Natural extension of median algebras

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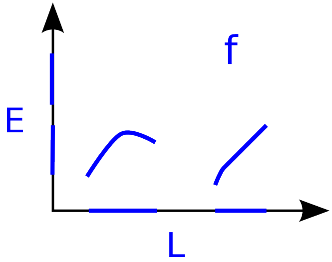
Back to the roots : canonical extension

Canonical extension $\mathbf{L}^\delta$ of a bounded DL $\mathbf{L}$ with topologies ι and δ :

- $\mathbf{L}^\delta$ is doubly algebraic.
- $\mathbf{L} \hookrightarrow \mathbf{L}^\delta$.
- $\mathbf{L}$ is dense in $\mathbf{L}^\delta_\iota$.
- $\mathbf{L}$ is dense and discrete in $\mathbf{L}^\delta_\delta$.

JÓNSSON-TARSKI (1951), GEHRKE and JÓNSSON (1994)..., GEHRKE and HARDING (2011), GEHRKE and VOSMAER (2011), DAVEY and PRIESTLEY (2011)...

A tool to extend maps in a canonical way comes with the topology δ



f^δ can be defined by order and continuity properties.

Leads to canonical extension of lattice-based algebras.

Tool used to obtain canonicity of logics. JÓNSSON (1994).

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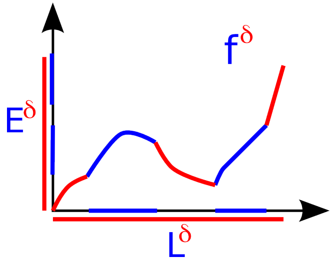

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It is possible to generalize canonical extension to non lattice-based algebras

Step 1	**Step 2**
Define the natural extension of an **algebra**	Define the natural extension of a **map**
DAVEY, GOUVEIA, HAVIAR and PRIESTLEY (2011)	A partial solution

We adopt the settings of natural dualities

A finite algebra **M**

A discrete alter-ego topological structure $\underset{\sim}{M}$

We assume that $\underset{\sim}{M}$ yields a duality for $\mathcal{A}$. We focus on objects.

Algebra	**Topology**
$\mathbf{M}$	$\underset{\sim}{M}$
$\mathcal{A} = \mathbb{ISP}(\mathbf{M})$	$\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{M})$
$\mathbf{A}$	$\mathbf{A}^* = \mathcal{A}(\mathbf{A}, \mathbf{M}) \leq_c \underset{\sim}{M}^{\mathbf{A}}$
$\underset{\sim}{X}_* = \mathcal{X}(\underset{\sim}{X}, \underset{\sim}{M}) \leq \mathbf{M}^X$	$\underset{\sim}{X}$

$$(\mathbf{A}^*)_* \simeq \mathbf{A}$$

Natural extension of an algebra can be constructed from its dual

Canonical extension	**Natural extension**
$\mathbf{L}^{\delta}$ is the algebra of **order-preserving maps** from $\mathbf{L}^{*}$ to $\underset{\sim}{2}$.	$\mathbf{A}^{\delta}$ is the algebra of **structure preserving maps** from $\mathbf{A}^{*}$ to $\underset{\sim}{M}$.

DAVEY, GOUVEIA, HAVIAR and PRIESTLEY (2011)

The variety of median algebras will perfectly illustrate the construction

Median algebras are the $(\cdot,\cdot,\cdot)$-subalgebras of the distributive lattices where

$$(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z).$$

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$\mathcal{B}$	Boolean algebras	$\mathbb{ISP}(\mathbf{2})$	$\mathbf{2} = \langle\{0,1\}, \vee, \wedge, \neg, 0, 1\rangle$
$\mathcal{D}$	Bounded DL	$\mathbb{ISP}(\mathbf{2})$	$\mathbf{2} = \langle\{0,1\}, \vee, \wedge, 0, 1\rangle)$
$\mathcal{A}$	Median algebra	$\mathbb{ISP}(\mathbf{2})$	$\mathbf{2} = \langle\{0,1\}, (\cdot,\cdot,\cdot)\rangle$

On $\{0,1\}$, operation $(\cdot,\cdot,\cdot)$ is the majority function.

There is a natural duality for median algebras

$$\underset{\sim}{2} := \langle \{0,1\}, \leq, \cdot^{\bullet}, 0, 1, \iota \rangle.$$

Algebra	**Topology**
2	$\underset{\sim}{2}$
$\mathcal{A} = \mathbb{ISP}(\mathbf{2})$ is the variety of **median algebras**	$\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{2})$ is the category of **bounded strongly complemented PRIESTLEY spaces**

Proposition (ISBELL (1980), WERNER (1981))

1. *Structure $\underset{\sim}{2}$ yields a logarithmic duality for median algebras.*
2. *Operation $\cdot^{\bullet}$ is an involutive order reversing homeomorphism such that $0^{\bullet} = 1$ and $x \leq x^{\bullet} \to x = 0$.*

We may associate orders to a median algebra

Let $a \in \mathbf{A} = \langle A, (\cdot,\cdot,\cdot)\rangle$. Define $\leq_a$ on A by

$$b \leq_a c \quad \text{if} \quad (a, b, c) = b.$$

Then $\leq_a$ is a $\wedge$-semilattice order on A with $b \wedge_a c = (a, b, c)$.

Semillatices obtained in this way are the *median semilattices*.

Proposition

In a median semilattice, ***principal*** *ideals are* ***distributive lattices****.*

GRAU (1947), BIRKHOFF and KISS (1947), SHOLANDER (1952, 1954), . . . , ISBELL (1980), BANDELT and HEDLÍKOVÁ (1983). . .

Natural extension completes everything it can complete

$\mathbf{A}^{\delta} \equiv$ the algebra of $\{\leq, 0, 1, \cdot^{\bullet}\}$-preserving maps from $\mathbf{A}^{*}$ to $\underset{\sim}{2}$.

Proposition

Let $a, b \in \mathbf{A}$

1. $\langle \mathbf{A}^{\delta}, \wedge_a \rangle$ *is a bounded complete $\wedge_a$-semilattice which is an extension of $\langle \mathbf{A}, \wedge_a \rangle$.*
2. $(b]_{\langle \mathbf{A}^{\delta}, \wedge_a \rangle}$ *is a canonical extension of $(b]_{\langle \mathbf{A}, \wedge_a \rangle}$.*

We can define $\mathbf{A}$ in $\mathbf{A}^\delta$ in a purely topological language

$$\mathcal{X}_p(\mathbf{A}^*, \underset{\sim}{2}) := \bigcup \{\mathcal{X}(\underset{\sim}{F}, \underset{\sim}{2}) \mid \underset{\sim}{F} \leq_c \mathbf{A}^*\}.$$

Consider the topology δ generated by the family Δ of the

$$O_f = \{x \in \mathbf{A}^\delta \mid x \supseteq f\}, \quad f \in \mathcal{X}_p(\mathbf{A}^*, \underset{\sim}{2}).$$

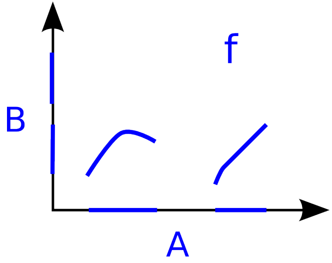
Lemma

1. *Δ is a topological basis of δ.*
2. *$\mathbf{A}$ is dense and discrete in $\mathbf{A}^\delta_\delta$.*

The lemma generalizes to any **logarithmic** dualities.

We use the topology δ to canonically extend maps to multimaps

$f : \mathbf{A} \to \mathbf{B}$



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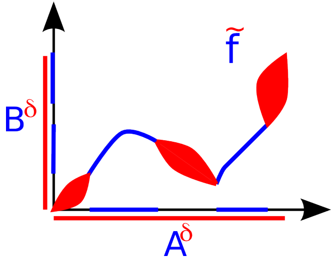

$\widetilde{f} : \mathbf{A}^{\delta} \to \Gamma(\mathbf{B}^{\delta}_{\iota})$

We define the multi-extension of $f : \mathbf{A} \to \mathbf{B}$

$$\bar{f} : \mathbf{A} \to \Gamma(\mathbf{B}^{\delta}_{\iota}) : a \mapsto \{f(a)\}.$$

$\mathbf{A}$ is dense in $\mathbf{A}^{\delta}_{\delta}$ and $\Gamma(\mathbf{B}^{\delta}_{\iota})$ is a complete lattice.

Definition

The *multi-extension* $\widetilde{f}$ *of* f is defined by

$$\widetilde{f} : \mathbf{A}^{\delta}_{\delta} \to \Gamma(\mathbf{B}^{\delta}_{\iota}) : x \mapsto \mathrm{limsup}_{\delta} \bar{f}(x),$$

In other words, for any $F \Subset \mathbf{B}^{*}$,

$$\widetilde{f}(x)\restriction_{F} = \bigcap \{ \overline{\{f(a)\restriction_{F} \mid a \in V\}} \mid V \in \delta_{x} \},$$

where the closure is computed in $\mathbf{B}^{\delta}_{\iota}$.

The multi-extension is a continuous map

We say that f is *smooth* if $\#\widetilde{f}(x) = 1$ for any $x \in \mathbf{A}^\delta$.

Let $\sigma{\downarrow}$ be the *co-Scott* topology on $\Gamma(\mathbf{B}_\iota^\delta)$.

Proposition (Generalizes to logarithmic dualities)

1. *For any $a \in \mathbf{A}$, $\widetilde{f}(a) = \{f(a)\}$.*
2. *The map $\widetilde{f} : \mathbf{A}^\delta \to \Gamma(\mathbf{B}_\iota^\delta)$ is $(\delta, \sigma{\downarrow})$.*
3. *If $f' : \mathbf{A}^\delta \to \Gamma(\mathbf{B}_\iota^\delta)$ satisfies 1 and 2 then $\widetilde{f}(x) \subseteq f'(x)$ for every $x \in \mathbf{A}^\delta$.*

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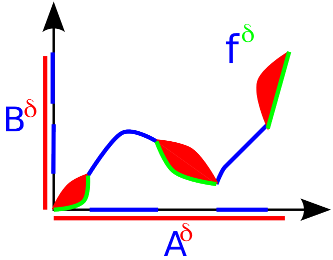
3. *If $f' : \mathbf{A}^\delta \to \Gamma(\mathbf{B}^\delta_\iota)$ satisfies 1 and 2 then $\widetilde{f}(x) \subseteq f'(x)$ for every $x \in \mathbf{A}^\delta$.*
4. *f is smooth if and only if it admits an (δ, ι)-continuous extension, namely $f^\delta : \mathbf{A}^\delta \to \mathbf{B}^\delta : x \mapsto f^\delta(x) \in \widetilde{f}(x)$.*
5. *If f is not smooth, there is no extension $f' : \mathbf{A}^\delta \to \mathbf{B}^\delta$ of f and that is (δ, ι)-continuous that satisfies $f'(x) \in \widetilde{f}(x)$.*

We can use $\leq_a$ to turn the multi-extension into an extension



Definition

Let $b \in B$. The map $f_b^\delta : \mathbf{A}^\delta \to \mathbf{B}^\delta$ is defined by

$$f_b^\delta(x) = \bigwedge_b \widetilde{f}(x).$$

f^δ is a continuous map

Proposition

1. *The map f^δ_b is $(\delta, \iota_b \uparrow)$-continuous.*
2. *If $f : A \to A$ respects $\wedge_a$ on finite subsets then f^δ_b respects $\wedge_a$ on any set.*
3. *For a median algebra, being a bounded DL is a property preserved by natural extension.*
4. *For a median algebra, being a Boolean algebra is a property preserved by natural extension.*

Among open questions/further work

- How to canonically extend maps if the duality fails to be logarithmic ?
- Use continuity properties to study preservation of equations.
- Determine the links with profinite extension.
- Do something clever with that.