Combinatorial properties of singular cardinals

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August 2013

Overview

- Singular cardinals

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- Singular cardinals
- Consistency results

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- Large cardinals

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- Conbinatorial principles

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- Fact: if κ, λ are infinite, then $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$.

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- A cardinal κ is **singular** if $\mathrm{cf}(\kappa) < \kappa$.
- For example, $\aleph_n$ is regular for every n, and $\aleph_\omega$ is singular.

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- GCH implies SCH.
- Addressing these questions gave rise to **consistency results**.

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  The only constraints:
  - $\kappa < \lambda$ implies $2^\kappa \leq 2^\lambda$,
  - Kőnig's lemma.

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The Singular Cardinal Problem: Describe a complete set of rules for the behavior of the exponential function $\kappa \mapsto 2^\kappa$ for singular cardinals κ.

Using forcing to add new subsets of a cardinal κ

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Information about $V[G]$ can be obtained while working in V via a relation definable in V, called the *forcing relation*, "$p \Vdash \phi$".

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Let G be $Add(\kappa, 1)$-generic over V, and set $f^* = \bigcup_{f \in G} f$. Then $f^* : \kappa \to \{0, 1\}$ is a total function and

$$a =_{def} \{\alpha < \kappa \mid f^*(\alpha) = 1\}$$

is a new subset of κ. I.e. $a \in V[G] \setminus V$.

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- Prikry forcing: changes cofinality without collapsing cardinals; requires large cardinals.

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Remark

An alternative way to define these large cardinals is via elementary embeddings of the set theoretic universe.

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The strategy: add subsets to a large cardinal, then singularize it.

Extender based forcing

Alternative way: start with a singular κ; say $\kappa = \sup_n \kappa_n$; and blow up its powerset to some regular λ in a Prikry fashion via **extender based forcing**.

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- In particular, this forcing uses *extenders*; an extender is a system of ultrafilters.
- No need to add subsets of κ in advance, so can keep GCH below κ, as opposed to the above forcings.

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Theorem

(S.) Starting from a supercompact cardinal κ, there is a forcing which simultaneously singularizes κ and increases its powerset, while maintaining GCH below κ.

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- No bounded subsets of κ are added.
- In the final model, GCH holds below κ, and $2^\kappa > \kappa^+$. So SCH fails at κ.
- Collapses κ^+ and actually an interval of cardinals (unlike the classical extender based forcing).

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- $\square^*_\kappa$ is a weakening which allows up to κ guesses for each club.
- $\kappa^{<\kappa} = \kappa \to \square^*_\kappa$; so we focus on the case κ singular.

Lemma

In the Hybrid Prikry model, we have $\square^*_\kappa$.

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A point $\gamma < \mu$ of cofinality between ω and κ is a **good point** iff there exists an unbounded $A \subseteq \gamma$, such that $\langle f_\alpha(n) \mid \alpha \in A \rangle$ is strictly increasing for all large n. If A is club in γ, then γ is a **very good point**.

A scale is (**very**) **good** iff modulo the club filter on μ, almost every point of cofinality between $\mathrm{cf}(\kappa)$ and κ is (very) good.

Scales

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Lemma

When forcing with Hybrid Prikry, scales in $\prod_n \kappa^{+n+1}$ from V generate scales $\prod_n \kappa$ in the generic extension.

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