On some topological properties of pointfree function rings

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Outline of the talk

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- Spatial setting

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- Frames and $\mathcal{L}(\mathbb{R})$

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- Dini properties

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- Dini properties
- The Stone-Weierstrass property

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Dini's theorem

Let X be a compact Hausdorff space, $f_n \in C(X)$ ($n \in \mathbb{N}_0$) and $f \in C(X)$. If $(f_n)_n$ is increasing (i.e. $f_n \leq f_{n+1}$) and $(f_n)_n$ converges to f pointwise, then $(f_n)_n$ converges to f uniformly.

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The Stone-Weierstrass theorem

Let X be a compact Hausdorff space. Every separating unital $\mathbb{R}$-subalgebra A of $C(X)$ which separates points is dense in $C(X)$ w.r.t. the uniform topology.

From theorems to properties

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The Weak Dini Property (wDP)

A topological space X is said to satisfy the *Weak Dini Property* (wDP) if every increasing sequence $(f_n)_n$ in $C(X)$ which converges to $f \in C(X)$ pointwise, converges to f uniformly.

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A topological space is said to have the *Stone-Weierstrass Property* (SWP) if every separating unital $\mathbb{R}$-subalgebra A of $C^*(X)$ which separates points is dense in $C^*(X)$ w.r.t. the uniform topology.

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The category **Frm**

objects: complete lattices L (top e, bottom 0) such that

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dual category: $\mathbf{Loc} := \mathbf{Frm}^{\mathrm{op}}$: locales

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dual equivalence: $\mathbf{Sob} \simeq \{$spatial frames$\}^{\mathrm{op}} = \{$spatial locales$\}$

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Option 2

Define $\mathcal{L}(\mathbb{R})$ to be the frame with generators

$$\text{all pairs } (p, q) \text{ with } p, q \in \mathbb{Q}$$

subject to the following relations:

- $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$
- $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$
- $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$
- $e = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$

The function rings $\mathfrak{R}(L)$ and $\mathfrak{R}^*(L)$

The pointfree counterpart to $C(X)$

For a frame L, let

$$\mathfrak{R}(L) := \mathbf{Frm}(\mathcal{L}(\mathbb{R}), L)$$

with the following operations on it:

- for $\diamond \in \{+, \cdot, \vee, \wedge\}$,

$$(\alpha \diamond \beta)(p, q) := \bigvee \{\alpha(r, s) \wedge \beta(t, u) \mid \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\}$$

- $(-\alpha)(p, q) := \alpha(-q, -p)$
- for each $r \in \mathbb{Q}$ a 0-ary operation $\mathbf{r}$ defined by

$$\mathbf{r}(p, q) := \begin{cases} e & \text{if } p < r < q \\ 0 & \text{otherwise} \end{cases}$$

Properties of $\mathfrak{R}(L)$

Fact

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- archimedean: $\alpha, \beta \geq 0$ and $n\alpha \leq \beta$ (all n) imply $\alpha = 0$
- f-ring: $|\alpha\beta| = |\alpha||\beta|$, with $|\alpha| := \alpha \vee (-\alpha)$

The pointfree counterpart to $C^*(X)$

For a frame L, let

$$\mathfrak{R}^*(L) := \{\alpha \in \mathfrak{R}(L) \mid |\alpha| \leq \mathbf{n}, \text{ some } n\}$$

Fact

$\mathfrak{R}^*(L)$ is an l-subring of $\mathfrak{R}(L)$ and hence also a strong unital archimedean f-ring.

The uniform topology: pointfree case

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Definition

For a frame L the *uniform topology* on $\mathfrak{R}(L)$ is the topology having

$$V_n(\alpha) := \{\gamma \in \mathfrak{R}(L) \mid |\alpha - \gamma| < \frac{1}{\mathbf{n}}\}, \qquad \text{all } n \in \mathbb{N}_0$$

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Definition

For a frame L the *uniform topology* on $\mathfrak{R}^*(L)$ is the subspace topology it inherits from the uniform topology on $\mathfrak{R}(L)$.

Pointwise convergence = convergence everywhere: spatial case

Definition

For a topological space X, a net $(f_\eta)_{\eta\in D}$ and $f \in C(X)$ we say that $(f_\eta)_{\eta\in D}$ *converges to* f *everywhere*, and write $(f_\eta)_{\eta\in D} \to f$, if

$$\forall x \in X, \forall m \in \mathbb{N}_0, \exists \eta_0 \in D, \forall \eta \in D : \eta \geq \eta_0 \Rightarrow |f(x) - f_\eta(x)| < \frac{1}{m}$$

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Definition

- A net $(f_\eta)_{\eta\in D}$ is called *increasing* if

$$\forall \eta, \mu \in D : \eta \leq \mu \Rightarrow f_\eta \leq f_\mu.$$

- A net $(f_\eta)_{\eta\in D}$ is called *decreasing* if

$$\forall \eta, \mu \in D : \eta \leq \mu \Rightarrow f_\eta \geq f_\mu$$

Convergence everywhere for increasing/decreasing nets: spatial case

So for $(f_\eta)_{\eta\in D}$ increasing and with $f_\eta \leq f$ for all $\eta \in D$:

$$
\begin{aligned}
&(f_\eta)_{\eta\in D} \to f \\
\Leftrightarrow &\forall m \in \mathbb{N}_0 : \bigcup_{\eta_0\in D} \bigcap_{\eta\in D, \eta\geq\eta_0} \{x \in X \mid |f(x) - f_\eta(x)| < \frac{1}{m}\} = X \\
\Leftrightarrow &\forall m \in \mathbb{N}_0 : \bigcup_{\eta_0\in D} \{x \in X \mid f(x) - f_{\eta_0}(x) < \frac{1}{m}\} = X \\
\Leftrightarrow &\forall m \in \mathbb{N}_0 : \bigcup_{\eta_0\in D} \{x \in X \mid (1 - m(f(x) - f_{\eta_0}(x))) > 0\} = X \\
\Leftrightarrow &\forall m \in \mathbb{N}_0 : \bigcup_{\eta_0\in D} \{x \in X \mid (1 - m(f(x) - f_{\eta_0}(x))) \vee 0 \neq 0\} = X
\end{aligned}
$$

The cozero part of a frame - completely regular frames

Notation: in $\mathcal{L}(\mathbb{R})$, for every $p \in \mathbb{Q}$

$$(-, p) := \bigvee \{(q, p) \mid q \in \mathbb{Q}, q < p\}$$

$$(p, -) := \bigvee \{(p, q) \mid q \in \mathbb{Q}, q > p\}$$

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Definition

For a frame L and $\alpha \in \mathfrak{R}(L)$,

$$\mathrm{coz}(\alpha) := \alpha((-, 0) \vee (0, -))$$

is called the *cozero element* determined by α.
$\mathrm{Coz}L := \{\mathrm{coz}(\alpha) \mid \alpha \in \mathfrak{R}(L)\}$ is called the *cozero part of L*.

The cozero part of a frame - completely regular frames

Fact

For any frame L, $\mathrm{Coz}(L)$ is a sub-σ-frame of L.

Definition

L a frame, $a, b \in L$:

- $a \prec b$ (a rather below b) $\equiv a^* \vee b = e$
- $a \prec\!\prec b$ (a well below b) $\equiv$ exists $(a_r)_{r \in \mathbb{D}}$ such that $a_0 = a$, $a_1 = b$, and $a_r \prec a_s$ wehenver $r < s$
- *L completely regular* $\equiv a = \bigvee\{x \in L \mid x \prec\!\prec a\}$ for all $a \in L$

Fact

A frame L is completely regular if and only if it is ($\bigvee$)-generated by $\mathrm{Coz}L$.

Convergence everywhere for increasing nets: pointfree case

Definition

Let L be a frame, $\alpha \in \mathfrak{R}(L)$ and $(\alpha_\eta)_{\eta\in D}$ a net in $\mathfrak{R}(L)$. Then we say that $(\alpha_\eta)_{\eta\in D}$ *increases everywhere to* α, and we write $(\alpha_\eta)_{\eta\in D} \uparrow \alpha$ if $(\alpha_\eta)_{\eta\in D}$ is increasing, $\alpha_\eta \leq \alpha$ for all $\eta \in D$, and

$$\forall m \in \mathbb{N}_0 : \bigvee_{\eta\in D} \mathrm{coz}((\mathbf{1} - m(\alpha - \alpha_\eta))^+) = e.$$

Notation: for $\gamma \in \mathfrak{R}(L)$, we write $\gamma^+ := \gamma \vee 0$

Convergence everywhere for decreasing nets: pointfree case

Definition

Let L be a frame, $\alpha \in \mathfrak{R}(L)$ and $(\alpha_\eta)_{\eta \in D}$ a net in $\mathfrak{R}(L)$. Then we say that $(\alpha_\eta)_\eta$ *decreases everywhere to* α, and we write $(\alpha_\eta)_{\eta \in D} \downarrow \alpha$ if $(\alpha_\eta)_{\eta \in D}$ is decreasing, $\alpha_\eta \geq \alpha$ for all $\eta \in D$, and

$$\forall m \in \mathbb{N}_0 : \bigvee_{\eta \in D} \operatorname{coz}((\mathbf{1} - m(\alpha_\eta - \alpha))^+) = e.$$

Notation: for $\gamma \in \mathfrak{R}(L)$, we write $\gamma^+ := \gamma \vee 0$

The weak Dini Property

Definition (wDP)

For a frame L, we say that L satisfies the *weak Dini property* or (wDP) if for any $\alpha \in \mathfrak{R}(L)$ and any sequence $(\alpha_n)_n$ in $\mathfrak{R}(L)$ which increases everywhere to α, the sequence $(\alpha_n)_n$ converges to α in the uniform topology on $\mathfrak{R}(L)$.

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Remark: note that (wDP) is equivalent to the statement with 'increasing' $\rightarrow$ 'decreasing'

Pointfree pseudo-compactness

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Theorem (Banaschewski-Gilmour)

For any frame L, the following are equivalent:

(1) L is pseudo-compact.

(2) Any sequence $a_0 \prec\prec a_1 \prec\prec a_2 \prec\prec \dots$ such that $\bigvee a_n = e$ in L terminates, that is, $a_k = e$ for some k.

(3) The σ-frame $\mathrm{Coz}L$ is compact.

Characterizing (wDP)

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- show that $(\alpha \wedge \mathbf{n})_n \uparrow \alpha$

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- show that $(\alpha \wedge \mathbf{n})_n \uparrow \alpha$
- by (wDP), $(\alpha \wedge \mathbf{n})_n$ converges to α w.r.t. the uniform topology
- so $\alpha - \alpha \wedge \mathbf{n} \leq \mathbf{1}$ for some n, hence α is bounded

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- invoking pseudo-compactness, form $(n_m)_m$ such that

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- then $\forall m \in \mathbb{N}_0 : \alpha - \alpha_{n_m} \leq \frac{1}{\mathbf{m}}$
- so, since $(\alpha_n)_n$ is increasing, $(\alpha_n)_n$ converges to α w.r.t. the uniform topology □

The Strong Dini Poperty

Definition (sDP)

For a frame L, we say that L satisfies the *strong Dini property* or (sDP) if for any $\alpha \in \mathcal{R}L$ and any net $(\alpha_\eta)_{\eta \in D}$ in $\mathcal{R}L$ which increases everywhere to α, the net $(\alpha_\eta)_{\eta \in D}$ converges to α in the uniform topology on $\mathcal{R}L$.

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Remark: note that (sDP) is equivalent to the statement with 'increasing' $\rightarrow$ 'decreasing'

Characterizing (sDP)

Theorem

For a frame L the following assertions are equivalent:

(1) L satisfies (sDP).

(2) Every cover L consisting of cozero elements has a finite subcover.

(3) The completely regular coreflection of L compact.

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(2) $\Leftrightarrow$ (3): clear

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- $\bigvee_{\eta \in D} \mathrm{coz}((\mathbf{1} - m\alpha_\eta)^+) = e$
- using (2), pick $\eta_0 \in D$ with $\mathrm{coz}((\mathbf{1} - m\alpha_{\eta_0})^+) = e$

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- then $\alpha_{\eta_0}\leq\frac{1}{\mathbf{m}}$

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- using (2), pick $\eta_0 \in D$ with $\mathrm{coz}((\mathbf{1} - m\alpha_{\eta_0})^+) = e$
- then $\alpha_{\eta_0} \leq \frac{\mathbf{1}}{\mathbf{m}}$
- remember $(\alpha_\eta)_{\eta\in D}$ is decreasing
- so $(\alpha_\eta)_\eta$ converges to α w.r.t. the uniform topology □

Characterizing (sDP)

<u>$(1) \Rightarrow (2)$:</u>

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- so

$$e = \mathrm{coz}\left(\sum_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})^+\right) = \bigvee_{(a,n)\in\eta} \mathrm{coz}(n\alpha_a - \mathbf{1})^+ \leq \bigvee_{(a,n)\in\eta} a$$

$\square$

Variant on a theme: the κ-Dini Property

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For an infinite cardinal number κ, a frame L is called *initially κ-compact* if every cover of L of cardinality at most κ admits a finite subcover.

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For an infinite cardinal number κ, a frame L is called *initially κ-compact* if every cover of L of cardinality at most κ admits a finite subcover.

Note: for $\kappa = \aleph_0$, initially κ-compact means countably compact.

Definition (κ-DP)

For a frame L and an infinite cardinal number κ, we say that L satisfies the *κ-Dini property* or (κ-DP) if for any $\alpha \in \mathfrak{R}(L)$ and any net $(\alpha_\eta)_{\eta \in D}$ in $\mathfrak{R}(L)$ with cardinality of D at most κ and which increases everywhere to α, the net $(\alpha_\eta)_{\eta \in D}$ converges to α in the uniform topology on $\mathfrak{R}(L)$.

Characterizing (κ-DP)

Corollary

For a frame L and an infinite cardinal number κ, the following assertions are equivalent:

(1) L satisfies (κ-DP).

(2) Every cover L consisting of cozero elements and of cardinality at most κ has a finite subcover.

(3) The completely regular coreflection of L is initially κ-compact.

Characterizing (κ-DP)

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Proof:

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Proof:

Note that for κ infinite and $\mathrm{Card}(F) \leq \kappa$:

$$\mathrm{Card}(\mathfrak{P}_{\mathrm{fin}}(F \times \mathbb{N}_0)) = \mathrm{Card}(\bigcup_{n\in\mathbb{N}} (F \times \mathbb{N}_0)^n) \leq \kappa.$$

□

Some terminology

Definition

(κ an infinite cardinal number) A frame L is called

- *Lindelöf* if every cover of L admits a countable subcover
- *quasi-Lindelöf* if every cover of L consisting of cozero elements admits a countable subcover
- *initially κ-Lindelöf*, if every cover of L of cardinality at most κ admits a countable subcover
- *initially κ-quasi-Lindelöf*, if every cover of L consisting of cozero elements and of cardinality at most κ admits a countable subcover

Characterizing (sDP) and (κ-DP)

Proposition

For a frame L, the following assertions are equivalent:

(1) L satisfies (sDP).

(2) L is quasi-Lindelöf and L satisfies (wDP).

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For a frame L and an infinite cardinal number κ, the following assertions are equivalent:

(1) L satisfies (κ-DP).

(2) L is initially κ-quasi-Lindelöf and L satisfies (wDP).

Characterizing quasi-Lindelöfness

Theorem

For a frame L, the following assertions are equivalent:

(1) L is quasi-Lindelöf.

(2) The completely regular coreflection of L is Lindelöf.

(3) For any net $(\alpha_\eta)_{\eta\in D}$ in $\mathfrak{R}(L)$ and any $\alpha \in \mathfrak{R}(L)$ such that $(\alpha_\eta)_{\eta\in D} \uparrow \alpha$ (resp. $(\alpha_\eta)_{\eta\in D} \downarrow \alpha$), there exists an increasing sequence $(\eta_n)_n$ in D such that $(\alpha_{\eta_n})_n \uparrow \alpha$ (resp. $(\alpha_{\eta_n})_n \downarrow \alpha$).

Characterizing initially κ-quasi-Lindelöfness

Theorem

For a frame L and an infinite cardinal number κ, the following assertions are equivalent:

(1) L is initially κ-quasi-Lindelöf.

(2) The completely regular coreflection of L is initially κ-Lindelöf.

(3) For any net $(\alpha_\eta)_{\eta\in D}$ in $\mathfrak{R}(L)$ with cardinality of D at most κ and any $\alpha \in \mathfrak{R}(L)$ such that $(\alpha_\eta)_{\eta\in D} \uparrow \alpha$ (resp. $(\alpha_\eta)_{\eta\in D} \downarrow \alpha$), there exists an increasing sequence $(\eta_n)_n$ in D such that $(\alpha_{\eta_n})_n \uparrow \alpha$ (resp. $(\alpha_{\eta_n})_n \downarrow \alpha$).

A final characterization of (sDP), (κ-DP) and (wDP)

Definition

(κ an infinite cardinal number) A frame L is called

- *almost-compact* if for every cover F of L, there exists $S \subseteq F$ finite such that
$$(\bigvee S)^* = 0.$$
- *initially κ-almost-compact* if for every cover F of L of cardinalty at most κ, there exists $S \subseteq F$ finite such that
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Definition

(κ an infinite cardinal number) A frame L is called

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A final characterization of (sDP), (κ-DP) and (wDP)

Proposition

Every quasi-almost-compact frame satisfies (sDP). For any infinite cardinal number κ, every initially κ-quasi-almost-compact frame satisfies (κ-DP).

A final characterization of (sDP), (κ-DP) and (wDP)

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Proof:

- assume $(\alpha_\eta)_{\eta \in D} \downarrow 0$ in $\mathfrak{R}(L)$ (with $\mathrm{Card}(D) \leq \kappa$)

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Proof:

- assume $(\alpha_\eta)_{\eta\in D} \downarrow 0$ in $\mathfrak{R}(L)$ (with $\mathrm{Card}(D) \leq \kappa$)
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$$\bigvee_{\eta\in D} \mathrm{coz}((\mathbf{1} - m\alpha_\eta)^+) = e$$

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Every quasi-almost-compact frame satisfies (sDP). For any infinite cardinal number κ, every initially κ-quasi-almost-compact frame satisfies (κ-DP).

Proof:

- assume $(\alpha_\eta)_{\eta \in D} \downarrow 0$ in $\mathfrak{R}(L)$ (with $\mathrm{Card}(D) \leq \kappa$)
- so

$$\bigvee_{\eta \in D} \mathrm{coz}((\mathbf{1} - m\alpha_\eta)^+) = e$$

- by (κ-)quasi-almost-compactness, there exists $\eta_0 \in D$ such that

$$\underbrace{(\mathrm{coz}((\mathbf{1} - m\alpha_{\eta_0})^+))}_{a:=}{}^* = 0$$

A final characterization of (sDP), (κ-DP) and (wDP)

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i.e.
$$0 \leq \alpha_{\eta_0} \leq \frac{1}{\mathbf{m}}$$
- so $(\alpha_n)_n$ converges to 0 w.r.t. the uniform topology

□

A final characterization of (sDP), (κ-DP) and (wDP)

... to make terminology consistent:

Definition

(κ an infinite cardinal number) a frame L is called

- *quasi-compact* if every cover of L consisting of cozero elements admits a finite subcover
- *initially κ-quasi-compact* if every cover of L consisting of cozero elements and of cardinality at most κ admits a finite subcover

A final characterization of (sDP), (κ-DP) and (wDP)

Corollary

For a frame L the following assertions are equivalent:

(1) L satisfies (sDP).

(2) L is quasi-compact.

(3) The completely regular coreflection of L is compact.

(4) L is quasi-almost-compact.

A final characterization of (sDP), (κ-DP) and (wDP)

Corollary

For a frame L and an infinite cardinal number κ, the following assertions are equivalent:

(1) L satisfies (κ-DP).

(2) L is initially κ-quasi-compact.

(3) The completely regular coreflection of L is initially κ-compact.

(4) L is initially κ-quasi-almost-compact.

A final characterization of (SDP), (κ-DP) and (wDP)

Corollary

For a completely regular frame L the following assertions are equivalent:

(1) L satisfies (sDP).

(2) L is compact.

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A final characterization of (SDP), (κ-DP) and (wDP)

Corollary

For a completely regular frame L and an infinite cardinal number κ, the following assertions are equivalent:

(1) L satisfies *(κ-DP)*.

(2) L is initially κ-compact.

(3) L is initially κ-almost-compact.

A final characterization of (SDP), (κ-DP) and (wDP)

Corollary

For a completely regular frame L and an infinite cardinal number κ, the following assertions are equivalent:

(1) L satisfies (wDP).

(2) L is countably-compact.

(3) L is countably-almost-compact.

(4) L is pseudo-compact.

The Pointfree Stone-Weierstrass theorem

Definition (Banaschewski)

Let L be a completely regular frame. An $\mathbb{R}$-subalgebra A of $\mathfrak{R}^*(L)$ is called *separating* if $coz[A] = \{coz(\alpha) \mid \alpha \in A\}$ ($\bigvee$-)generates L.

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Definition

A completely regular frame L is said to have the *Stone-Weierstrass Property* or(SWP) if every separating unital $\mathbb{R}$-subalgebra of $\mathfrak{R}^*(L)$ is dense in $\mathfrak{R}^*(L)$ with respect to the uniform topology.

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Pointfree Stone-Weierstrass Theorem (Banaschewski)

All compact completely regular frames satisfy (SWP).

Characterizing (SWP)?

Definition

For a frame L, we call an l-subring A of $\mathfrak{R}^*(L)$ a *K-ring* of L if

- A is complete with respect to the natural uniformity
- A contains the constant functions
- A is separating

We write $\mathbf{KRg}(L)$ for the lattice of K-rings of L, considered as a category.

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We write $\mathbf{KRg}(L)$ for the lattice of K-rings of L, considered as a category.

Theorem (Banaschewski-S.)

For a completely regular frame L, $\mathbf{KRg}(L)$ is equivalent to the category the category $\Delta(\mathbf{K} \downarrow L)$ of all compactifications of L.

Characterizing (sWP)

Corollary

For a completely regular frame L, the following assertions are equivalent:

(1) L satisfies (sWP)

(2) $\mathfrak{R}^*(L)$ is the only K-ring of L

(3) β_L is (upto isomorphisms fixing L) the only compactification of L.

Thanks for your attention!