Computing Spectra via Dualities in the MTL hierarchy

Diego Valota
Department of Computer Science
University of Milan
valota@di.unimi.it

11th ANNUAL CECAT WORKSHOP IN POINTFREE MATHEMATICS

- Spectra Problems;
- Spectral Duality for Finite Gödel Algebras;
  - **Free** Spectrum of Gödel Algebras (D'Antona and Marra. 2006), (Horn,1969);
  - **Fine** Spectrum of Gödel Algebras;
- Generalizations
  - Hierarchy of Monoidal T-norms based Logics;
  - (Free) Spectra of (Subvarieties) of WNM Algebras.

Let $\mathcal{C}$ be a class of structures.

- **spectrum** of $\mathcal{C}$: $Spec(\mathcal{C}) = \{k \mid k = |C|, C \in \mathcal{C}\}$, that is the set of cardinalities of structures occurring in $\mathcal{C}$;
- **fine spectrum** of $\mathcal{C}$: $Fine_{\mathcal{C}}(k)$, that is the function counting non-isomorphic k-element structures in $\mathcal{C}$,

Let $\mathcal{C}$ be a class of structures.

- **spectrum** of $\mathcal{C}$: $Spec(\mathcal{C}) = \{k \mid k = |C|, C \in \mathcal{C}\}$, that is the set of cardinalities of structures occurring in $\mathcal{C}$;
- **fine spectrum** of $\mathcal{C}$: $Fine_{\mathcal{C}}(k)$, that is the function counting non-isomorphic k-element structures in $\mathcal{C}$,

when $\mathcal{C}$ is a variety of algebras we can also define:

- **free spectrum** of $\mathcal{C}$: $Free_{\mathcal{C}}(k) = |\mathbf{F}_{\mathcal{C}}(k)|$, that is the function computing the sizes of the free k-generated algebra $\mathbf{F}_{\mathcal{C}}(k)$ in $\mathcal{C}$.

Let $\mathcal{C}$ be a class of structures.

- **spectrum** of $\mathcal{C}$: $Spec(\mathcal{C}) = \{k \mid k = |C|, C \in \mathcal{C}\}$, that is the set of cardinalities of structures occurring in $\mathcal{C}$;
- **fine spectrum** of $\mathcal{C}$: $Fine_{\mathcal{C}}(k)$, that is the function counting non-isomorphic k-element structures in $\mathcal{C}$,

when $\mathcal{C}$ is a variety of algebras we can also define:

- **free spectrum** of $\mathcal{C}$: $Free_{\mathcal{C}}(k) = |\mathbf{F}_{\mathcal{C}}(k)|$, that is the function computing the sizes of the free k-generated algebra $\mathbf{F}_{\mathcal{C}}(k)$ in $\mathcal{C}$.

Dedekind's Problem:

to find the number $M(n)$ of **monotone Boolean functions** with n variables, that are functions obtained using only conjunctions and disjunctions.

Let $\mathcal{C}$ be a class of structures.

- **spectrum** of $\mathcal{C}$: $Spec(\mathcal{C}) = \{k \mid k = |C|, C \in \mathcal{C}\}$, that is the set of cardinalities of structures occurring in $\mathcal{C}$;
- **fine spectrum** of $\mathcal{C}$: $Fine_{\mathcal{C}}(k)$, that is the function counting non-isomorphic k-element structures in $\mathcal{C}$,

when $\mathcal{C}$ is a variety of algebras we can also define:

- **free spectrum** of $\mathcal{C}$: $Free_{\mathcal{C}}(k) = |\mathbf{F}_{\mathcal{C}}(k)|$, that is the function computing the sizes of the free k-generated algebra $\mathbf{F}_{\mathcal{C}}(k)$ in $\mathcal{C}$.

Dedekind's Problem:

to find the number $M(n)$ of **monotone Boolean functions** with n variables, that are functions obtained using only conjunctions and disjunctions.

Let L_n be the free distributive lattice on n generators. The lattice of monotone Boolean functions is isomorphic to L_n.
Hence, $M(n) = |L_n|$.

Let $\mathcal{C}$ be a class of structures.

- **spectrum** of $\mathcal{C}$: $Spec(\mathcal{C}) = \{k \mid k = |C|, C \in \mathcal{C}\}$, that is the set of cardinalities of structures occurring in $\mathcal{C}$;
- **fine spectrum** of $\mathcal{C}$: $Fine_{\mathcal{C}}(k)$, that is the function counting non-isomorphic k-element structures in $\mathcal{C}$,

when $\mathcal{C}$ is a variety of algebras we can also define:

- **free spectrum** of $\mathcal{C}$: $Free_{\mathcal{C}}(k) = |\mathbf{F}_{\mathcal{C}}(k)|$, that is the function computing the sizes of the free k-generated algebra $\mathbf{F}_{\mathcal{C}}(k)$ in $\mathcal{C}$.

Dedekind's Problem:

n	$M(n)$
0	2
1	3
2	6
3	20
4	168
5	7581
6	7828354
7	2414682040998
8	56130437228687557907788

to find the number $M(n)$ of **monotone Boolean functions** with n variables, that are functions obtained using only conjunctions and disjunctions.

Let L_n be the free distributive lattice on n generators. The lattice of monotone Boolean functions is isomorphic to L_n.
Hence, $M(n) = |L_n|$.

Gödel algebras are Heyting algebras (=Tarski-Lindenbaum algebras of intuitionistic propositional calculus) satisfying the prelinearity equation:

$$(x \to y) \vee (y \to x) = \top$$

Gödel algebras are Heyting algebras (=Tarski-Lindenbaum algebras of intuitionistic propositional calculus) satisfying the prelinearity equation:

$$(x \to y) \vee (y \to x) = \top$$

A *commutative integral bounded residuated lattice* is an algebra
$\mathbf{A} = (A, \wedge, \vee, \odot, \to, \bot, \top)$ of type $(2, 2, 2, 2, 0, 0)$ such that
$(A, \wedge, \vee, \bot, \top)$ is a bounded lattice,
$(A, \odot, \top)$ is a commutative monoid,
and the *residuation* equivalence, $x \odot y \leq z$ if and only if $x \leq y \to z$, holds.

An **MTL algebra** $\mathbf{A} = (A, \wedge, \vee, \odot, \to, \bot, \top)$ is a commutative integral bounded residuated lattice satisfying the **prelinearity** equation,

$$(x \to y) \vee (y \to x) = \top$$

.

A **Gödel Algebra** $\mathbf{A} = (A, \wedge, \vee, \to, \bot, \top)$ is an **idempotent** MTL Algebra.

Gödel logic can be semantically defined as a many-valued logic.
Let $\textsc{Form}$ be the set of formulas over propositional variables $x_1, x_2, , \ldots$ in the language $\vee, \wedge, \rightarrow, \neg, \bot$.

An assignment is a function $\mu : \textsc{Form} \rightarrow [0,1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha, \beta \in \textsc{Form}$,

$$\begin{aligned}
\mu(\alpha \wedge \beta) &= \mathit{min}\{\mu(\alpha), \mu(\beta)\},\\
\mu(\alpha \vee \beta) &= \mathit{max}\{\mu(\alpha), \mu(\beta)\},\\
\mu(\alpha \rightarrow \beta) &= \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases}\\
\mu(\neg\alpha) &= \mu(\alpha \rightarrow \bot),\\
\mu(\bot) &= 0,\\
\mu(\top) &= 1.
\end{aligned}$$

A tautology is a formula α such that $\mu(\alpha) = 1$ for every assignment μ (denoted $\vDash \alpha$).

We write $\vdash \alpha$ to mean that α is derivable from the axioms of Gödel logic using *modus ponens* as the only deduction rule.
Gödel logic is complete with respect to the many-valued semantics defined above: in symbols, $\vdash \alpha$ if and only if $\vDash \alpha$.

Stone's Duality. Every Boolean algebra is isomorphic to the Boolean algebra of all clopen sets in a compact totally disconnected Hausdorff topological space.

Representation for Finite Boolean Algebras. Every finite Boolean algebra is isomorphic to the Boolean algebra of all subsets of a finite set.

Stone's Duality. Every Boolean algebra is isomorphic to the Boolean algebra of all clopen sets in a compact totally disconnected Hausdorff topological space.

Representation for Finite Boolean Algebras. Every finite Boolean algebra is isomorphic to the Boolean algebra of all subsets of a finite set.

Priestley's Duality. The category of bounded distributive lattices and bounded lattices homomorphisms, is dually equivalent to the category of Priestley spaces and continuous order-preserving maps.

Birkhoff's Duality. The category of finite distributive lattices and complete lattice homomorphisms, is dually equivalent the category of finite posets and open maps.

D.H.J. de Jongh and A.S. Troelstra.
On the connection of partially ordered sets with some pseudo-boolean algebras.
Indagationes Mathematicae (Proceedings), 69:317 – 329, 1966.

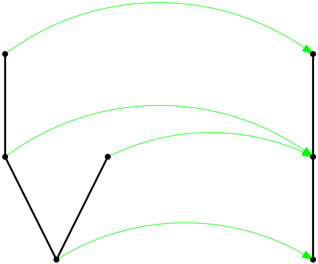
Esakia's Duality. Every Heyting algebra is isomorphic to the Heyting algebra of all clopen sets in a compact totally order-disconnected space (Priestley space) such that the downset of each clopen subset is also clopen.

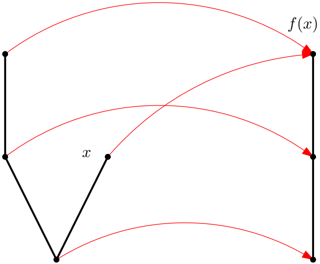
Representation for Finite Gödel Algebras. Every finite Gödel algebra is isomorphic to the Gödel algebra of all *subforests* of a *finite forest*.

A **forest** is a finite poset F such that for every $x \in F$, $\downarrow x$ is a chain. A **tree** is a forest with a bottom element.

An order preserving map $f : F \to F'$ is **open** when, for every $x \in F$

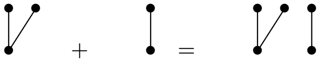
$$f(\downarrow x) = \downarrow f(x).$$



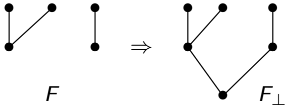


Product of Trees

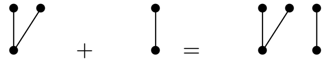
The coproduct of two forests $F + G$ is the disjoint union of F and G.



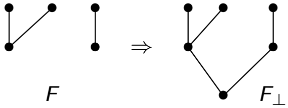
$F_\perp$ is the tree obtained appending a fresh minimum at the finite forest F.



The coproduct of two forests $F + G$ is the disjoint union of F and G.



$F_\perp$ is the tree obtained appending a fresh minimum at the finite forest F.

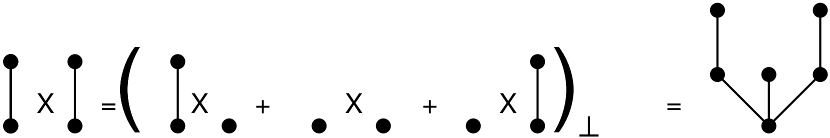


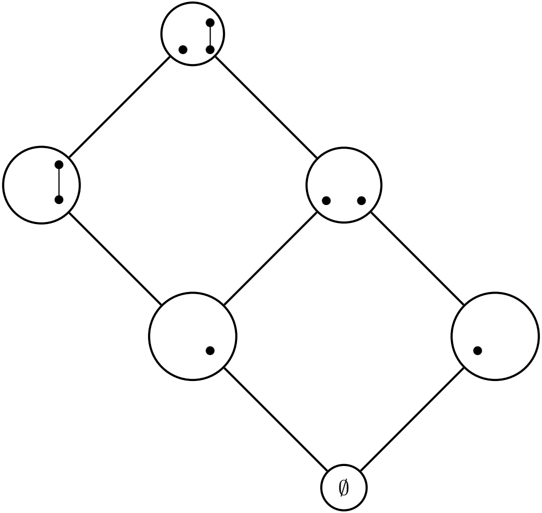
Let F,F' and G be forests in F.
We define the product of forests by the following rules,

(P1) $F \times F' \cong F'$ when $|F| = 1$;

(P2) $G \times (F + F') \cong (G \times F) + (G \times F')$;

(P3) $F_\perp \times F'_\perp \cong ((F_\perp \times F') + (F \times F') + (F \times F'_\perp))_\perp$.





Let **Sub**(F) be the collection of all the subforests of F.
Then,
$\langle \mathbf{Sub}(F), \cap, \cup, \rightarrow, \emptyset, F \rangle$ is a (Gödel) algebra of subforests where:

$$H \rightarrow K = F \setminus \uparrow (H \setminus K),$$

for all $H, K \in \mathbf{Sub}(F)$.

A nonempty subset F of A is called an *upper-set* when for all $x, y \in A$, if $x \leq y$ and $x \in F$, then $y \in F$. If $x \odot y \in F$ for all $x, y \in F$, then F is a **filter** of **A**. We call $\bigwedge_{x \in F} x$ the *generator* of the filter F. A filter F of A is **prime** if $F \neq A$ and for all $x, y \in A$, either $x \to y \in F$ or $y \to x \in F$.

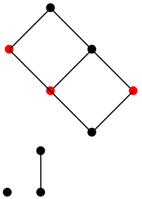
We call the poset **Prime(A)** of prime filters of finite Gödel algebra **A** ordered by reverse inclusion, the **prime spectrum** of **A**.

A nonempty subset F of A is called an *upper-set* when for all $x, y \in A$, if $x \leq y$ and $x \in F$, then $y \in F$. If $x \odot y \in F$ for all $x, y \in F$, then F is a **filter** of $\mathbf{A}$. We call $\bigwedge_{x \in F} x$ the *generator* of the filter F. A filter F of A is **prime** if $F \neq A$ and for all $x, y \in A$, either $x \to y \in F$ or $y \to x \in F$.

We call the poset **Prime**(**A**) of prime filters of finite Gödel algebra **A** ordered by reverse inclusion, the **prime spectrum** of **A**.

Proposition (Horn, 1969)

Let **A** be a finite Gödel algebra, then **Prime**(**A**) is a forest.

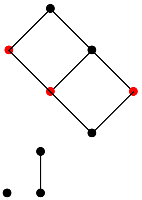


A nonempty subset F of A is called an *upper-set* when for all $x, y \in A$, if $x \leq y$ and $x \in F$, then $y \in F$. If $x \odot y \in F$ for all $x, y \in F$, then F is a **filter** of **A**. We call $\bigwedge_{x \in F} x$ the *generator* of the filter F. A filter F of A is **prime** if $F \neq A$ and for all $x, y \in A$, either $x \to y \in F$ or $y \to x \in F$.

We call the poset **Prime**(**A**) of prime filters of finite Gödel algebra **A** ordered by reverse inclusion, the **prime spectrum** of **A**.

Proposition (Horn, 1969)

Let **A** be a finite Gödel algebra, then **Prime**(**A**) is a forest.



For every morphism $h\colon \mathbf{A} \to \mathbf{B}$ of Gödel algebras, $\mathbf{Prime}(h)\colon \mathbf{Prime}(\mathbf{B}) \to \mathbf{Prime}(\mathbf{A})$ is the open map sending each prime filter F in $\mathbf{Prime}(B)$ to the prime filter in $\mathrm{Spec}(A)$ defined as follows:

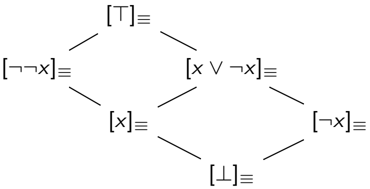
$$(\mathbf{Prime}(h))(F) = \{a \in A \mid h(a) \in F\}.$$

As usual, $\varphi, \psi \in \mathrm{FORM}_n$ are called **logically equivalent** $\varphi \equiv \psi$, if both $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$ hold.

The quotient set $\mathrm{FORM}_n/\equiv$ endowed with operations $\wedge, \vee, \to, \top, \bot$ induced from the corresponding logical connectives becomes a Gödel algebra with top and bottom element $\top$ and $\bot$, respectively.

The specific Gödel algebra $\mathcal{G}_n = \mathrm{FORM}_n/\equiv$ is, by construction, the **Lindenbaum algebra** of Gödel logic over the language $\{x_1, \ldots, x_n\}$.

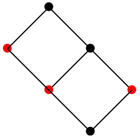
The free 1-generated Gödel algebra $\mathcal{G}_1$:

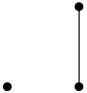


Lindenbaum algebras are isomorphic to free algebras, and then $\mathcal{G}_n$ is the free n-generated Gödel algebra.
Since the variety of Gödel algebras is locally finite, every finite Gödel algebra can be obtained as a quotient of a free n-generated Gödel algebra.

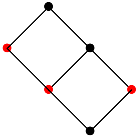
Free Spectrum

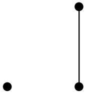
The free Gödel Algebra on one generator $\mathcal{G}_1$ and its prime spectrum.





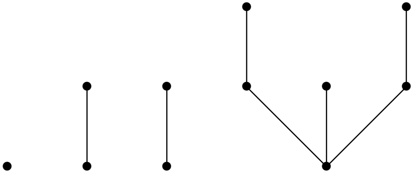
The free Gödel Algebra on one generator $\mathcal{G}_1$ and its prime spectrum.





In any variety, the free n-generated algebra $\mathcal{G}_n$ is the coproduct of n copies of the free 1-generated algebra.
Dually, we can describe the prime spectrum of $\mathcal{G}_n$ with

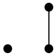
$$\mathbf{Prime}(\mathcal{G}_n) = \prod^{n} \mathbf{Prime}(\mathcal{G}_1).$$



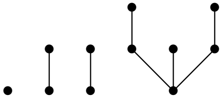
$$\mathbf{Prime}(\mathcal{G}_2) = \mathbf{Prime}(\mathcal{G}_1) \times \mathbf{Prime}(\mathcal{G}_1).$$

Free Spectrum for Finite Gödel Algebras (D'Antona and Marra. 2006), (Horn. 1969).

$$H_1 = \{\bullet\},\ \mathbf{Prime}(\mathcal{G}_1) = H_1 + (H_1)_\perp,$$



Free Spectrum for Finite Gödel Algebras (D'Antona and Marra. 2006), (Horn. 1969).

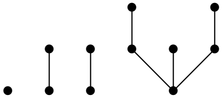


$H_1 = \{\bullet\}$, $\textbf{Prime}(\mathcal{G}_1) = H_1 + (H_1)_\perp$,

$$\textbf{Prime}(\mathcal{G}_2) = H_1 + 2 \cdot H_2 + (H_1 + 2 \cdot H_2)_\perp$$

,

Free Spectrum for Finite Gödel Algebras (D'Antona and Marra. 2006), (Horn. 1969).



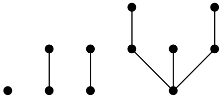
$H_1 = \{\bullet\}$, $\mathbf{Prime}(\mathcal{G}_1) = H_1 + (H_1)_\perp$,

$$\mathbf{Prime}(\mathcal{G}_2) = H_1 + 2 \cdot H_2 + (H_1 + 2 \cdot H_2)_\perp$$

,

$$\mathbf{Prime}(\mathcal{G}_n) = H_n + (H_n)_\perp \qquad H_0 = \{\emptyset\} \qquad H_n = \sum_{i=0}^{n-1} \binom{n}{i} (H_i)_\perp .$$

Free Spectrum for Finite Gödel Algebras (D'Antona and Marra. 2006), (Horn. 1969).



$H_1 = \{\bullet\}$, $\mathbf{Prime}(\mathcal{G}_1) = H_1 + (H_1)_\perp$,

$$\mathbf{Prime}(\mathcal{G}_2) = H_1 + 2 \cdot H_2 + (H_1 + 2 \cdot H_2)_\perp$$

,

$$\mathbf{Prime}(\mathcal{G}_n) = H_n + (H_n)_\perp \qquad H_0 = \{\emptyset\} \qquad H_n = \sum_{i=0}^{n-1} \binom{n}{i} (H_i)_\perp .$$

$$|\mathcal{G}_n| = c_n^2 + c_n, \qquad c_0 = 1 \qquad c_n = \prod_{i=0}^{n-1} (c_i + 1)^{\binom{n}{i}} .$$

$|\mathcal{G}_1| = 6 \quad |\mathcal{G}_2| = 342 \quad |\mathcal{G}_3| = 147186159382 \quad |\mathcal{G}_4| = 2.05740183252e + 64$

(Aguzzoli and Gerla, 2008), (Aguzzoli, Bova and Gerla, 2012)

Fine Spectrum

when dealing with ordered structures, *"the fine spectrum problem is usually hopeless"* (Quackenbush, 1982).

Fine Spectrum of Gödel Algebras

when dealing with ordered structures, *"the fine spectrum problem is usually hopeless"* (Quackenbush, 1982).

k	
1	1
2	1
3	1
4	2
5	2
6	3
7	3
8	5
9	6
10	8
11	8
12	12

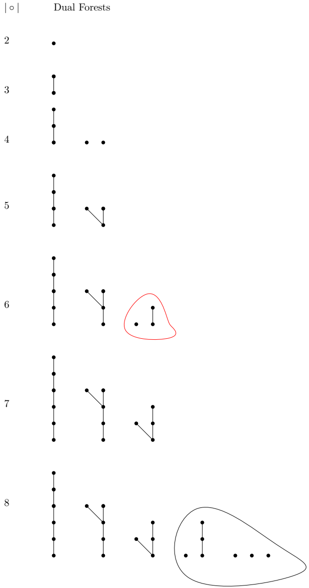
(Belohlávek and Vychodil. 2014)

Fine Spectrum of Gödel Algebras

when dealing with ordered structures, *"the fine spectrum problem is usually hopeless"* (Quackenbush, 1982).

k	
1	1
2	1
3	1
4	2
5	2
6	3
7	3
8	5
9	6
10	8
11	8
12	12

(Belohlávek and Vychodil. 2014)



- $|\mathbf{Sub}(F)| = k$ if and only if $|\mathbf{Sub}(F_\perp)| = k + 1$.
- $|\mathbf{Sub}(F)| = n$ and $|\mathbf{Sub}(F')| = m$ if and only if $|\mathbf{Sub}(F + F')| = n \times m$.

- $|\mathbf{Sub}(F)| = k$ if and only if $|\mathbf{Sub}(F_\perp)| = k+1$.
- $|\mathbf{Sub}(F)| = n$ and $|\mathbf{Sub}(F')| = m$ if and only if $|\mathbf{Sub}(F + F')| = n \times m$.

$$\mathrm{fact}(k) := \{(n_1, \ldots, n_t) \mid k = n_1 \times \cdots \times n_t, n_1 \leq \cdots \leq n_t, t > 1\}.$$

- $|\mathbf{Sub}(F)| = k$ if and only if $|\mathbf{Sub}(F_\perp)| = k+1$.
- $|\mathbf{Sub}(F)| = n$ and $|\mathbf{Sub}(F')| = m$ if and only if $|\ \mathbf{Sub}(F+F')\ | = n \times m$.

$\mathrm{fact}(k) := \{(n_1, \ldots, n_t) \mid k = n_1 \times \cdots \times n_t, n_1 \leq \cdots \leq n_t, t > 1\}$.

Define the following set of forests, $H_1 = \{\emptyset\} \qquad H_k = P_k \cup Z_k$

$$P_k = \{F_\perp | F \in H_{k-1}\}$$

$$Z_k = \{F_1 \sqcup \cdots \sqcup F_t \mid F_1 \in P_{n_1}, \ldots, F_t \in P_{n_t}, (n_1, \ldots, n_t) \in \mathrm{fact}(k)\}$$

- $|\mathbf{Sub}(F)| = k$ if and only if $|\mathbf{Sub}(F_\perp)| = k+1$.
- $|\mathbf{Sub}(F)| = n$ and $|\mathbf{Sub}(F')| = m$ if and only if $|\ \mathbf{Sub}(F+F')\ | = n \times m$.

$\mathrm{fact}(k) := \{(n_1, \ldots, n_t) \mid k = n_1 \times \cdots \times n_t, n_1 \leq \cdots \leq n_t, t > 1\}$.

Define the following set of forests, $H_1 = \{\emptyset\}$ $\qquad H_k = P_k \cup Z_k$

$$P_k = \{F_\perp | F \in H_{k-1}\}$$
$$Z_k = \{F_1 \sqcup \cdots \sqcup F_t \mid F_1 \in P_{n_1}, \ldots, F_t \in P_{n_t}, (n_1, \ldots, n_t) \in \mathrm{fact}(k)\}$$

H_k is the set of of forests corresponding to non-isomorphic k-elements Gödel algebras.

- $|\mathbf{Sub}(F)| = k$ if and only if $|\mathbf{Sub}(F_\bot)| = k+1$.
- $|\mathbf{Sub}(F)| = n$ and $|\mathbf{Sub}(F')| = m$ if and only if $|\,\mathbf{Sub}(F+F')\,| = n \times m$.

$$\mathrm{fact}(k) := \{(n_1,\dots,n_t) \mid k = n_1 \times \cdots \times n_t, n_1 \le \cdots \le n_t, t > 1\}.$$

Define the following set of forests, $H_1 = \{\emptyset\}$ $\qquad H_k = P_k \cup Z_k$

$$P_k = \{F_\bot | F \in H_{k-1}\}$$

$$Z_k = \{F_1 \sqcup \cdots \sqcup F_t \mid F_1 \in P_{n_1},\dots,F_t \in P_{n_t},(n_1,\dots,n_t) \in \mathrm{fact}(k)\}$$

H_k is the set of of forests corresponding to non-isomorphic k-elements Gödel algebras.

$Fine_{\mathbb{G}}(k) = f(k) + pr(k) \times g(k)$ with,

$$f(1) = 1 \qquad f(k) = Fine_{\mathbb{G}}(k-1)$$

$$g(k) = \sum_{(n_1,\dots,n_t)\in \mathrm{fact}(k)} f(n_1) \times \cdots \times f(n_t)$$

$pr(k)$ is the polynomial time function deciding if k is a prime number.
That is, $pr(k) = 0$ when k is prime, $pr(k) = 1$ otherwise.

- $|\mathbf{Sub}(F)| = k$ if and only if $|\mathbf{Sub}(F_\perp)| = k+1$.
- $|\mathbf{Sub}(F)| = n$ and $|\mathbf{Sub}(F')| = m$ if and only if $\mid \mathbf{Sub}(F+F') \mid = n \times m$.

$\mathrm{fact}(k) := \{(n_1, \ldots, n_t) \mid k = n_1 \times \cdots \times n_t, n_1 \leq \cdots \leq n_t, t > 1\}$.

Define the following set of forests, $H_1 = \{\emptyset\}$ $\quad H_k = P_k \cup Z_k$

$$P_k = \{F_\perp | F \in H_{k-1}\}$$

$$Z_k = \{F_1 \sqcup \cdots \sqcup F_t \mid F_1 \in P_{n_1}, \ldots, F_t \in P_{n_t}, (n_1, \ldots, n_t) \in \mathrm{fact}(k)\}$$

H_k is the set of of forests corresponding to non-isomorphic k-elements Gödel algebras.

$Fine_{\mathbb{G}}(k) = f(k) + pr(k) \times g(k)$ with,

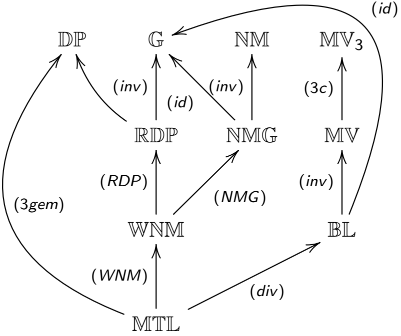
$$f(1) = 1 \qquad f(k) = Fine_{\mathbb{G}}(k-1)$$

$$g(k) = \sum_{(n_1,\ldots,n_t)\in\mathrm{fact}(k)} f(n_1) \times \cdots \times f(n_t)$$

$Spec(\mathbb{G}) =$
$= \{k \in \mathbb{N}^+ \mid Fine_{\mathbb{G}}(k) > 0\} =$
$= \mathbb{N}^+$.

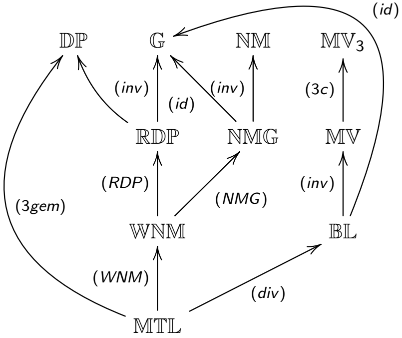
$pr(k)$ is the polynomial time function deciding if k is a prime number.
That is, $pr(k) = 0$ when k is prime, $pr(k) = 1$ otherwise.

MTL hierarchy



A **WNM Algebra** is an MTL algebra satisfying the *Weak Nilpotent Minimum* equation:

$$(x \odot y)' \vee ((x \wedge y) \to (x \odot y)) = \top.$$

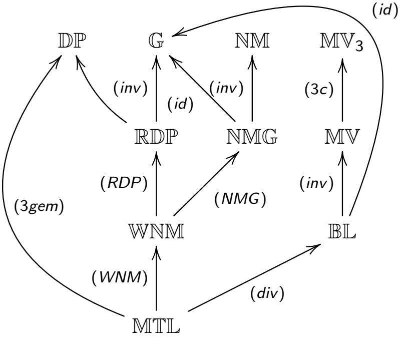


A **WNM Algebra** is an MTL algebra satisfying the *Weak Nilpotent Minimum* equation:

$$(x \odot y)' \vee ((x \wedge y) \to (x \odot y)) = \top.$$

A **Gödel Algebra** is an *idempotent* MTL Algebra.

A **NM Algebra** is an *involutive* WNM algebra.



A **WNM Algebra** is an MTL algebra satisfying the *Weak Nilpotent Minimum* equation:

$$(x \odot y)' \vee ((x \wedge y) \to (x \odot y)) = \top.$$

A **Gödel Algebra** is an *idempotent* MTL Algebra.

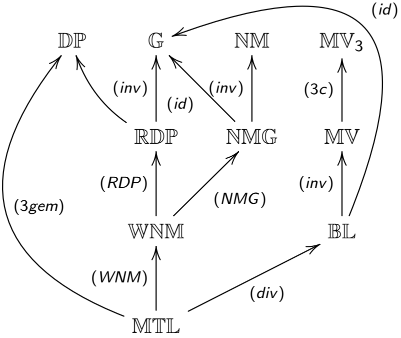
A **NM Algebra** is an *involutive* WNM algebra.

A **RDP Algebra** is an MTL Algebra satisfying:

$$\varphi'' \vee (\varphi \to \varphi') = \top$$

A **NMG Algebra** is an WNM algebra satisfying:

$$(\varphi'' \to \varphi) \vee (\varphi \wedge \psi \to \psi \odot \psi) = \top$$



A **WNM Algebra** is an MTL algebra satisfying the *Weak Nilpotent Minimum* equation:

$$(x \odot y)' \vee ((x \wedge y) \to (x \odot y)) = \top.$$

A **Gödel Algebra** is an *idempotent* MTL Algebra.

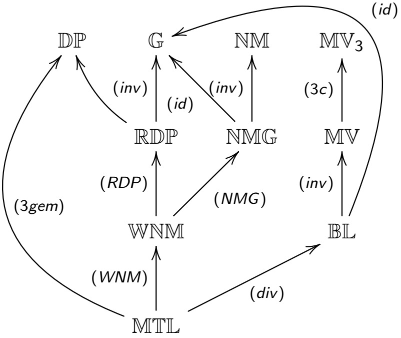
A **NM Algebra** is an *involutive* WNM algebra.

A **RDP Algebra** is an MTL Algebra satisfying:

$$\varphi'' \vee (\varphi \to \varphi') = \top$$

A **NMG Algebra** is an WNM algebra satisfying:

$$(\varphi'' \to \varphi) \vee (\varphi \wedge \psi \to \psi \odot \psi) = \top$$

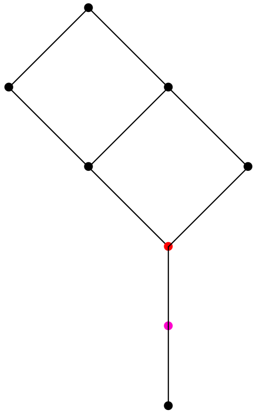


BL Algebras are MTL algebras satisfying divisibility:

$$x \wedge y = x \odot (x \to y)$$

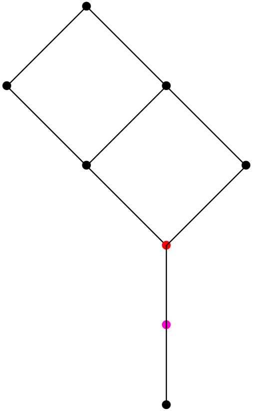
Given the set $I(A) = \{x \in A \mid x \odot x = x\}$ of idempotent elements of an RDP-algebra $\mathbf{A}$, it is possible to describe the prime spectrum of $\mathbf{A}$ in terms of the prime spectrum of the Gödel algebra:

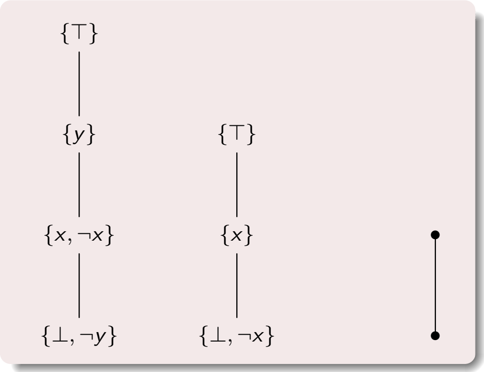
$$\mathbf{A}_G = (I(\mathbf{A}), \wedge, \vee, \odot, \to, \bot, \top).$$



Given the set $I(A) = \{x \in A \mid x \odot x = x\}$ of idempotent elements of an RDP-algebra $\mathbf{A}$, it is possible to describe the prime spectrum of $\mathbf{A}$ in terms of the prime spectrum of the Gödel algebra:

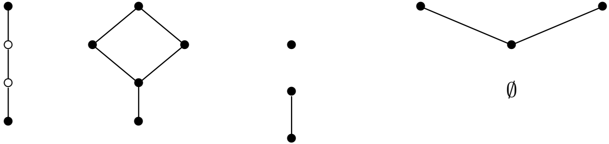
$$\mathbf{A}_G = (I(\mathbf{A}), \wedge, \vee, \odot, \rightarrow, \bot, \top).$$





Proposition

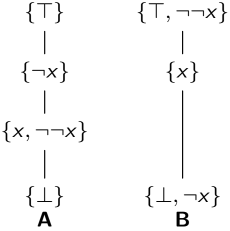
Let $\mathbf{A}$ be a finite directly indecomposable RDP-algebra. Then, the prime spectrum of $\mathbf{A}$ is order isomorphic to $\mathbf{Prime}(\mathbf{A}_G)$.



A **B** $\mathbf{Prime}^+(\mathbf{A})$ $\mathbf{Prime}^+(\mathbf{B})$

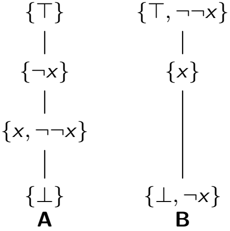
$\mathbf{Prime}^+(\mathbf{A}) = \langle \mathbf{Prime}(\mathbf{A}_G), 2 \rangle \qquad \mathbf{Prime}^+(\mathbf{B}) = \langle \mathbf{Prime}(\mathbf{B}_G), 0 \rangle$

Finite NMG Algebras

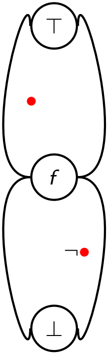


Prime(A) = Prime(B)

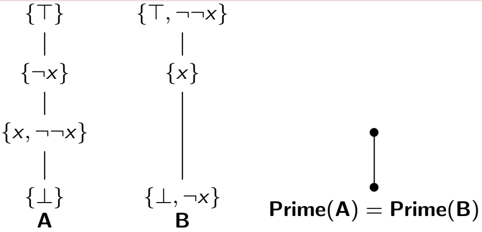
Finite NMG Algebras

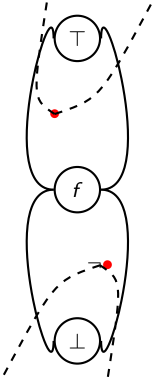


$\mathbf{Prime}(\mathbf{A}) = \mathbf{Prime}(\mathbf{B})$

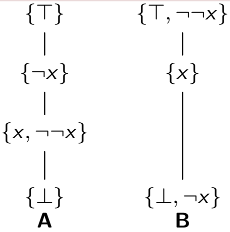


Finite NMG Algebras

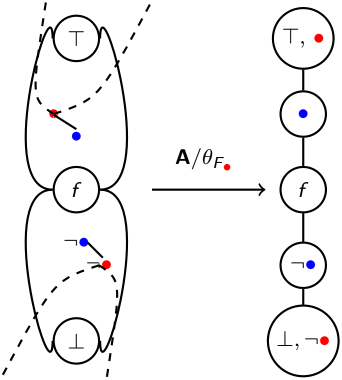




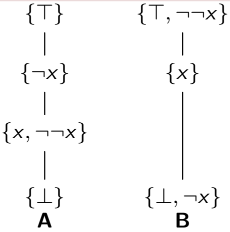
Finite NMG Algebras



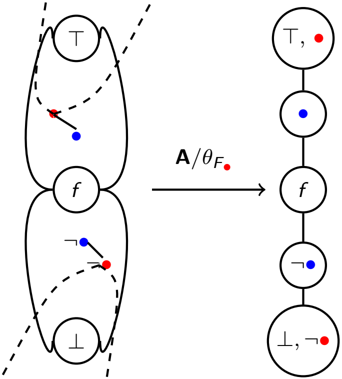
Prime(**A**) $=$ **Prime**(**B**)



Finite NMG Algebras



Prime(**A**) = **Prime**(**B**)



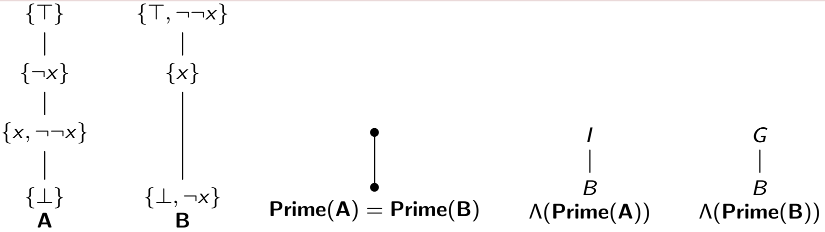
For every filter $F_\bullet$ in **Prime**(**A**) we define a label:

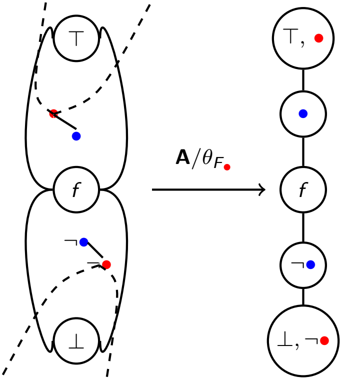
$$\Lambda(F_\bullet) = \begin{cases} B & \text{if } \bullet = m \text{ and } f \notin \mathbf{A}; \\ I & \text{if } \bullet \text{ is involutive, } \textit{or} \\ & \text{if } \bullet = m \text{ and } f \notin \mathbf{A}; \\ G & \text{otherwise.} \end{cases}$$

In this way, in case **A** is d.i., we obtain a labelled tree

$$(\Lambda(\mathbf{Prime}(\mathbf{A})), \leq).$$

Finite NMG Algebras





For every filter $F_\bullet$ in **Prime(A)** we define a label:

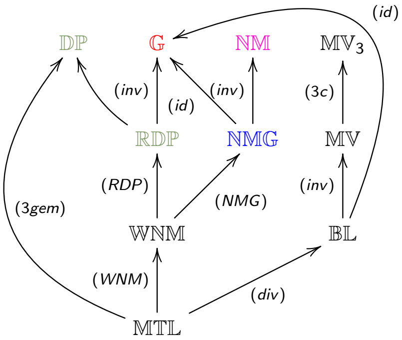
$$\Lambda(F_\bullet) = \begin{cases} B & \text{if } \bullet = m \text{ and } f \notin \mathbf{A}; \\ I & \text{if } \bullet \text{ is involutive, } \textit{or} \\ & \text{if } \bullet = m \text{ and } f \notin \mathbf{A}; \\ G & \text{otherwise.} \end{cases}$$

In this way, in case **A** is d.i., we obtain a labelled tree

$$(\Lambda(\mathbf{Prime}(\mathbf{A})), \leq).$$

Finite WNM Algebras

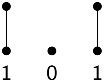
- The category of finite Gödel Algebras is dually equivalent to the category of forests and order-preserving open maps (Horn. 1969).

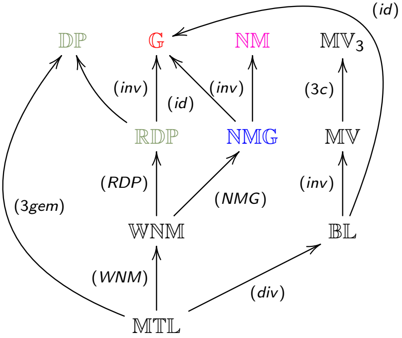


Finite WNM Algebras

- The category of finite Gödel Algebras is dually equivalent to the category of forests and order-preserving open maps (Horn. 1969).

- The category of finite NM Algebras is dually eq. to the category of *labelled* forests (Aguzzoli, Busaniche, Marra. 2006).

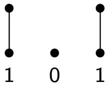




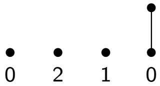
Finite WNM Algebras

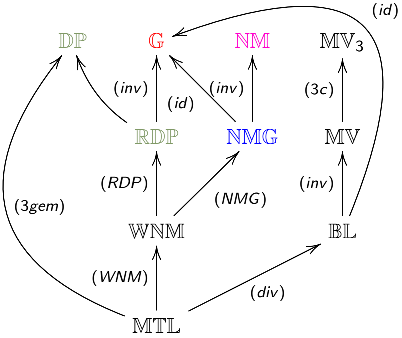
- The category of finite Gödel Algebras is dually equivalent to the category of forests and order-preserving open maps (Horn. 1969).

- The category of finite NM Algebras is dually eq. to the category of *labelled* forests (Aguzzoli, Busaniche, Marra. 2006).

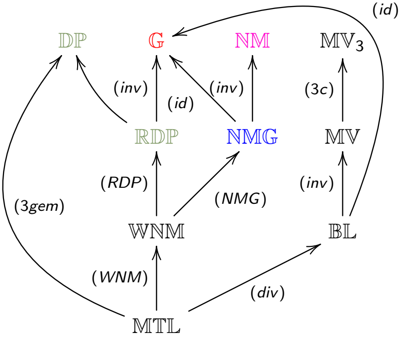


- The category of finite RDP Algebras is dually eq. to the category of *hall* forests (Bova and V. 2010).

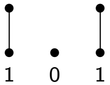




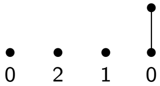
- The category of finite Gödel Algebras is dually equivalent to the category of forests and order-preserving open maps (Horn. 1969).



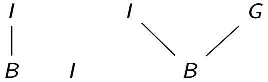
- The category of finite NM Algebras is dually eq. to the category of *labelled* forests (Aguzzoli, Busaniche, Marra. 2006).



- The category of finite RDP Algebras is dually eq. to the category of *hall* forests (Bova and V. 2010).

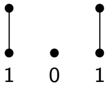


- The category of finite NMG Algebras is dually eq. to the category of *BIG-forests* (V.????).

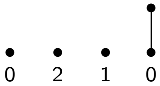


- The category of finite Gödel Algebras is dually equivalent to the category of forests and order-preserving open maps (Horn. 1969).

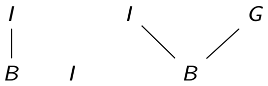
- The category of finite NM Algebras is dually eq. to the category of *labelled* forests (Aguzzoli, Busaniche, Marra. 2006).

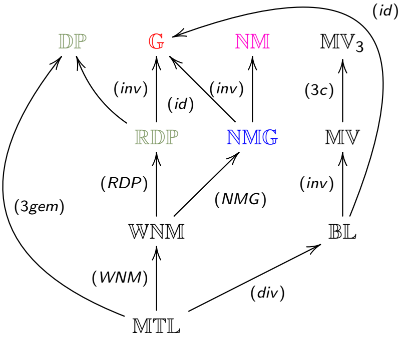


- The category of finite RDP Algebras is dually eq. to the category of *hall* forests (Bova and V. 2010).

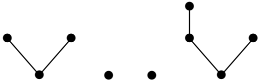


- The category of finite NMG Algebras is dually eq. to the category of *BIG-forests* (V.????).

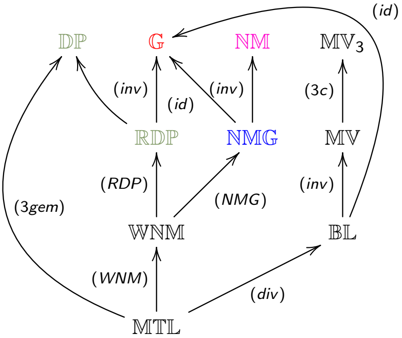




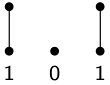
WNM algebras?



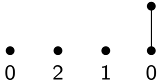
- The category of finite Gödel Algebras is dually equivalent to the category of forests and order-preserving open maps (Horn. 1969).



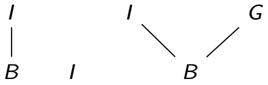
- The category of finite NM Algebras is dually eq. to the category of *labelled* forests (Aguzzoli, Busaniche, Marra. 2006).



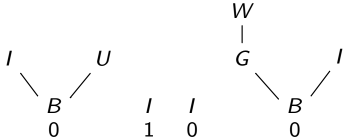
- The category of finite RDP Algebras is dually eq. to the category of *hall* forests (Bova and V. 2010).



- The category of finite NMG Algebras is dually eq. to the category of *BIG-forests* (V.????).



WNM algebras?



Free Spectrum

	$n = 1$	$n = 2$	$n = 3$
$\lvert\mathbb{F}_n(B)\rvert$	4	16	256
$\lvert\mathbb{F}_n(G)\rvert$	6	342	137186159382
$\lvert\mathbb{F}_n(NM)\rvert$	48	3149280000	$\sim 2.79 \cdot 10^{70}$
$\lvert\mathbb{F}_n(NMG)\rvert$	72	738910317600	$\sim 1.02 \cdot 10^{104}$
$\lvert\mathbb{F}_n(RDP)\rvert$	72	94556160000	$\sim 4.06 \cdot 10^{71}$
$\lvert\mathbb{F}_n(WNM)\rvert$	1200	$\sim 1.26324 \cdot 10^{34}$	...

Computed Spectra

The number of k-element Gödel algebras with $1 \leq k \leq 150$.

k	$Fine_{\mathbb{G}}(k)$	k	$Fine_{\mathbb{G}}(k)$	k	$Fine_{\mathbb{G}}(k)$	k	$Fine_{\mathbb{G}}(k)$	k	$Fine_{\mathbb{G}}(k)$
1	1	31	136	61	1484	91	7390	121	25519
2	1	32	162	62	1620	92	7987	122	27003
3	1	33	170	63	1679	93	8123	123	27347
4	2	34	193	64	1868	94	8668	124	29103
5	2	35	199	65	1892	95	8730	125	29249
6	3	36	248	66	2122	96	9627	126	31501
7	3	37	248	67	2122	97	9627	127	31501
8	5	38	279	68	2338	98	10318	128	33559
9	6	39	291	69	2390	99	10528	129	33965
10	8	40	344	70	2631	100	11439	130	36075
11	8	41	344	71	2631	101	11439	131	36075
12	12	42	406	72	2990	102	12418	132	38925
13	12	43	406	73	2990	103	12418	133	39018
14	15	44	466	74	3238	104	13387	134	41140
15	17	45	493	75	3341	105	13713	135	41878
16	23	46	545	76	3651	106	14573	136	44455
17	23	47	545	77	3675	107	14573	137	44455
18	31	48	646	78	4063	108	15947	138	47442
19	31	49	655	79	4063	109	15947	139	47442
20	41	50	740	80	4492	110	17085	140	50619
21	44	51	763	81	4608	111	17333	141	51164
22	52	52	860	82	4952	112	18646	142	53795
23	52	53	860	83	4952	113	18646	143	53891
24	69	54	986	84	5541	114	20119	144	57988
25	73	55	1002	85	5587	115	20223	145	58206
26	85	56	1132	86	5993	116	21604	146	61196
27	91	57	1163	87	6102	117	21955	147	61974
28	109	58	1272	88	6636	118	23227	148	65460
29	109	59	1272	89	6636	119	23296	149	65460
30	136	60	1484	90	7354	120	25455	150	69922

Sequence A130841 of the *On-Line Encyclopedia of Integer Sequences*.

Gödel Logic and Finite Forests:

- O. M. D'Antona and V. Marra. Computing Coproducts of Finitely Presented Gödel Algebras. *Annals of Pure and Applied Logic*, 142(1-3):202–211, 2006.
- A. Horn. Logic with Truth Values in a Linearly Ordered Heyting Algebra. *The Journal of Symbolic Logic*, 34(3):395–408, 1969.

Representations of Free Algebras:

- S. Aguzzoli, S. Bova, and B. Gerla. Free Algebras and Functional Representation. In P. Cintula, P. Hajek, and C. Noguera, editors, *Handbook of Mathematical Fuzzy Logic*. College Publications, 2012.
- S. Bova, D. Valota. Finite RDP-algebras: duality, coproducts and logic. *Journal of Logic and Computation* 22:417–450, 2012.
- S. Aguzzoli, S. Bova, and D. Valota. Free Weak Nilpotent Minimum Algebras *Soft Computing*, 21(1):79–95, 2017.
- S. Aguzzoli and B. Gerla. Normal Forms and Free Algebras for Some Extensions of MTL. *Fuzzy Sets and Systems*, 159(10):1131–1152, 2008.
- D. Valota, Representations for Logics and Algebras related to Revised Drastic Product t-norm,*Submitted*

Spectra Problems

- J. Berman and P.M. Idziak.*Generative complexity in algebra*. Number 828 in Memoirs of the American Mathematical Society 175. American Mathematical Society, 2005.
- A. Durand, N.D. Jones, J.A. Makowsky, and M. More. Fifty years of the spectrum problem: survey and new results. *The Bulletin of Symbolic Logic*, 18(4):505–553, 2012.
- D. Valota, Spectra of Gödel Algebras,*Submitted*.