Duality for sheaves of distributive-lattice-ordered algebras over stably compact spaces

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BLAST

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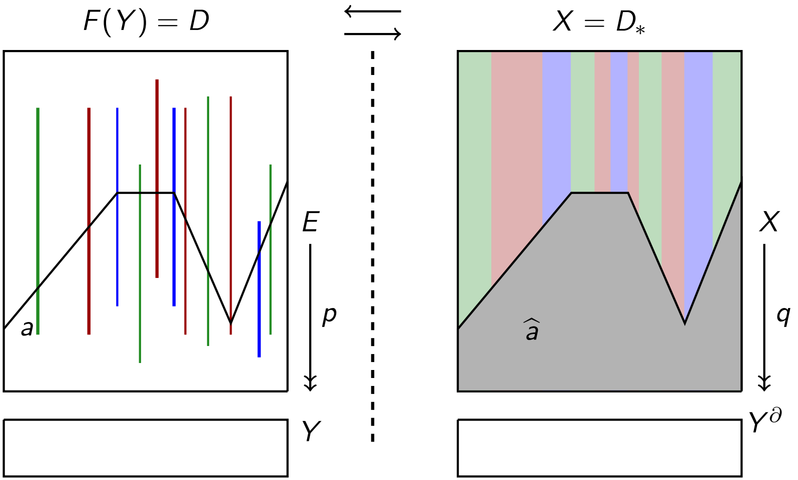
This talk in a picture



Definition of étale space

- Let $\mathcal{V}$ be a variety of abstract algebras, (Y, ρ) a topological space.
- Let $(A_y)_{y \in Y}$ be a Y-indexed family of $\mathcal{V}$-algebras.
- Let $E := \bigsqcup_{y \in Y} A_y$, with $p : E \twoheadrightarrow Y$ the natural surjection.
- Suppose τ is a topology on E such that $p : (E, \tau) \twoheadrightarrow (Y, \rho)$ is a local homeomorphism: any point has an open neighbourhood on which p has a right inverse.
- $p : (E, \tau) \twoheadrightarrow (Y, \rho)$ is called an étale space of $\mathcal{V}$-algebras.

Sheaf from an étale space

- Let $p : (E, \tau) \twoheadrightarrow (Y, \rho)$ be an étale space of $\mathcal{V}$-algebras.
- For any $U \in \rho$, write $F(U)$ for the set of local sections over U:

$$F(U) := \{s : U \to E \text{ continuous s.t. } p \circ s = \mathbf{id}_U\}.$$

- Note: $F(U)$ is a $\mathcal{V}$-algebra (being a subalgebra of $\prod_{y \in U} A_y$).
- If $U \subseteq V$, there is a natural restriction map $F(V) \to F(U)$.
- F is called the sheaf associated with p.

Definition of sheaf

- In general, a sheaf F on Y consists of the data:
  - For each open U, a $\mathcal{V}$-algebra $F(U)$ ("local sections");
  - For each open $U \subseteq V$, a $\mathcal{V}$-homomorphism $()|_U : F(V) \to F(U)$ ("restriction maps");

  such that the appropriate diagrams commute, satisfying the following patching property:
  - For any open cover $(U_i)_{i \in I}$ of an open set U, $(s_i)_{i \in I}$ a "compatible family" of local sections, i.e., $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$.
  - there exists a unique $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.
- $F(Y)$ is called the algebra of global sections of the sheaf F.

Sheaves vs. étale spaces

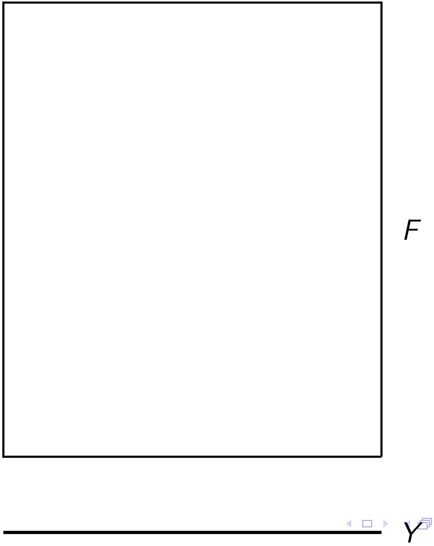
Fact

Any sheaf arises from an étale space, and vice versa.

Boolean product representation

- Let A be an abstract algebra.
- A Boolean product representation of A is a sheaf F on a Boolean space Y such that A is isomorphic to the algebra of global sections of F.
- Equivalent: a subdirect embedding $A \rightarrowtail \prod_{y \in Y} A_y$ satisfying:
  - (Open equalizers) For any $a, b \in A$, the equalizer $\|a = b\| := \{y \in Y \mid a_y = b_y\}$ is open;
  - (Patch) For K clopen in Y, $a, b \in A$, there exists $c \in A$ such that $a|_K = c|_K$ and $b|_{K^c} = c|_{K^c}$.

Boolean product, pictorially



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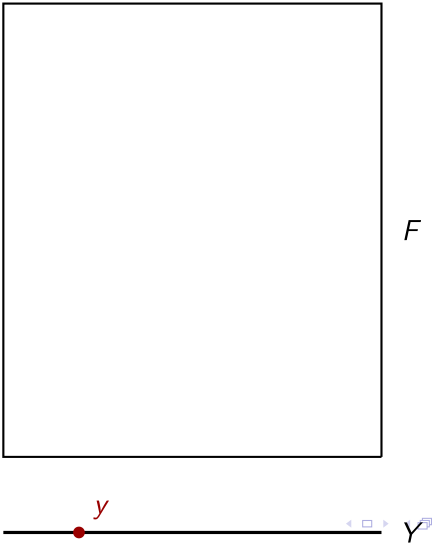

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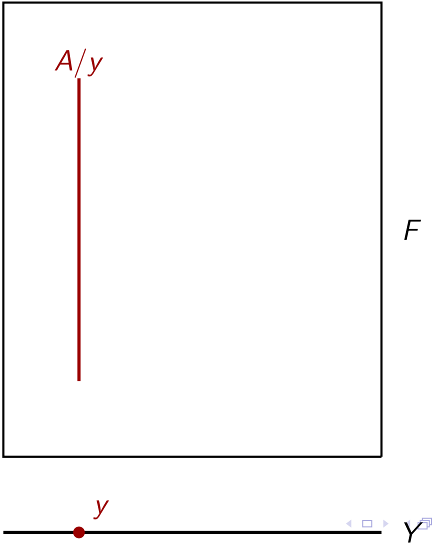

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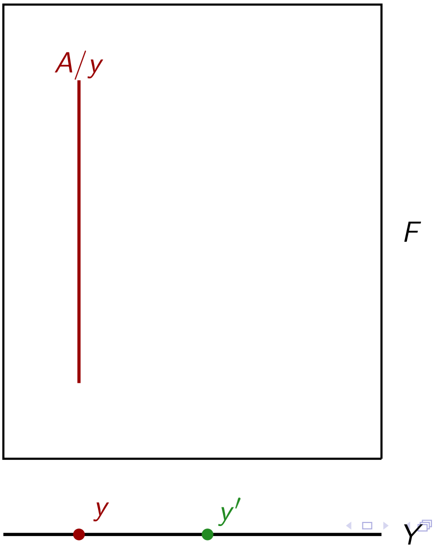

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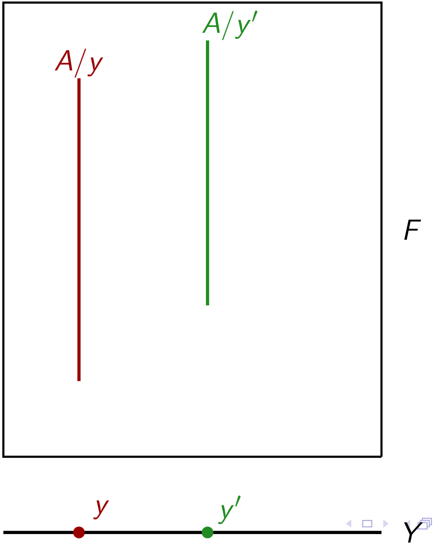

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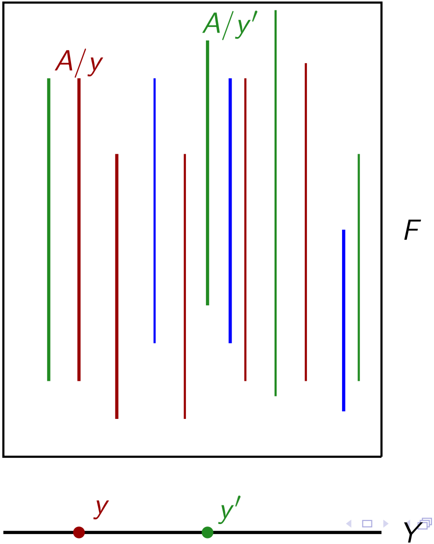

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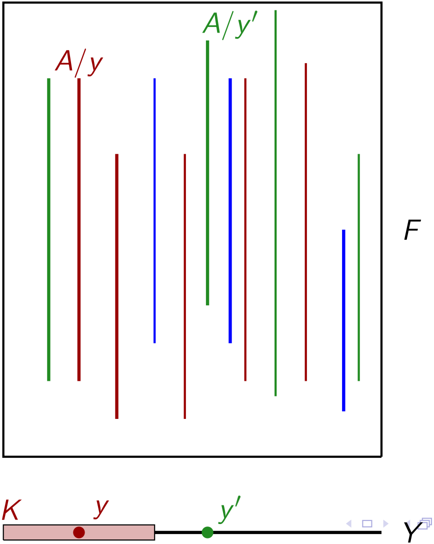

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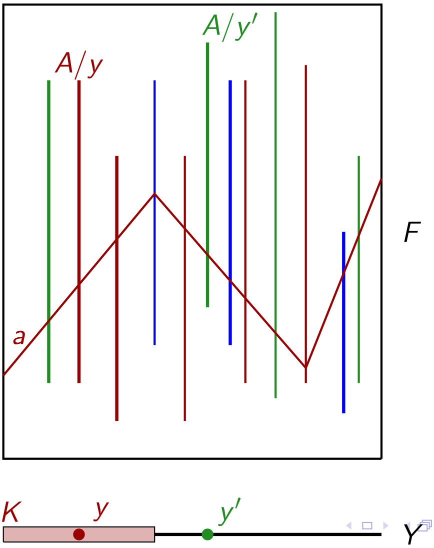

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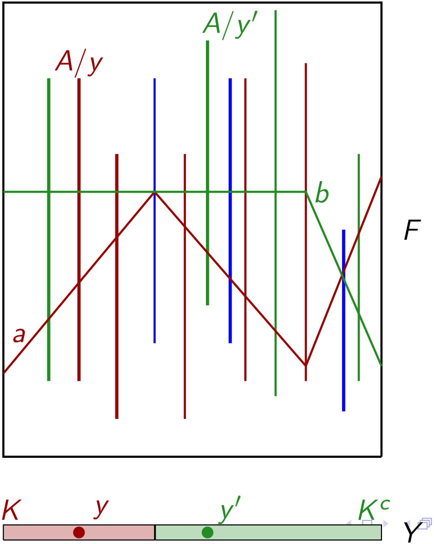

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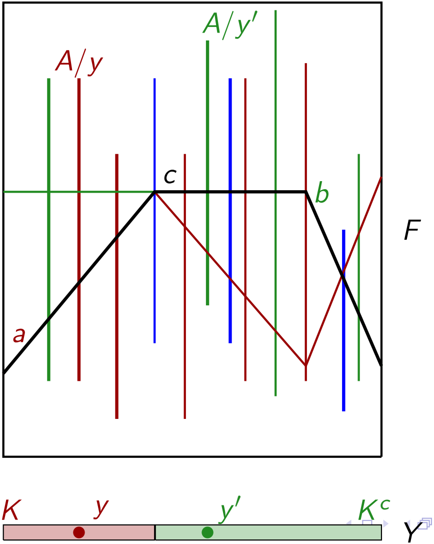


Lattices of congruences

Theorem (Comer 1971, Burris & Werner 1980)

Boolean product representations of A are in a natural one-to-one correspondence with relatively complemented distributive lattices of permuting congruences on A.

Boolean sum decompositions

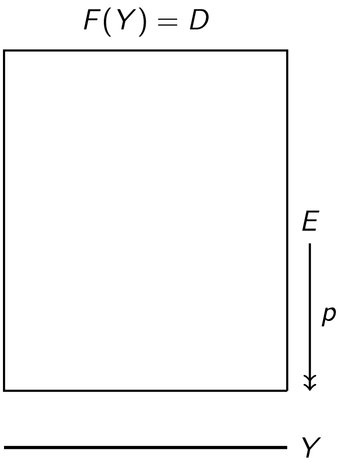
- Let D be a distributive lattice.

Theorem (Gehrke 1991)

Boolean product representations $D \rightarrowtail \prod_{y \in Y} D_y$ are in a natural one-to-one correspondence with Boolean sum decompositions of the Stone dual space X of D into the Stone dual spaces $(X_y)_{y \in Y}$ of the lattices $(D_y)_{y \in Y}$.

- Also see [Hansoul & Vrancken-Mawet 1984] for a version for the Priestley dual spaces.

Dual characterization, pictorially



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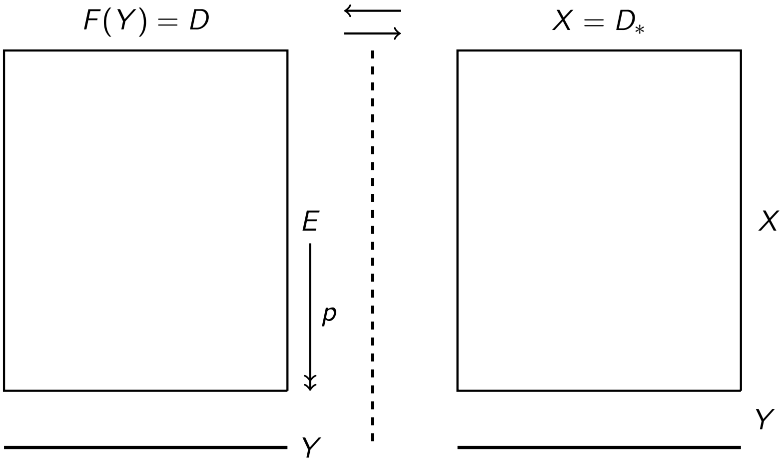

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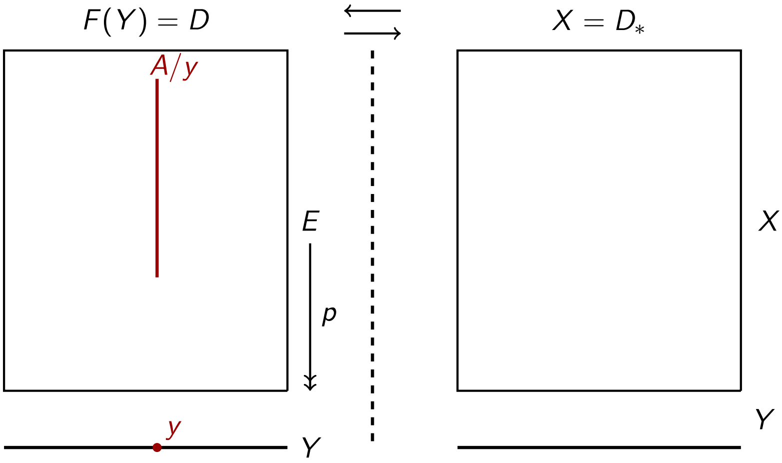

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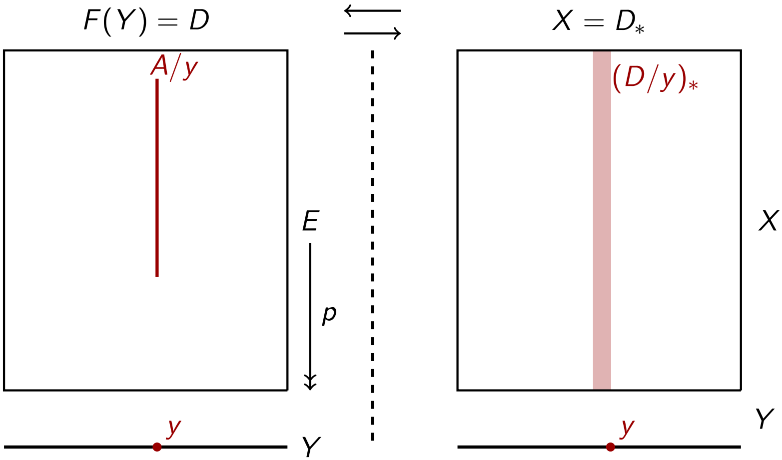

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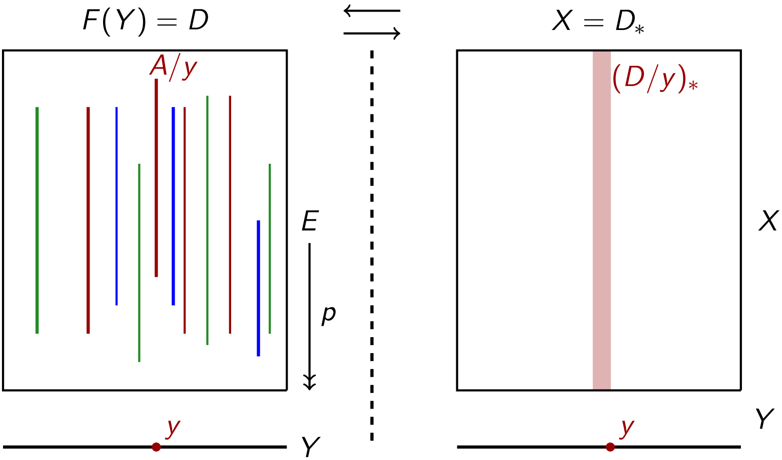

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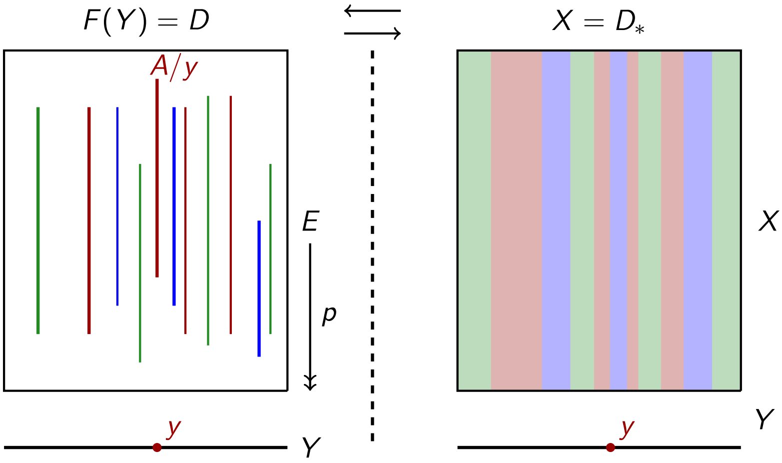

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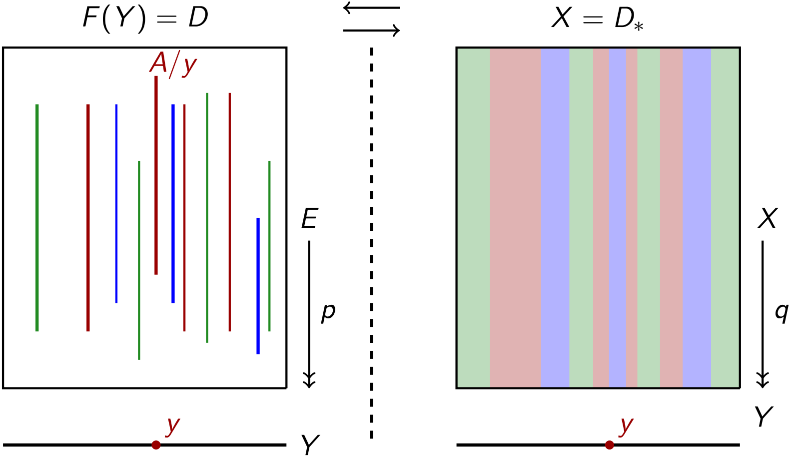

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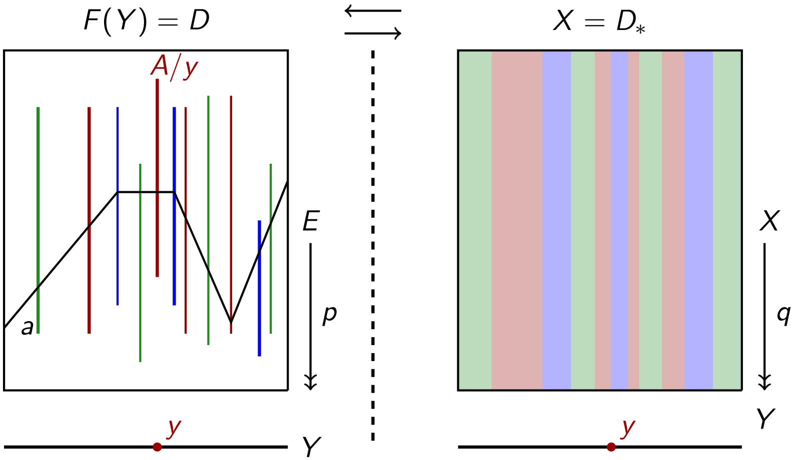

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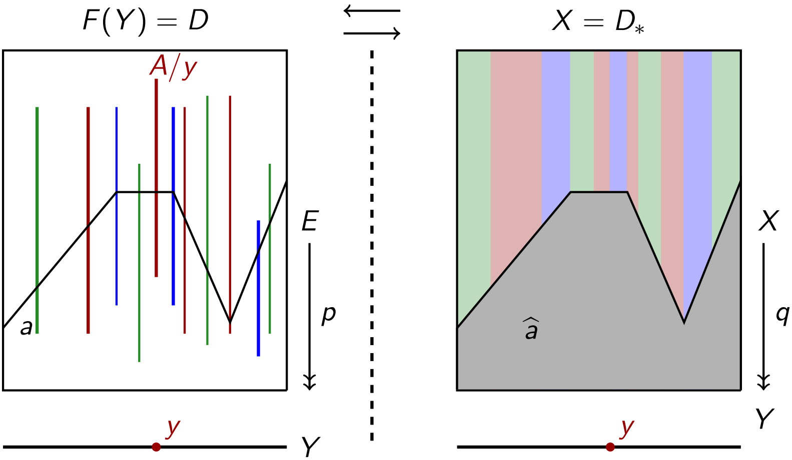
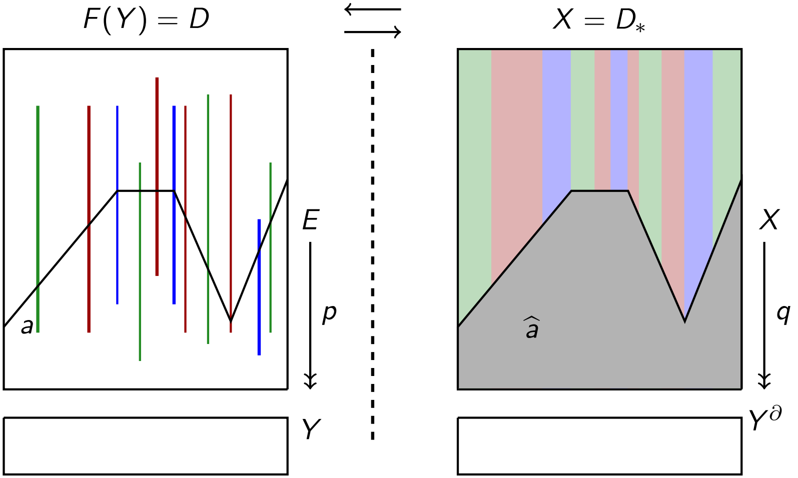


This talk in a picture



Question

- What if Y is no longer a Boolean space?

Motivation

- Many interesting sheaf representations use a base space which is spectral or compact Hausdorff.
- Stably compact spaces form a common generalization of these two classes.

Stably compact spaces

- "Generalisation of compact Hausdorff to T_0-setting"

Definition

Stably compact space =

- T_0,
- Sober,
- Locally compact,
- Intersection of compact saturated is compact.

De Groot dual and patch topology

- For any topological space (Y, ρ), define its de Groot dual

$$\rho^{\partial} := \langle U \subseteq Y \mid Y \setminus U \text{ is compact saturated in } \rho \rangle_{\text{top}}$$

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- Fact: $\leq$ is a closed subspace of $(Y \times Y, \rho^p \times \rho^p)$.

De Groot dual and patch topology

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- Let $y \leq y' \iff y' \in \overline{\{y\}}$, the specialization order of ρ.
- Fact: $\leq$ is a closed subspace of $(Y \times Y, \rho^p \times \rho^p)$.
- So $(Y, \rho^p, \leq)$ is a compact ordered space (Nachbin 1965).

Compact ordered spaces

- Conversely, given a compact ordered space $(Y, \pi, \leq)$, let $\pi^{\downarrow}$ the topology of open down-sets.
- Then $(Y, \pi^{\downarrow})$ is a stably compact space, and $(\pi^{\downarrow})^{\partial} = \pi^{\uparrow}$.

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- Conversely, given a compact ordered space $(Y, \pi, \leq)$, let $\pi^{\downarrow}$ the topology of open down-sets.
- Then $(Y, \pi^{\downarrow})$ is a stably compact space, and $(\pi^{\downarrow})^{\partial} = \pi^{\uparrow}$.

Fact

The categories of stably compact spaces and compact ordered spaces are isomorphic.

Representing stably compact spaces

Example (Open basis presentation)

- X stably compact space
- D lattice-basis of open sets for X
- Define $U \, R \, V$ iff there exists compact saturated $K \subseteq X$ such that $U \subseteq K \subseteq V$
- Fact: X can be recovered as the space of "round prime ideals" of R.

Spectral spaces with retractions

Fact (Johnstone, 1982)

A topological space X is stably compact iff

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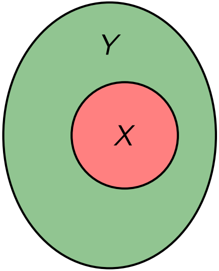
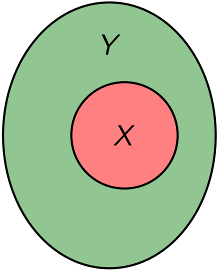
A topological space X is stably compact iff there exists a spectral space Y

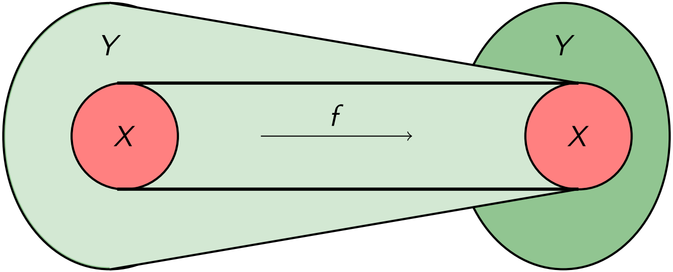




Spectral spaces with retractions

Fact (Johnstone, 1982)

A topological space X is stably compact iff there exists a spectral space Y and a continuous retraction of Y onto X.



Duality for spectral spaces with continuous maps

- Fact: $\mathbf{DL}_j \cong^{op} \mathbf{SpecSp}_c$
- Here, $\mathbf{SpecSp}_c$: spectral spaces with continuous maps,
- and $\mathbf{DL}_j$: distributive lattices with j-morphisms:

Definition

A relation $H \subseteq D \times E$ between distributive lattices D and E is called a j-morphism iff:

- $\geq \circ H \circ \geq = H$
- $a \, H \bigvee B \iff \forall b \in B \; aHb$
- $\bigwedge A \, H \, b \iff \forall a \in A \; aHb$
- If $\bigvee A \, H \, b$ then $\exists B \subseteq_\omega H[A]$ such that $b \leq \bigvee B$.

Duality for stably compact spaces

Definition

A join-strong proximity lattice is a pair (D, R) where D is a distributive lattice, $R^{-1} : D \to D$ is a j-morphism, and $R \circ R = R$.

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Proof.

- Stably compact spaces are retracts of spectral spaces.

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Proof.

- Stably compact spaces are retracts of spectral spaces.
- Therefore, duals of stably compact spaces are retracts of distributive lattices in the category $\mathbf{DL}_j$.

□

The case of MV-algebras

Theorem (Gehrke, Marra, vG 2012)

The Priestley dual space X of the distributive lattice underlying an MV-algebra A decomposes as a stably compact sum over the base space Y of prime MV ideals of A.

Stably compact sum decompositions

Definition

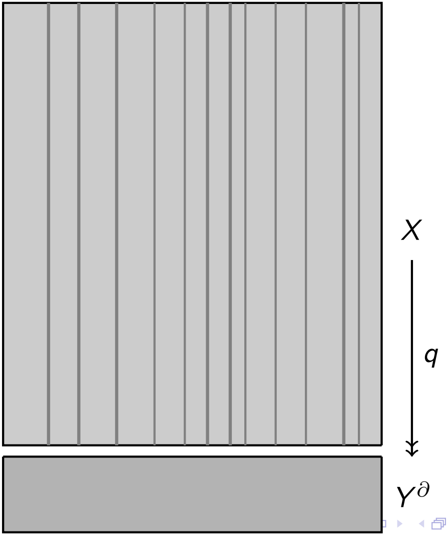
A stably compact sum decomposition of a Priestley space X is a continuous surjection $q : X \twoheadrightarrow Y^\partial$, with Y stably compact, satisfying the following dual patching property:

(P) Let $(U_i)_{i=1}^n$ be any finite cover of Y by ρ^∂-open sets, and let $(\widehat{a}_i)_{i=1}^n$ be any finite collection of clopen downsets of X such that

$$\widehat{a}_i \cap q^{-1}(U_i \cap U_j) = \widehat{a}_j \cap q^{-1}(U_i \cap U_j)$$

holds for any $i, j \in \{1, \ldots, n\}$. Then the set $\bigcup_{i=1}^n (\widehat{a}_i \cap q^{-1}(U_i))$ is a clopen downset in X.

Property (P), pictorially



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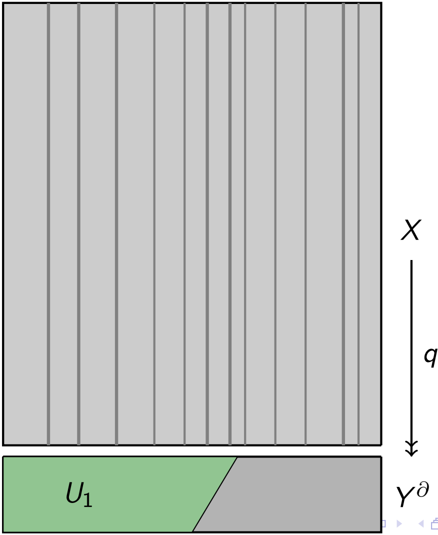

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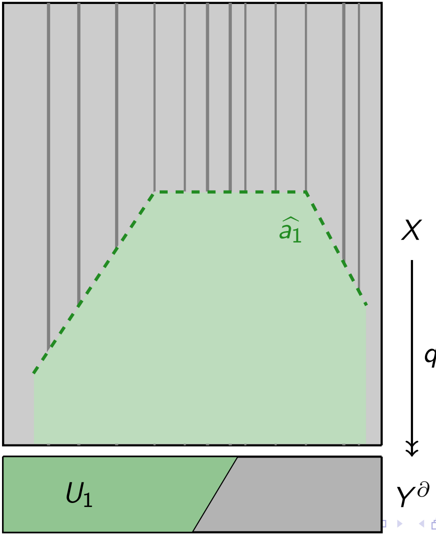

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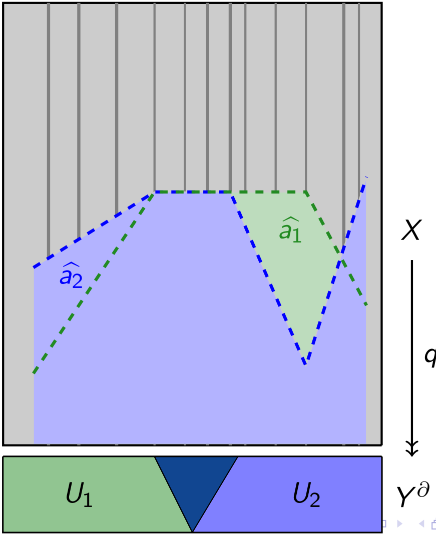

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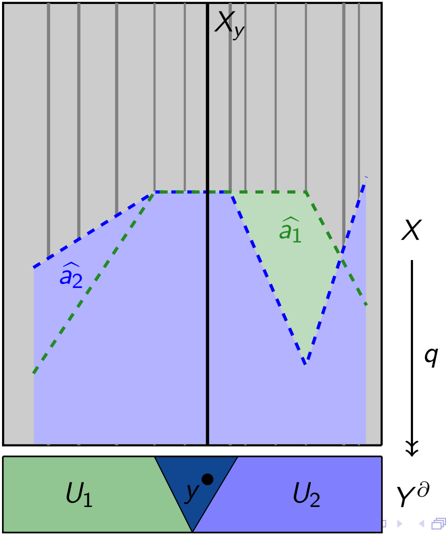

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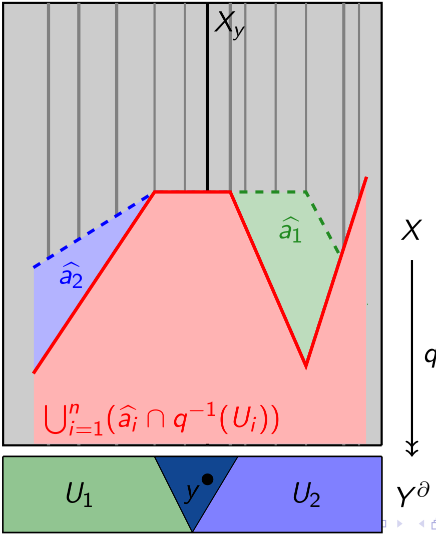

Spectral sum yields sheaf

Theorem (Gehrke, Marra, vG 2012)

If X is the Priestley space of a distributive lattice A, then any stably compact sum decomposition $q : X \twoheadrightarrow Y^\partial$ yields a sheaf representation of A over Y.

Example

For an MV-algebra A, there are two natural stably compact sum decompositions of the dual space X, each of which yields a sheaf representation of A: one over its prime, the other over its maximal spectrum.

Fitted sheaves

- Question: which sheaves can be captured by such a decomposition?

Fitted sheaves

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- Let B a basis for the base space Y.
- Call a sheaf F fitted for B if, for each $U \in B$, the restriction map $F(Y) \to F(U)$ is surjective.
- ("Fitted for $\mathcal{O}(Y)$" = "flabby" or "flasque"...)

Lattices of congruences, revisited

- Let F be a sheaf of distributive lattices on a topological space Y which is fitted for a lattice basis B for Y with $A := F(Y)$.
- For $U \in B$, define $\theta_F(U) := \ker(F(Y) \twoheadrightarrow F(U))$.

Proposition

The function $\theta_F : B^{\mathrm{op}} \to \mathrm{Con}_{\mathrm{DL}}(A)$ is a lattice homomorphism, and any two congruences in the image of θ_F permute.

Sheaf yields decomposition map

- Given a sheaf F fitted for B, lift this lattice homomorphism $\theta_F : B^{\text{op}} \to \text{Con}_{\text{DL}}(A)$ to $\overline{\theta_F} : \mathcal{O}(Y^\partial) \to \text{Con}_{\text{DL}}(A)$.
- Note that $\text{Con}_{\text{DL}}(A) \cong \mathcal{O}(X)$, where X is the Priestley dual space of the distributive lattice A.
- By pointless duality, we obtain a continuous map $q : X \to Y^\partial$.

Lifting to frames

Lemma (Lifting)

Suppose that B is a lattice basis for the open sets of a stably compact space Y and that $h : B^{\mathrm{op}} \to F$ is a lattice homomorphism from B^{op} into a frame F. Then the function $\overline{h} : \mathcal{O}(Y^{\partial}) \to F$ defined by

$$\overline{h}(W) := \bigvee \{h(U) \mid U \in B,\ U^c \subseteq W\}$$

is a frame homomorphism.

- Proof based on strong proximity lattice of (O, K)-pairs by Jung & Sünderhauf (1996).

Proof of lifting lemma

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- So, if $U \in B$ and $U^c \subseteq \bigcup_{i\in I} W_i$, by compactness pick finite cover $\mathcal{F} \subseteq \{V \in \mathcal{O}(Y^\partial) \mid \exists i \in I, U \in B : V \subseteq U^c \subseteq W_i\}$.

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- For each $V \in \mathcal{F}$, pick $U_V \in B$, $i_V \in I$, with $V \subseteq (U_V)^c \subseteq W_{i_V}$ and $U^c \subseteq \bigcup \mathcal{F}$.

Proof of lifting lemma

- To show: $\overline{h}(W) := \bigvee\{h(U) \mid U \in B,\ U^c \subseteq W\}$ preserves $\bigvee$.
- Enough: $\overline{h}(\bigcup_{i \in I} W_i) \leq \bigvee_{i \in I} \overline{h}(W_i)$.
- From the fact that B is a basis, deduce that $W_i = \bigcup\{V \in \mathcal{O}(Y^\partial) \mid \exists U \in B : V \subseteq U^c \subseteq W_i\}$.
- So, if $U \in B$ and $U^c \subseteq \bigcup_{i \in I} W_i$, by compactness pick finite cover $\mathcal{F} \subseteq \{V \in \mathcal{O}(Y^\partial) \mid \exists i \in I, U \in B : V \subseteq U^c \subseteq W_i\}$.
- For each $V \in \mathcal{F}$, pick $U_V \in B$, $i_V \in I$, with $V \subseteq (U_V)^c \subseteq W_{i_V}$ and $U^c \subseteq \bigcup \mathcal{F}$.
- Then $h(U) \leq h(\bigcap_{V \in \mathcal{F}} U_V) \leq \bigvee_{V \in \mathcal{F}} h(U_V) \leq \bigvee_{i \in I} \overline{h}(W_i)$.

The decomposition map

- Let $\overline{\theta_F} : \mathcal{O}(Y^\partial) \to \mathrm{Con}_{\mathbf{DL}}(A)$ be the frame homomorphism associated to a sheaf F.

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- This shows that an analogue of the function k from MV-algebras[1] is available in the context of any fitted sheaf representation!

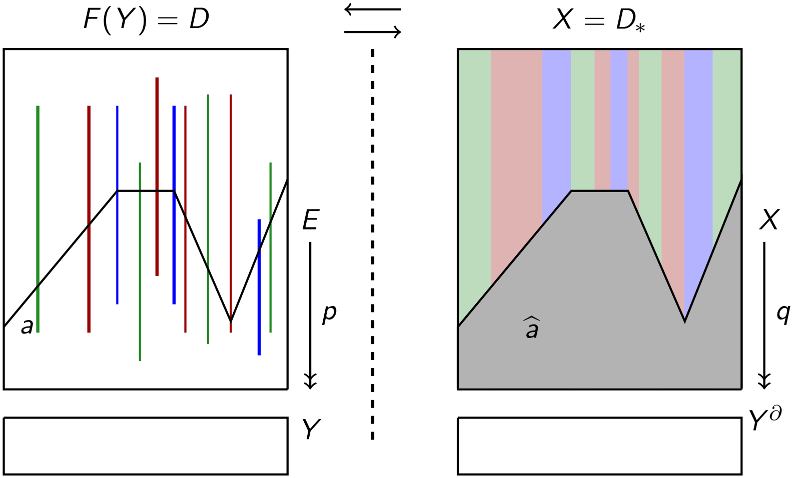
[1]See Mai Gehrke's talk yesterday afternoon.

Main theorem

Theorem (Gehrke, vG 2013)

A fitted sheaf representation of a distributive lattice A over a stably compact space Y yields a stably compact sum decomposition of the Priestley dual space X of A over Y^∂.

This talk in a picture



Analogy with Boolean case

Sheaf over Boolean space	Sheaf over stably compact space
Rel. comp. distributive lattice of congruences	Strong proximity lattice of congruences
Boolean sum decomposition	Stably compact sum decomposition

Further work

- To retrieve the topology of the dual space from the topologies on the subspaces and on the base space;
- To apply these results to more general and to other classes of lattice-ordered algebras.

Duality for sheaves of distributive-lattice-ordered algebras over stably compact spaces

Sam van Gool
(joint work with Mai Gehrke)

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6 August 2013

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Chapman University, Orange, CA