathcal{S})$,

$$\bigsqcap e(F) \leqslant \bigsqcup e(J) \implies F \cap J \neq \emptyset.$$

- $\bigsqcup\bigsqcap$*-dense. of* $\mathcal{S}$.

The canonical extension of a unital semilattice

Theorem

Let $\mathcal{S} \in \mathbb{S}_\wedge$. *Let*

$$\mathcal{S}^\delta = \{ \mathcal{F} \in \mathrm{Filt}(\mathrm{Filt}(\mathcal{S})) \mid \mathcal{F} \text{ is a meet of directed joins of elements from}$$

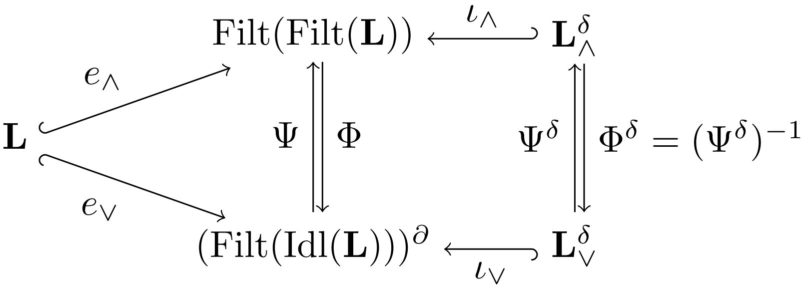
Then

- e *is an* $\mathbb{S}_\wedge$*-embedding of* $\mathcal{S}$ *into* $\mathcal{S}^\delta$*;*
- *Meets in* $\mathrm{Filt}^2(\mathcal{S})$ *and in* $\mathcal{S}^\delta$ *are given by* $\bigcap$ *and directed joins by* $\bigcup$*.*
- $(\overline{e}, \mathcal{S}^\delta)$ *is a canonical extension of* $\mathcal{S}$ *(*$\overline{e}$ *denotes* e *with codomain restricted to* $\mathcal{S}^\delta$*).*
- $\mathcal{S}^\delta$ *coincides with the Galois-closed sets for the polarity* $(\mathrm{Filt}(\mathcal{S}), \mathrm{Idl}(\mathcal{S}), R)$*, where* $F \; R \; J$ *iff* $F \cap J = \emptyset$*.*

Final statement a CONSEQUENCE of earlier ones.

Canonical extensions of bounded lattices

Φ and Ψ define an adjunction. Restriction maps Φ^δ and Ψ^δ set up mutually inverse isomorphisms—they are the maps from the usual polarity $(\mathrm{Filt}(\mathbf{L}), \mathrm{Idl}(\mathbf{L}), R)$.



Back to semilattices: incarnations of $\mathrm{Filt}^2(\mathcal{S})$

Duality strikes again!
Hofmann–Mislove–Stralka duality for $\mathcal{S}_\wedge$: Let

$$\begin{aligned} \mathcal{S}_\wedge &= \mathbb{ISP}(\mathbf{2}) & \mathbf{2} &= \langle \{0,1\}; \wedge, 1\rangle, \\ \mathcal{S}_{\wedge\mathcal{T}} &= \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_{\mathcal{T}}) & \underset{\sim}{\mathbf{2}}_{\mathcal{T}} &= \langle \{0,1\}; \wedge, 1, \mathcal{T}\rangle. \end{aligned}$$

Then $\mathcal{S}_{\wedge\mathcal{T}}$ gives the category of compact 0-dimensional topological semilattices *alias* algebraic lattices with maps preserving $\bigsqcup$ and $\bigwedge$. The hom-functors $\mathsf{D} = \mathcal{S}_\wedge(-, \mathbf{2})$ and $\mathsf{E} = \mathcal{S}_{\wedge\mathcal{T}}(-, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ yield a full duality between $\mathcal{S}_\wedge$ and $\mathcal{S}_{\wedge\mathcal{T}}$.

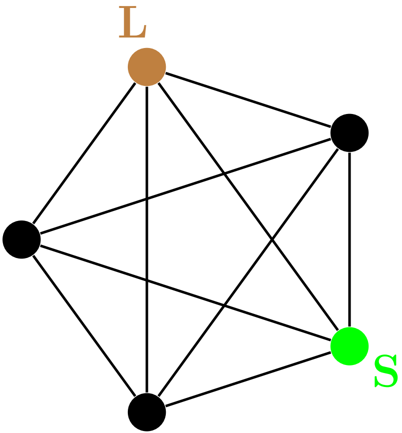
Via characteristic functions we can identify

$\mathrm{Filt}^2(\mathcal{S})$ and $(\mathsf{D}(\mathsf{D}(\mathcal{S})^\flat))^\flat$ —denote this by $\widehat{\mathcal{S}}$.

Canonical extensions, functorially

Easy to show that

- a $\mathcal{S}_\wedge$ morphism $f\colon \mathcal{S} \to \mathcal{T}$ lifts to $\widehat{f}\colon \widehat{\mathcal{S}} \to \widehat{\mathcal{T}}$ preserving $\bigsqcup$ and $\bigwedge$;
- by restriction, a $\mathcal{S}_\wedge$ morphism $f\colon \mathcal{S} \to \mathcal{T}$ lifts to $f^\delta\colon \mathcal{S}^\delta \to \mathcal{T}^\delta$;
- by looking at both semilattice reducts, a bounded lattice morphism $f\colon \mathbf{L} \to \mathbf{K}$ lifts to a complete lattice homomorphism $f^\delta\colon \mathbf{L}^\delta \to \mathbf{K}^\delta$.



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Canonical extensions of distributive lattices, with Choice

If $\mathbf{L} \in \mathfrak{D}$, then in [ZFC] we know, by Priestley duality, that the canonical extension $\mathbf{L}^\delta$ is tthe up-set lattice of its prime filters under $\subseteq$) and so is a complete ring of sets. As such, it is

(1) **completely distributive**;

(2) **superalgebraic**: it satisfies one, and hence all, of the equivalent conditions:

  (a) the completely join-prime elements are join-dense;

  (b) $\mathbf{L}^\delta$ is a frame and the completely join-irreducibles are join-dense;

  (c) [splitting pairs] $a \nleqslant b$ in $\mathbf{L}^\delta$ implies $\exists p, \kappa(p)$ such that $a \nleqslant p$, $b \nleqslant \kappa(p)$ and ${\downarrow}p \cup {\uparrow}\kappa(p) = \mathbf{L}^\delta$;

(3) **weakly atomic**.

Canonical extensions of bounded lattices, with and without Choice

For sure, existence and uniqueness of $\mathbf{L}^\delta$ for a bounded lattice $\mathbf{L}$ need only [ZF].

FACTS:
Let $\mathbf{L}$ be a bounded lattice.

- $\mathbf{L} \in \mathfrak{D}$ implies $\mathbf{L}^\delta \in \mathfrak{D}$ (Gehrke& Harding, 2001, implicitly).
- $\mathbf{L} \in \mathfrak{D}$ implies $\mathbf{L}^\delta$ a frame (Gehrke, 2011)

QUESTION: What choice principles are required in order that $\mathbf{L}^\delta$ should have the properties of a complete ring of sets, for $\mathbf{L} \in \mathfrak{D}$? What can be said for general bounded lattices?

ANSWERS: (Erné, 2012)

Separating filters and ideals

Given a bounded lattice $\mathbf{L}$ let $\mathcal{F} = \mathrm{Filt}(\mathbf{L})$ and $\mathcal{I} = \mathrm{Idl}(\mathbf{L})$.
Given $F \in \mathcal{F}$ and $I \in \mathcal{I}$ with $F \cap I = \emptyset$, there exist $P \in \mathcal{F}$ and $Q \in \mathcal{I}$ such that $F \subseteq P$, $I \subset Q$ and

$Q = L \setminus P$	Prime Separation Property
$F \cap Q = \emptyset$, $P \cap I = \emptyset$ and $P \cup Q = L$	Normal Separation Property.

Lemma

[ZF] $\mathbf{NSP} \implies \mathbf{PSP} \implies \mathbf{L}$ *distributive.*

Theorem

In $[\mathbf{Z}F]$*:*

(i) *A bounded lattice* $\mathbf{L}$ *satisfies* $(\mathbf{PSP})$ *iff* $\mathbf{L}^{\delta}$ *is superalgebraic.*

(ii) *A bounded lattice* $\mathbf{L}$ *satisfies* $(\mathbf{NSP})$ *iff* $\mathbf{L}^{\delta}$ *satisfies the choice-free formulation of complete distributivity.*

Equivalents of the Ultrafilter Principle

Theorem

The Ultrafilter Principle is equivalent to each of the following:

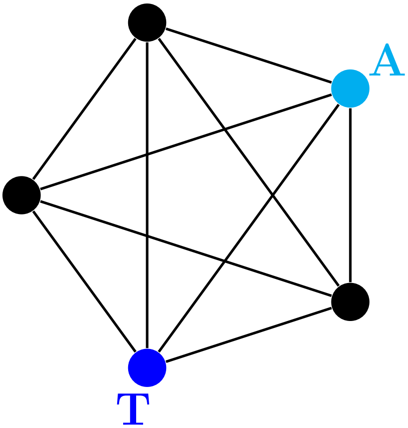
(1) *Any bounded distributive lattice satisfies* $(\mathbf{PSP})$.

(2) *Any bounded distributive lattice satisfies* $(\mathbf{NSP})$.

(3) *Any bounded distributive lattice has a canonical extension with property* $)\star)$.

(4) *Any Boolean lattice has a canonical extension with property* $(\star)$.

Here the property $(\star)$ *may be any of:*
superalgebraic, spatial frame, algebraic, weakly atomic;
completely distributive, or versions of this restricted to families of 2-element sets or of finite sets.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Residually finite (pre)-varieties

Suppose $\mathfrak{M}$ is a set (not necessarily finite) of finite algebras of common type and let $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$. Except that the term is usually used for varieties, this is **residual finiteness**.
Let

$$\mathfrak{M}_{\mathcal{T}} = \langle \bigcup \{ M \in \mathfrak{M} \} M; R, \mathcal{T} \rangle.$$

,where R is a set of finitary algebraic relations. Let $\mathfrak{X}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}^+(\mathfrak{M}_{\mathcal{T}})$. This is just like the set-up for a multisorted natural duality except that now don't assume $\mathfrak{M}$ is finite.
We have well-defined hom-functors

$$\mathrm{D}\colon \mathcal{A} \to \mathfrak{X}_{\mathcal{T}} \quad \text{and} \quad \mathrm{E}\colon \mathfrak{X}_{\mathcal{T}} \to \mathcal{A}.$$

Moreover ED and DE are embeddings, and given by (multisorted) evaluation maps.

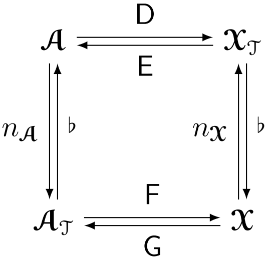
The natural extension functor, for a class $\mathbb{ISP}(\mathfrak{M})$ of algebras

Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathfrak{M})$ (as above) and let $\boldsymbol{\mathcal{A}}_{\mathfrak{T}} = \mathbb{ISP}(\mathfrak{M}_{\mathfrak{T}})$. Then there exists a covariant functor $n_{\boldsymbol{\mathcal{A}}} \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}_{\mathfrak{T}}$ with the following properties:

- $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ is a Boolean-topological algebra whose algebra reduct belongs to $\boldsymbol{\mathcal{A}}$;
- $n_{\boldsymbol{\mathcal{A}}}$ is a reflector into a (non-full) subcategory of $\boldsymbol{\mathcal{A}}_{\mathfrak{T}}$ and is left-edjoint to the forgetful functor $^{\flat}$ from $\boldsymbol{\mathcal{A}}_{\mathfrak{T}}$ to $\boldsymbol{\mathcal{A}}$.

But where does this comes from?
We haven't involved the hom functors D and E yet . . .

The natural extension via paired adjunctions



Here the lower adjunction works just the same way as the upper one, except that the topology has been moved from $\underset{\sim}{\mathfrak{M}}$ to $\boldsymbol{\mathfrak{M}}$. So $\mathcal{A}_{\mathcal{T}} = \mathbb{IS}_{c}\mathbb{P}^{+}(\boldsymbol{\mathfrak{M}}_{\mathcal{T}})$ and $\mathcal{X} = \mathbb{ISP}(\boldsymbol{\mathfrak{M}})$.

So what is this natural extension gadget?

The **profinite completion** $\mathrm{Pro}_{\mathcal{A}}(\mathbf{A})$ of an algebra $\mathcal{A}$ in a finitely generated variety of the form $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ is the projective limit of the finite quotients of $\mathbf{A}$. The residual finiteness assumption implies that each $\mathbf{A} \in \mathcal{A}$ has a **profinite completion** $\mathrm{Pro}_{\mathcal{A}}(\mathbf{A})$. If $\mathcal{A}$ is not a variety , we just restrict to finite quotients which belong to $\mathcal{A}$.
FACT: there is a canonical embedding $\mu_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{Pro}_{\mathcal{A}}(\mathbf{A})$.

[Profinite objects in a category, are, loosely, those which are built from (discretely topologised) finite ones by means of filtered colimits. In context of algebras, profinite objects should be viewed as topological algebras.]

Theorem
Let $\mathcal{A} = \mathbb{ISP}(\mathfrak{M}$ *be a residually finite pre-variety. Then* $n_{\mathcal{A}}(\mathbf{A})$ *and* $\mathrm{Pro}_{\mathcal{A}}(\mathbf{A})$ *are isomorphic as topological algebras.*

Examples and comments

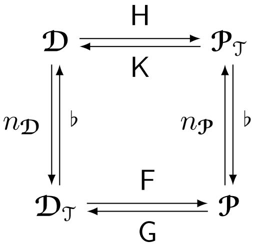
The Paired Adjunctions Theorem gives access to a way of finding $n_{\mathcal{A}}(\mathbf{A})$, especially if we have $\mathfrak{M}$ finite and a set R yielding a natural duality on $\mathcal{A}$. Note profinite completions can be hard to describe explicitly.

- Take $\mathcal{A} = \mathcal{D} = \mathbb{ISP}(\mathbf{2})$. Then $n_{\mathcal{D}}(\mathbf{L})$, as calculated from the Paired Adjunctions Theorem, is exactly the canonical extension $\mathbf{L}^{\delta}$.Well known that $\mathbf{L}^{\delta} \cong \mathrm{Pro}_{\mathcal{D}}(\mathbf{L})^{\flat}$.
- If $\mathcal{A}$ is a finitely generated lattice-based variety of finite type, then $\mathbf{A}^{\delta} \cong \mathrm{Pro}_{[} \mathcal{A}^{\flat}$ (Harding, 2006 + Gouveia 2009). $n_{\mathcal{A}}(\mathbf{A})$ provides another description.
- What happens for semilattices? Consider $\mathcal{S}_{\wedge}$. Then HMS duality tells us that $n_{CSw}(\mathcal{S})^{\flat} \cong \mathrm{Filt}^{2}(\mathcal{S})$. As a topological algebra, $n_{\mathcal{S}_{\wedge}}(\mathcal{S}) \cong \mathrm{Pro}_{\mathcal{S}_{\wedge}}$ is known as the **Bohr compactification** of $\mathcal{S} \in \mathcal{S}_{\wedge}$.
  Note that HMS duality is special in that the algebra persona and the alter ego persona are the same, apart from the topology. (cf. Pontryagin duality for abelian groups, where same phenomenon occurs–this is a rare instance of a natural

Two dualities in partnership: Priestley duality and Banaschewski duality

$\mathcal{D} := \mathbb{ISP}(\mathbf{2})$, $\mathcal{P} := \mathbb{ISP}(\underset{\sim}{\mathbf{2}})$ (posets),

$\mathcal{P}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, $\mathcal{D}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}^{+}(\mathbf{2}_{\mathcal{T}})$ (Boolean-topological DLs)



Here the top adjunction gives Priestley duality. The bottom one gives the duality between $\mathcal{P}$ and $\mathcal{D}_{\mathcal{T}}$ (Banaschewski, 1976). Symbol $^{\flat}$ denotes the functor forgetting topology.

From algebras to structures

The term **Bohr compactification** suggests we are thinking in terms of structures rather than algebras. Indeed we should. As hinted in first talk, natural duality theory extends to this wider setting.

Compatibility of a pair of structures on the same set

- $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$ are compatible structures on the same finite set M (operations, relations and partial operations allowed).
  No presumption that $\mathbf{M}$ is "algebraic" and $\underset{\sim}{\mathbf{M}}$ "relational".
- Compatibility: the structure of $\underset{\sim}{\mathbf{M}}$ is preserved by the operations and partial operations of $\mathbf{M}$ and the relations are substructures.

This notion is symmetric.

The natural extension functor for structures

It all goes through, with a little care. Natural extension functor works as before and we get a Paired Adjunctions Theorem. However the identification of the natural extension with a profinite completion is NOT available when $\mathcal{A}$ is class of structures (i.e. we have relations as well as operations in the type).

A common framework

The natural extension functor for classes of structures $\mathbb{ISP}(\mathfrak{M})$ should be seen as providing a common umbrella for assorted results seem variously as belonging to algebra or to topology:

- Profinite completions in context of a residually finite variety, with an explicit description if the variety has a natural duality.
- Stone-Czech compactification of a discrete space, and ordered Stone-Czech compactification of a poset.
- Hybrid algebraic/relational examples, ...

A dual equivalence on the cheap: Hofmann–Mislove–Stralka duality for semilattices

$\mathcal{S} = \mathbb{ISP}(\mathbf{2})$	$\mathsf{SL} = \wedge, 1 -$ semilattices
$\mathcal{Z} = \mathcal{S}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}(\mathbf{2}_{\mathcal{T}})$	STONE-SL

(Here we have two categories rather than four.)

On finite, discretely topologised, objects the topology does no work, so

$$\mathcal{Z}_{\mathsf{fin}} \text{ "is" } \mathcal{S}_{\mathsf{fin}}.$$

With this identification the evaluation maps are just identities. SO we have a dual equivalence at the finite level.

HMS duality, continued

Easy:

$\mathcal{S}$ built from $\mathcal{S}_{\mathsf{fin}}$ by taking directed (cofiltered) limits,

$\mathfrak{Z}$ built from $\mathfrak{Z}_{\mathsf{fin}}$ by taking projective limits (filtered colimits).

and the limits/colimits are preserved by the functors.

Marshall Stone

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