On the filter theory of residuated lattices

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Commutative bounded integral residuated lattices (residuated lattices, in short) form a large class of algebras which contains e.g. algebras that are algebraic counterparts of some propositional many-valued and fuzzy logics:

MTL-algebras, i.e. algebras of the monoidal *t*-norm based logic;

BL-algebras, i.e. algebras of Hájek's basic fuzzy logic;

MV-algebras, i.e. algebras of the Łukasiewicz infinite valued logic.

Moreover,

Heyting algebras, i.e. algebras of the intuitionistic logic.

Residuated lattices = algebras of a certain general logic that contains the mentioned non-classical logics as particular cases.

The deductive systems of those logics correspond to the filters of their algebraic counterparts.

A commutative bounded integral residuated lattice is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ satisfying the following conditions.

(i) $(M; \odot, 1)$ is a commutative monoid.

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.

(iii) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$, for any x, y, $z \in M$.

In what follows, by a residuated lattice we will mean a commutative bounded integral residuated lattice.

We define the unary operation (negation) "$^-$" on M by $x^- := x \rightarrow 0$ for any $x \in M$.

A residuated lattice M is

an MTL-algebra if M satisfies the identity of pre-linearity
(iv) $(x \to y) \vee (y \to x) = 1$;

involutive if M satisfies the identity of double negation
(v) $x^{--} = x$;

an Rl-monoid (or a bounded commutative GBL-algebra) if M satisfies the identity of divisibility
(vi) $(x \to y) \odot x = x \wedge y$;

a BL-algebra if M satisfies both (iv) and (vi);

an MV-algebra if M is an involutive BL-algebra;

a Heyting algebra if the operations "$\odot$" and "$\wedge$" coincide on M.

Lemma

Let M be a residuated lattice. Then for any x, y, $z \in M$ we have:

(i) $x \leq y \implies y^{-} \leq x^{-}$,

(ii) $x \odot y \leq x \wedge y$,

(iii) $(x \to y) \odot x \leq y$,

(iv) $x \leq x^{--}$,

(v) $x^{---} = x^{-}$,

(vi) $x \leq y \implies y \to z \leq x \to z$,

(vii) $x \leq y \implies z \to x \leq z \to y$,

(viii) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,

(ix) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$.

If M is a residuated lattice and $\emptyset \neq F \subseteq M$ then F is called a filter of M if for any $x, y \in F$ and $z \in M$:

1. $x \odot y \in F$;
2. $x \leq z \implies z \in F$.

If $\emptyset \neq F \subseteq M$ then F is a filter of M if and only if for any $x, y \in M$

3. $x \in F,\ x \to y \in F \implies y \in F$,

that means if F is a deductive system of M.

Denote by $\mathcal{F}(M)$ the set of all filters of a residuated lattice M. Then $(\mathcal{F}(M), \subseteq)$ is a complete lattice in which infima are equal to the set intersections.

If $B \subseteq M$, denote by $\langle B \rangle$ the filter of M generated by B. Then for $\emptyset \neq B \subseteq M$ we have

$$\langle B \rangle = \{z \in M : \ z \geq b_1 \odot \cdots \odot b_n, \ \text{where} \ \ n \in \mathbb{N},\ b_1, \ldots, b_n \in B\}.$$

If M is a residuated lattice, $F \in \mathcal{F}(M)$ and $B \subseteq M$, put

$$E_F(B) := \{x \in M : \ x \vee b \in F \text{ for every } b \in B\}.$$

Theorem

Let M be a residuated lattice, $F \in \mathcal{F}(M)$ and $B \subseteq M$. Then $E_F(B) \in \mathcal{F}(M)$ and $F \subseteq E_F(B)$.

$E_F(B)$ will be called the extended filter of a filter F associated with a subset B.

Theorem

If M is a residuated lattice, $B \subseteq M$ and $\langle B \rangle$ is the filter of M generated by B, then $E_F(B) = E_F(\langle B \rangle)$ for any $F \in \mathcal{F}(M)$.

Let L be a lattice with 0. An element $a \in L$ is pseudocomplemented if there is $a^* \in L$, called the pseudocomplement of a such that $a \wedge x = 0$ iff $x \leq a^*$, for each $x \in L$. A pseudocomplemented lattice is a lattice with 0 in which every element has a pseudocomplement.

Let L be a lattice and $a, b \in L$. If there is a largest $x \in L$ such that $a \wedge x \leq b$, then this element is denoted by $a \rightarrow b$ and is called the relative pseudocomplement of a with respect to b. A Heyting algebra is a lattice with 0 in which $a \rightarrow b$ exists for each $a, b \in L$.

Heyting algebras satisfy the infinite distributive law: If L is a Heyting algebra, $\{b_i : i \in I\} \subseteq L$ and $\bigvee_{i \in I} b_i$ exists then for each $a \in L$, $\bigvee_{i \in I}(a \wedge b_i)$ exists and $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I}(a \wedge b_i)$.

Based on the previous theorem, in the sequel we will investigate, without loss of generality, $E_F(B)$ only for $B \in \mathcal{F}(M)$.

Theorem

If M is a residuated lattice, then $(\mathcal{F}(M), \subseteq)$ is a complete Heyting algebra. Namely, if F, $K \in \mathcal{F}(M)$ then the relative pseudocomplement $K \to F$ of the filter K with respect to F is equal to $E_F(K)$.

Corollary

a) Every interval $[H, K]$ in the lattice $\mathcal{F}(M)$ is a Heyting algebra.

b) If F is an arbitrary filter of M and $K \in \mathcal{F}(M)$ such that $F \subseteq K$, then $E_F(K)$ is the pseudocomplement of K in the Heyting algebra $[F, M]$.

c) For $F = \{1\}$ and any $K \in \mathcal{F}(M)$ we have $E_{\{1\}}(K) = K^*$.

Theorem

Let M be a residuated lattice and $F, K, G, L, F_i, K_i \in \mathcal{F}(M)$, $i \in I$. Then:

1. $K \cap E_F(K) \subseteq F$;
2. $K \subseteq E_F(E_F(K))$;
3. $F \subseteq E_F(K)$;
4. $F \subseteq G \implies E_F(K) \subseteq E_G(K)$;
5. $F \subseteq G \implies E_K(G) \subseteq E_K(F)$;
6. $K \cap E_F(K) = K \cap F$;
7. $E_F(K) = M \iff K \subseteq F$;
8. $E_F(E_F(G)) \cap E_F(G) = F$;
9. $F \subseteq G \implies E_F(G) \cap G = F$;
10. $E_F(E_F(E_F(K))) = E_F(K)$.

Theorem

1. $K \subseteq L,\ E_F(K) = F \implies E_F(L) = F;$
2. $E_{E_M(L)}(K) = E_M(K \cap L);$
3. $E_{E_F(K)}(L) = E_{E_F(L)}(K);$
4. $E_F(K) = F \implies E_F(E_F(K)) = M;$
5. $\bigcap_{i \in I} E_{F_i}(K) = E_{\bigcap\{F_i \colon i \in I\}}(K);$
6. $E_F\left(\bigvee_{i \in I} K_i\right) = \bigcap_{i \in I} E_F(K_i).$

Now we will deal with the sets $E_F(K)$ where F and K, respectively, are fixed.
Let M be a residuated lattice and $K \in \mathcal{F}(M)$. Put

$$E(K) := \{E_F(K) : F \in \mathcal{F}(M)\}.$$

Theorem

If M is a residuated lattice and $K \in \mathcal{F}(M)$, then $(E(K), \subseteq)$ is a complete lattice which is a complete inf-subsemilattice of $\mathcal{F}(M)$.

One can show that $E(K)$, in general, is not a sublattice of $\mathcal{F}(M)$. We can do it in a more general setting for arbitrary Heyting algebras.

Let A be a complete Heyting algebra. If $d \in A$, put $E(d) := \{d \to x : x \in A\}$. Then, analogously as in a special case in the previous theorem, we can show that $E(d)$ is a complete lattice which is a complete inf-subsemilattice of A.

Proposition

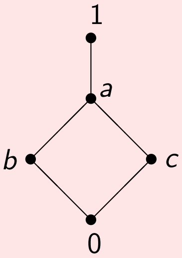
If A is a complete Heyting algebra and $a \in A$, then $E(a)$ need not be a sublattice of the lattice A.

Let A be any complete Heyting algebra such that subset $A \setminus \{1\}$ have a greatest element a and let there exist elements $b, c \in A$ such that $b < a$, $c < a$ and $b \vee c = a$. Then $a \to y = y$ for any $y < a$ and $a \to a = 1 = a \to 1$, hence $a \notin E(a)$, but $b, c \in E(a)$. Therefore in the lattice $E(a)$ we have $b \vee_{E(a)} c = 1$, that means $E(a)$ is not a sublattice of A.

Example 1

Consider the lattice A with the diagram in the figure.
Then A is a complete Heyting algebra with the relative pseudocomplements in the table.
We get $E(a) = \{0, b, c, 1\}$, but the lattice $E(a)$ is not a sublattice of A.

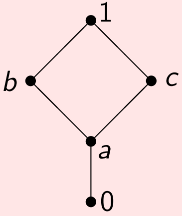
$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	c	1
b	c	1	1	c	1
c	b	1	b	1	1
1	0	a	b	c	1



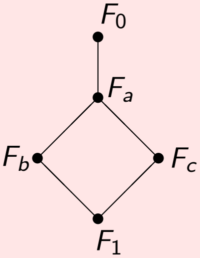
Example 2

Let M be the lattice in the figure. Then M is a Heyting algebra with the relative pseudocomplements in the table.

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1



If we put $\odot = \wedge$, then $M = (M;\ \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice. Since the filters of the residuated lattice M are precisely the lattice filters of M, we get $\mathcal{F}(M) = \{F_0,\ F_a,\ F_b,\ F_c,\ F_1\}$, where $F_0 = M = \{0,\ a,\ b,\ c,\ 1\}$, $F_a = \{a,\ b,\ c,\ 1\}$, $F_b = \{b,\ 1\}$, $F_c = \{c,\ 1\}$, $F_1 = \{1\}$. Hence the lattice $\mathcal{F}(M)$ is anti-isomorphic to the lattice M. (See the following figure.) Therefore, similarly as in Example 1, we have that $E(F_a) = \{F_1,\ F_b,\ F_c,\ F_0\}$ is not a sublattice of $\mathcal{F}(M)$.



Corollary

If M is a residuated lattice and $F \in \mathcal{F}(M)$, then $E(F)$ need not be a sublattice of the lattice $\mathcal{F}(M)$.

Let M be a residuated lattice and $F \in \mathcal{F}(M)$. Put

$$E_F := \{E_F(K) : K \in \mathcal{F}(M)\}.$$

Theorem

If M is a residuated lattice and $F \in \mathcal{F}(M)$, then E_F ordered by set inclusion is a complete lattice which is a complete inf-subsemilattice of $\mathcal{F}(M)$.

We can show that E_F (similarly as $E(K)$) need not be a sublattice of $\mathcal{F}(M)$. We can again do it in a more general setting for arbitrary Heyting algebras.

Let A be a complete Heyting algebra. If $a \in A$, put $E_a := \{x \to a : \; a \in A\}$. Then analogously as in a special case in the preceding theorem one can show that E_a is a complete inf-subsemilattice of A.

Proposition

If A is a complete Heyting algebra and $a \in A$, then E_a need not be a sublattice of the lattice A.

Let A be a complete Heyting algebra which contains elements a, b, c, d such that $a < b < d < 1$, $a < c < d < 1$, $b \wedge c = a$, $b \vee c = d$, d is the greatest element in $A \setminus \{1\}$ and a is the greatest element in $L \setminus \{b, c, d, 1\}$. The $d \notin E_a$, while $b, c \in E_a$. From this we get $b \vee_{E_a} c \neq b \vee_A c$, and so E_a is not a sublattice of the lattice A.

Example 3

Let us consider the Heyting algebra A from Example 1.
We get $E_0 = \{0, b, c, 1\}$, hence E_0 is not a sublattice of A.

Example 4

Let M be the residuated lattice from Example 2.
Then $E_{F_1} = \{F_1, F_b, F_c, F_0\}$, and hence E_{F_1} is not a sublattice of the lattice $\mathcal{F}(M)$.

Corollary

If M is a residuated lattice and $F \in \mathcal{F}(M)$, then E_F need not be a sublattice of $\mathcal{F}(M)$.

Now we will deal with further connections between two filters of residuated lattices. Let M be a residuated lattice and $F, K \in \mathcal{F}(M)$. Then F is called stable with respect to K if $E_F(K) = F$.

Proposition

Let M be a residuated lattice and $F, K, L \in \mathcal{F}(M)$.

1. F is stable with respect to F.
2. If $K \subseteq L$ and F is stable with respect to K, then F is also stable with respect to L.
3. F is stable with respect to K if and only if $E_F(E_F(K)) = M$.

Proposition

Let A be a Heyting algebra, $x, y \in A$ and $y < x$. Let $x, y \in [a, b]$, where $a, b \in A$, $a \leq b$ and the interval $[a, b]$ is a chain such that $v \geq a$ implies $v \geq b$ and $w \leq b$ implies $w \leq a$, for any $v, w \in A$. Then $x \rightarrow y = y$.

The following theorem is now an immediate consequence.

Theorem

Let M be a residuated lattice, $K, F, P, R \in \mathcal{F}(M)$, $F \subset K$ and $F, K \in [P, R]$, where $[P, R]$ is a chain and $S \supseteq P$ implies $S \supseteq R$ and $T \subseteq R$ implies $T \subseteq P$, for any $S, T \in \mathcal{F}(M)$. Then F is stable with respect to K.

Thank you for your attention.