Ideal extension of semigroups and their applications

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In this talk we introduce ideal extension for topological semigroups using a new method, then we investigate the compactification spaces of these structures.
As a consequence, we use this result to characterize compactification spaces for Brandt λ-extension of topological semigroups.

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Definition

Let S and T be disjoint topological semigroups, T having a zero element 0. A topological semigroup Ω is a topological extension of S by T if Ω contains S as an ideal and the Rees factor semigroup $\frac{\Omega}{S}$ is topologically isomorphic to T.

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- If Ω is an ideal extension of topological semigroup S by T and Ω', S' and T' are compactifications of Ω, S and T respectively, whether Ω' can naturally characterize by S' and T'.

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- In especial case, results of this type are known by some authors, say for topological tensor product of semigroups, Sherier products of semigroups.

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$$(P2) \quad Aos = \overline{A}s, \qquad (P3) \quad soA = s\overline{A}, \qquad (P4) \quad sot = st.$$

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Theorem

Let S and T be disjoint topological semigroups such that T has a zero. Let $\theta : T^* = T - \{0\} \to S$ be continuous partial homomorphism. Then $\Omega = S \cup T^*$ with multiplication $(P1, P2, P3, P4)$ is a topological extension of S by T. Conversely, every topological extension of topological semigroup S by topological semigroup T can be so constructed

Proof.

(Sketch)

- Clearly, Ω is an extension of S by T.

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- Ω is a topological semigroup with identity.
- Suppose τ be the equivalence relation generated by $\tau = \{(u, su') \mid s \in S, u, u' \in \Omega\}$
- $\rho_\Omega = \{(x, y) \in \Omega \times \Omega \mid (uxv, uyv) \in \tau, \textit{for all } u, v \in \Omega\}$. ρ_Ω is the largest congruence on $\Omega \times \Omega$ contained in τ, and $\frac{\Omega}{\rho_\Omega} \simeq \frac{\Omega}{S} \simeq T$.

□

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- Let (ψ, X) be a topological semigroup compactification of Ω and τ_X be the equivalence relation generated by $\{(x, \psi(s)y) \mid x, y \in X, s \in S\}$ and ρ_X be the closure of the largest congruence on $X \times X$ contained in τ_X.

Theorem

Let S and T be disjoint topological semigroups such that T has a zero and Ω be a topological extension of S by T. Let (ψ, X) be a topological semigroup compactification of Ω. Then $\frac{X}{\rho_X}$ is a topological semigroup compactification of $\frac{\Omega}{S} \simeq T$.

Theorem

Let S and T be disjoint topological semigroups such that T has a zero and Ω be a topological extension of S by T. Let $(\varepsilon_T, T^{\mathcal{P}})$ and $(\varepsilon_\Omega, \Omega^{\mathcal{P}})$ be the universal $\mathcal{P}$-compactifications of T and Ω respectively. Then $T^{\mathcal{P}} \simeq \frac{\Omega^{\mathcal{P}}}{\rho_{\Omega^{\mathcal{P}}}}$ if

i) $\mathcal{P}$ is invariant under homomorphism,

ii) universal $\mathcal{P}$-compactification is a topological semigroup.

Corollary

Let Ω be a topological extension of topological semigroup S by topological semigroup T. Let (ε_s, S^{sap}), $(\varepsilon_\Omega, \Omega^{sap})$ [resp. (ε_s, S^{ap}), $(\varepsilon_\Omega, \Omega^{ap})$] be the strongly almost periodic compactifications [resp. almost periodic compactifications] of S and Ω, respectively. Then $T^{sap} \simeq \frac{\Omega^{sap}}{\rho_{\Omega^{sap}}}$ [resp. $T^{ap} \simeq \frac{\Omega^{ap}}{\rho_{\Omega^{ap}}}$].

Question. If X_S and X_T are topological semigroup compactifications of S and T respectively , whether topological extension of X_S and X_T exist and is semigroup compactification of extension of S by T ?

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Theorem

Let S and T be disjoint topological semigroups such that T has a zero and Ω be a topological extension of S by T. Let (ψ_S, X_S) and (ψ_T, X_T) be topological semigroup compactifications of S and T respectively such that $X_S \cap X_T = \emptyset$. Then the following assertions hold.

- *a) Topological extension X_Ω of X_S by X_T exist.*
- *b) Topological center $\Lambda(\Omega)$ is a topological extension of $\Lambda(S)$ by $\Lambda(T)$.*
- *c) (ψ_Ω, X_Ω) is a topological semigroup compactification of Ω where $\psi_\Omega|_T = \psi_T, \psi_\Omega|_S = \psi_S$.*

Following theorem shows that topological semigroup compactifications of S and T can be constructed by topological semigroup compactifications of their topological extension.

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Theorem

Let S and T be disjoint topological semigroups such that T has a zero and Ω be a topological extension of S by T. Suppose (ψ_Ω, X_Ω) is a topological semigroup compactification of Ω. Then there are topological semigroups compactifications (ψ_S, X_S), (ψ_T, X_T) of S and T respectively such that X_Ω is a topological extension of X_S by X_T.

Applications.

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- An important class of semigroups which has been considered from various points of view is completely 0-simple semigroup and Brandt λ-extension.
- In following we use topological extension technique to characterizing compactification spaces of Brandt λ-extension .
- Let $G^0 = G \cup \{0\}$ [resp. G] be a group with zero [resp. group], E and F be arbitrary nonempty sets.

- Let P be a $E \times F$ matrix over G^0 [resp. G].

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- The set $S = G \times E \times F \cup \{0\}$ [resp. $S = G \times E \times F$] is a semigroup under the composition

$$(i, a, j) \circ (l, b, k) = \begin{cases} (i, ap_{jl}b, k) & if p_{jl} \neq 0 \\ o & otherwise \end{cases}$$

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- This semigroup is denoted by $S = M(G, P, E, F)$ and is called Rees $E \times F$ matrix semigroup over G^0 [resp. G] with the sandwich matrix P.

- In special case, if $P = I$ is an identity matrix, $S = G^0$ is semigroup with zero, and $E = F = I_\lambda$ is a set of cardinality $\lambda \geq 1$.

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- Define the semigroup operation on the set $B_\lambda(S) = M(S, I, I_\lambda, I_\lambda)$ by

$$(i, a, j) \circ (l, b, k) = \begin{cases} (i, ab, k) & \textit{if } j = l \\ 0, & \textit{if } j \neq l \end{cases}$$

and $(i, a, j).0 = 0.(i, a, j) = 0.0 = 0$ for all $a, b \in S, i, j, l, k \in I_\lambda$.

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- The semigroup $B_\lambda(S)$ is called Brandt λ-extension of S.

- Now let $i \to u_i$ and $j \to v_j$ be mappings of E and F to S such that $u_k.u_k = 1_S$, for all $k \in \lambda$.

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- Then mapping $\theta : B_\lambda(S)^* = B_\lambda(S) - \{0\} \to S$ by $\theta(i, s, j) = u_i s u_j$ is a partial homomorphism.

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- Then there exists a topological extension Ω of S^* by $B_\lambda(S)$ and $\frac{\Omega}{S^*} \simeq B_\lambda(S)$.

Corollary

Let S be a topological semigroup with zero and Ω be a topological extension of $S^* = S - \{0\}$ by $B_\lambda(S)$. Let (ψ, X) be a topological semigroup compactification of topological semigroup Ω. Then $\frac{X}{\rho_X}$ is a topological semigroup compactification of $B_\lambda(S)$.

Corollary

Let S be a topological semigroup with zero and Ω be a topological extension of $S^* = S - \{0\}$ by $B_\lambda(S)$. Suppose $(\varepsilon_{B_\lambda(S)}, B_\lambda(S)^P)$ and $(\varepsilon_\Omega, \Omega^P)$ are the universal P-compactifications of $B_\lambda(S)$ and Ω respectively. Then $B_\lambda(S)^P \simeq \frac{\Omega^P}{\rho_{\Omega^P}}$, if

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Thank you for your attention