Notes on orthoalgebras in categories

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Overview

We show a certain interval in the (canonical) orthoalgebra $\mathfrak{D}A$ of an object A in a category $\mathscr{K}$ arises from decompositions.

- What kind of category are we considering here?

- How can we obtain the orthoalgebra of decompositions of an object in such a category?

Categories $\mathscr{K}$

Consider a category $\mathscr{K}$ with finite products such that

- **I**. projections are <u>epimorphisms</u> and
- **II**.for any ternary product $(q_i : X_1 \times X_2 \times X_3 \longrightarrow X_i)_{i\in\{1,2,3\}}$, the following diagram is a <u>pushout</u> in $\mathscr{K}$:

$$
\begin{array}{ccc}
X_1 \times X_2 \times X_3 & \xrightarrow{(q_2,q_3)} & X_2 \times X_3 \\
{\scriptstyle (q_1,q_3)}\downarrow & & \downarrow{\scriptstyle r_{X_3}} \\
X_1 \times X_3 & \xrightarrow[p_{X_3}]{} & X_3
\end{array}
$$

where p_{X_3} and r_{X_3} are the second projections.

Decompositions

- An isomorphism $A \longrightarrow X_1 \times \cdots \times X_n$ in $\mathscr{K}$ is called an <u>n-ary decomposition</u> of A.

- For decompositions $f : A \longrightarrow X_1 \times X_2$ and $g : A \longrightarrow Y_1 \times Y_2$ of A, we say <u>f is equivalent to g</u> if there are isomorphisms $\gamma_i : X_i \longrightarrow Y_i$ $(i = 1, 2)$ such that the following diagram is commutative in $\mathscr{K}$

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & X_1 \times X_2 \\
{\scriptstyle id_A}\downarrow & & \downarrow{\scriptstyle \gamma_1 \times \gamma_2} \\
A & \xrightarrow[\quad g \quad]{} & Y_1 \times Y_2
\end{array}
$$

Notation. Given $A \in \mathscr{K}$,

$[(f_1, f_2)]$: equivalence class of $f : A \longrightarrow X_1 \times X_2$.

$\mathfrak{D}(A)$: all equivalence classes of all decompositions of A in $\mathscr{K}$.

Partial operation $\oplus$ on decompositions

For $[(f_1, f_2)]$ and $[(g_1, g_2)]$ in $\mathfrak{D}(A)$,

- $[(f_1, f_2)] \oplus [(g_1, g_2)]$ is <u>defined</u> if there is a ternary decomposition

$$(c_1, c_2, c_3) : A \longrightarrow C_1 \times C_2 \times C_3$$

of A such that

$$[(f_1, f_2)] = [(c_1, (c_2, c_3))] \text{ and } [(g_1, g_2)] = [(c_2, (c_1, c_3))].$$

In this case, define the <u>sum</u> by

$$[(f_1, f_2)] \oplus [(g_1, g_2)] = [((c_1, c_2), c_3)]$$

- Also, the equivalence classes $[(\tau_A, id_A)]$ and $[(id_A, \tau_A)]$ are distinguished elements $\mathbf{0}$ and $\mathbf{1}$ in $\mathfrak{D}(A)$, respectively, where $\tau_A : A \longrightarrow T$ is the unique map into the terminal object T.

Orthoalgebras in $\mathscr{K}$

The following is due to Harding.

- Proposition 1. The structure $(\mathfrak{D}(A), \oplus, \mathbf{0}, \mathbf{1})$ is an orthoalgebra.

An orthoalgebra is a partial algebra $(\mathcal{A}, \oplus, \mathbf{0}, \mathbf{1})$ such that for all $a, b, c \in \mathcal{A}$,

1. $a \oplus b = b \oplus a$
2. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
3. For every a in $\mathcal{A}$, there is a unique b such that $a \oplus b = \mathbf{1}$
4. If $a \oplus a$ is defined, then $a = \mathbf{0}$

Note.

$$\textbf{BAlg} \subsetneq \textbf{OML} \subsetneq \textbf{OMP} \subsetneq \textbf{OA}$$

Intervals in $\mathfrak{D}(A)$

For any decomposition

$$(h_1, h_2) : A \longrightarrow H_1 \times H_2$$

of A in $\mathscr{K}$, define the <u>interval</u> of (h_1, h_2) by

$$\mathfrak{L}_{[(h_1,h_2)]} = \{[(f_1, f_2)] \in \mathfrak{D}(A) \mid [(f_1, f_2)] \leqslant [(h_1, h_2)]\},$$

where $\leqslant$ is the induced order from the orthoalgebra $\mathfrak{D}(A)$, that is,

- $[(f_1, f_2)] \leqslant [(h_1, h_2)]$ means

$$[(f_1, f_2)] \oplus [(g_1, g_2)] = [(h_1, h_2)]$$

for some decomposition (g_1, g_2) of A in $\mathscr{K}$.

Intervals as decompositions

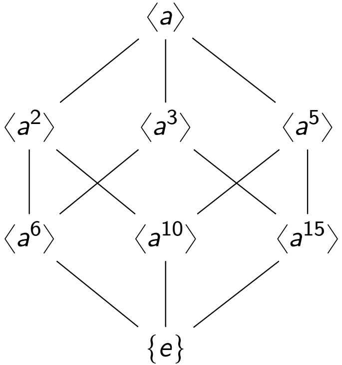
Proposition 2.(HY) For each decomposition

$$(h_1, h_2) : A \longrightarrow H_1 \times H_2$$

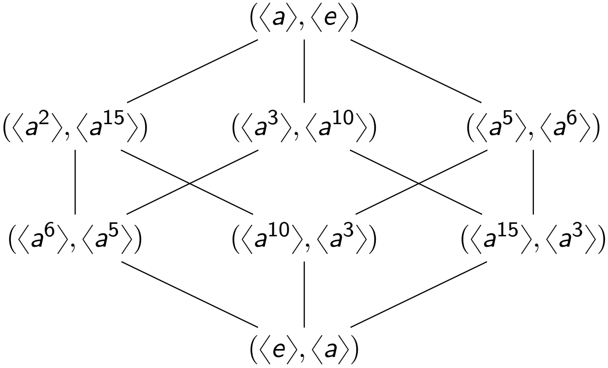
of an object A in $\mathscr{K}$, the interval $\mathfrak{L}_{[(h_1,h_2)]}$ is isomorphic to $\mathfrak{D}(H_1)$.

Example 1

- The category **Grp** of all groups and their maps satisfies all the necessary hypothesis. Consider a cyclic group $G = \langle a \rangle$ of order 30. Notice $|G| = 2 \cdot 3 \cdot 5$.



- $\mathfrak{D}(G)$ in the category **Grp**.



- The interval $\mathfrak{L}_{(\langle a^2\rangle,\langle a^{15}\rangle)}$ is a four element Boolean lattice. Also, we have the following:

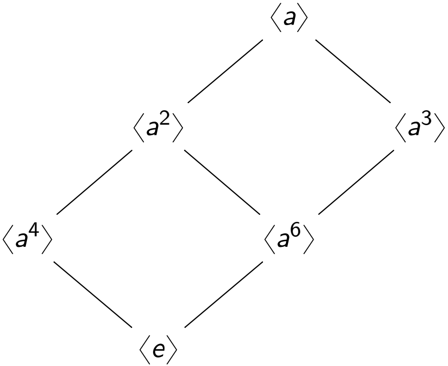
$$\mathfrak{D}(\langle a^2\rangle) \cong \{(\langle a^6\rangle,\langle a^{10}\rangle),(\langle a^{10}\rangle,\langle a^6\rangle),(\langle a^2\rangle,\langle e\rangle),(\langle e\rangle,\langle a^2\rangle)\}$$

Thus we obtain

$$\mathfrak{L}_{(\langle a^2\rangle,\langle a^{15}\rangle)} \cong \mathfrak{D}(\langle a^2\rangle)$$

Example 2

- Consider the cyclic group $G = \langle a \rangle$ with $|G| = 12 = 4 \cdot 3$.



Factor pairs : $\{(\langle a^4\rangle, \langle a^3\rangle), (\langle a^3\rangle, \langle a^4\rangle)(\langle e\rangle, \langle a\rangle), (\langle a\rangle, \langle e\rangle)\}$

(four-element Boolean lattice. Note that the poset is not isomorphic to $Sub(G)$)

$$\mathfrak{L}_{(\langle a^3\rangle, \langle a^4\rangle)} \cong \mathbf{2} \text{ and } \mathfrak{D}(\langle a^3\rangle) \cong \mathbf{2}$$

Proof (Sketch)

The essential part of the proof is to construct maps F and G

$$\mathfrak{L}_{[(h_1,h_2)]} \underset{\mathbf{G}}{\overset{\mathbf{F}}{\rightleftarrows}} \mathfrak{D}(H_1)$$

First define

$$\mathbf{G} : \mathfrak{D}(H_1) \longrightarrow \mathfrak{L}_{[(h_1,h_2)]}$$

by

$$[(m_1, m_2)] \rightsquigarrow [(m_1 h_1, (m_2 h_1, h_2))]$$

Conversely, seeking a map $\mathbf{F} : \mathfrak{L}_{[(h_1,h_2)]} \longrightarrow \mathfrak{D}(H_1)$, consider a binary decomposition $(f_1, f_2) : A \longrightarrow F_1 \times F_2$ in $\mathfrak{L}_{[(h_1,h_2)]}$.
Then there is an isomorphism $(c_1, c_2, c_3) : A \longrightarrow C_1 \times C_2 \times C_3$ in $\mathscr{K}$ such that

$$[(f_1, f_2)] = [(c_1, (c_2, c_3))] \text{ and } [(h_1, h_2)] = [((c_1, c_2), c_3)]$$

The latter implies that there is an isomorphism

$$(r_1, r_2) : H_1 \longrightarrow C_1 \times C_2$$

with $(r_1, r_2)h_1 = (c_1, c_2)$. Then define the map $\mathbf{F}$ by

$$[(f_1, f_2)] \rightsquigarrow [(r_1, r_2)]$$

It is known-that the correspondences **F** and **G** are indeed well-defined. Moreover, they are orthoalgebra homomorphisms that are inverses to each other.

Speculations

- Do we have more instances for the conditions I and II?
- Can we give some categorical conditions on morphisms so that $\mathfrak{D}(A)$ is an orthomodular poset? Moreover, can we also give some order/category-theoretic conditions on $Sub(A)$ in $\mathscr{K}$ such that

$$Sub(A) \longrightarrow \mathfrak{D}(A)$$

  is an orthomodular embedding (For example, $\mathbf{Hilb_K}$-like category)?

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Thank you