P_3-Isomorphisms for Graphs

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Received June 16, 1996

*Acknowledges partial support by PGSF scheme of New Zealand FRST, and hospitality of Vanderbilt University.
†Acknowledges partial support by PGSF scheme of New Zealand FRST, and hospitality of University of Otago.
‡Acknowledges partial support by NSF Grant # INT-9221418, and hospitality of University of Otago.

CCC 0364-9024/97/010035-17

Abstract: The P_3-*graph* of a finite simple graph G is the graph whose vertices are the 3-vertex paths of G, with adjacency between two such paths whenever their union is a 4-vertex path or a 3-cycle. In this paper we show that connected finite simple graphs G and H with isomorphic P_3-graphs are either isomorphic or part of three exceptional families. We also characterize all isomorphisms between P_3-graphs in terms of the original graphs. J Graph Theory **26:** 35–51, 1997

Keywords: *isomorphism, path graph, line graph,* P_3*-graph*

1. INTRODUCTION

All graphs in this paper are simple and finite, and any notation not found here may be found in Bondy and Murty [1]. In [2], Broersma and Hoede generalized the idea of line graphs to path graphs by defining adjacency as follows. Let P_k and C_k denote respectively a path and a cycle with k vertices. Let $\Pi_k(G)$ be the set of all P_k's in G, and let $P_k(G)$, the P_k*-graph of* G, be the graph on vertex set $\Pi_k(G)$ in which two P_k's are adjacent when their union is a P_{k+1} or C_k. Note that $P_2(G)$ is just the *line graph* $L(G)$. For a given k, there are two natural questions. First, which graphs are P_k-graphs? Second, if $P_k(G)$ is isomorphic to $P_k(H)$, are G and H necessarily isomorphic, and if not, how are they related? For $k = 2$, i.e., line graphs, the first problem, of characterization, was solved in different ways by several people, starting with Krausz [4] in 1943—see [3] for details. The second problem, of determination, was solved by Whitney [8] in 1932; he showed that, with four small exceptions, an edge isomorphism between connected graphs is induced by an isomorphism. (See also [3] for a simpler proof of this and for related material.)

In [2] Broersma and Hoede focused on the case $k = 3$ and characterized the graphs that are P_3-graphs; a problem with their characterization was resolved by Li and Lin [5]. Broersma and Hoede posed two questions about the hamiltonian properties of P_3-graphs that were answered by Yu [9], and also by Yuan and Lin [10]. Broersma and Hoede also showed that the answer to the determination problem for $k = 3$ was not as simple as for $k = 2$, by giving infinite families of nonisomorphic connected graphs with minimum degree 1 which are P_3-isomorphic but not isomorphic. However, it seemed possible that $P_3(G)$ might determine G if some kind of minimum degree condition were imposed. This idea was pursued by Li, who showed in [6] that if G and H are connected graphs that both have minimum degree at least 4, or have minimum degree at least 3 and satisfy some extra conditions, then every P_3-isomorphism from G to H is induced by an isomorphism. In [7] he extended this, with one exception, to all connected graphs of minimum degree at least 3. He also conjectured that it is true if G and H both have minimum degree at least 2.

In this paper we completely solve the determination problem for $k = 3$ by characterizing all P_3-isomorphisms from G to H, with no degree or connectedness constraints on G and H. In Section 2, we discuss some basic constructions for noninduced P_3-isomorphisms and the situation for disconnected graphs. Each of Sections 3, 4 and 5 discusses one of the three nontrivial families of noninduced P_3-isomorphisms, and we conclude in Section 6 with our Main Theorem giving a characterization of all P_3-isomorphisms between two graphs, at least one of which is connected. Our results show that Li's conjecture for connected graphs of minimum degree 2 is false, and even the weaker conjecture that two P_3-isomorphic graphs with minimum degree 2 must be isomorphic is false.

2. THORNS, DIAMONDS, SWAPS AND DISCONNECTED GRAPHS

In this section we establish much of the necessary terminology for our work. We discuss four simple ways of constructing P_3-isomorphisms which are not induced by isomorphisms. All depend on the presence of vertices of low degree (1 or 2). We also discuss the situation for disconnected graphs, and conclude that in examining P_3-isomorphisms from G to H we can restrict our attention to cases in which at least one of G or H is connected.

As usual we write $G \cong H$ to mean that G is isomorphic to H. A *P_k-isomorphism* from G to H is an isomorphism from $P_k(G)$ to $P_k(H)$. A P_2-isomorphism is also known as an *edge isomorphism.* If σ is an isomorphism from G to H, then σ *induces* a P_k-isomorphism σ^* from G to H, where $\sigma^*(a_1a_2\cdots a_k) = \sigma(a_1)\sigma(a_2)\cdots\sigma(a_k)$ for all $a_1a_2\cdots a_k \in \Pi_k(G)$. A P_k-isomorphism τ is *induced* if $\tau = \sigma^*$ for some isomorphism σ. If τ_i is a P_k-isomorphism from G_i to $H_i, i = 1$ and 2, then we say that τ_1 and τ_2 are *equivalent* if there are isomorphisms σ and ρ from G_1 to G_2 and H_1 to H_2, respectively, such that $\tau_1 = (\rho^*)^{-1} \circ \tau_2 \circ \sigma^*$. Loosely, two P_k-isomorphisms are equivalent if they are the same up to isomorphisms of the graphs concerned.

Now we focus on $k = 3$. For any graph G and any vertex a in G, let $N(a)$ denote the neighborhood of a in G and let $\deg(a)$ denote the degree of a, that is $|N(a)|$. We write $a \sim b$ if a and b are adjacent in G, and $a \not\sim b$ otherwise. For α in $\Pi_3(G)$, define $N(\alpha), \deg(\alpha), \alpha \sim \beta$ and $\alpha \not\sim \beta$ in $P_3(G)$ similarly. Let $m(\alpha)$ denote the middle vertex of α, let $S(a)$ be the set of all P_3's with middle vertex a, and define $a \vdash b$ to be the set of all P_3's in $S(a)$ with an end at b, where $a \vdash b$ is empty if $a \not\sim b$. We call $S(a)$ the *star of G at a*. The set $a \vdash b$, if nonempty, is called a *bundle* with a as its *middle* and b as its *base*. If $R \subseteq \Pi_3(G)$ then the P_3-isomorphism τ is said to *disperse* R if $m(\tau(\alpha)) \neq m(\tau(\beta))$ for some $\alpha, \beta \in R$.

Given two P_3's α and β in G, the permutation of $\Pi_3(G)$ which swaps α and β and fixes everything else is a P_3-isomorphism if $N(\alpha) = N(\beta)$. Below we examine four common situations in which this swap is not an induced P_3-isomorphism.

The first three situations arise from P_3's with one or both ends of degree 1. A vertex of degree 1 is also called *terminal,* and an edge is *terminal* if it has a terminal end. Suppose we are working with a graph G. Define an *i-thorn* to be a P_3 with exactly i ($i = 1$ or 2) terminal ends in G, and a *thorn* to be a 1- or 2-thorn. A P_3 in G is called *terminal* if it has degree 1 in $P_3(G)$; a terminal P_3 must be a 1-thorn with nonterminal end of degree 2. Let $T_i(G)$ be the set of i-thorns in G.

Our first swap involves 2-thorns. After defining it, we discuss the role of 2-thorns in more detail.

(1) The 2-thorns in G are precisely the isolated vertices of $P_3(G)$. Any swap of two 2-thorns is a P_3-isomorphism, which we call a *2-thorn swap,* or *T-swap* for short. It is induced when each 2-thorn is itself a component of G, or when both 2-thorns are subgraphs of a single component of G isomorphic to $K_{1,3}$; otherwise, it is not induced. For example, in Figure 1(a) swapping d_1dd_2 and c_1cc_2 is a noninduced T-swap.

In deriving structural relationships between two graphs based on a P_3-isomorphism between them, 2-thorns are almost no help. All P_3's of a connected graph G are 2-thorns if and only if the component is a *star,* i.e., $K_{1,n}$ for some $n \geq 0$. (When $n = 0$ or 1, the P_3-graphs of the stars $K_{1,0} = K_1$ and $K_{1,1} = K_2$ are the empty graph.) In characterizing P_3-isomorphisms, we therefore wish to ignore 2-thorns and star components as much as possible. We say that two P_3-isomorphisms τ_i from G_i to $H_i, i = 1$ and 2, are *T-related* if (i) G_1 and G_2 differ only in their star components, as do H_1 and H_2; (ii) $|T_2(G_1)| = |T_2(G_2)|$; and (iii) $\tau_1(\alpha) = \tau_2(\alpha)$ for every $\alpha \in \Pi_3(G_1) - T_2(G_1) = \Pi_3(G_2) - T_2(G_2)$. Loosely, T-related P_3-isomorphisms act on graphs with the same number of 2-thorns and behave identically for non-2-thorns. As a special

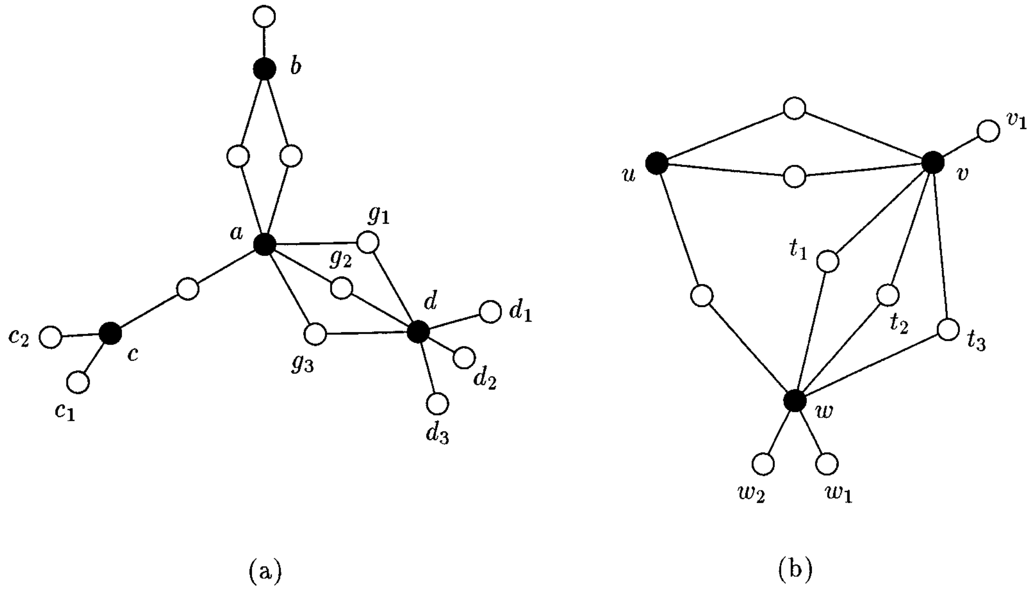

FIGURE 1. Illustrating swaps, diamond inflations and Whitney type P_3-isomorphisms.

case, two P_3-isomorphisms τ_1 and τ_2, both from G to H, are T-related precisely when $\tau_2^{-1} \circ \tau_1$ is the identity or a composition of T-swaps. However, the definition is more general than this, as one or both of the pairs G_1, G_2 and H_1, H_2 may be nonisomorphic.

Our second and third types of swap involve 1-thorns.

(2) Consider two 1-thorns abc and abd where $\deg(a) \geq 2$ and $\deg(c) = \deg(d) = 1$. Since $N(abc) = N(abd) = a \vdash b$, swapping abc and abd gives a P_3-isomorphism, which we call a *bundle 1-thorn swap,* or *B-swap* for short. It is induced when a, c and d are the only neighbors of b; otherwise it is not induced. For example, in Figure 1(a) swapping g_1dd_1 and g_1dd_2 is a noninduced B-swap.

(3) Suppose $abcde$ is a P_5 in G with both abc and cde terminal 1-thorns, i.e., $\deg(a) = \deg(e) = 1$ and $\deg(c) = 2$. We call abc and cde a *split 1-thorn pair.* Since $N(abc) = N(cde) = \{bcd\}$, swapping abc and cde gives a P_3-isomorphism, which we call a *split 1-thorn swap,* or *S-swap* for short. It is induced if $G \cong P_5$; otherwise it is not induced. For example, in Figure 1(b) swapping v_1vt_1 and t_1ww_1 is an S-swap.

The last type of swap arises from certain P_3's with both ends of degree 2. For distinct $a, b \in V(G)$ let $D_{a,b}$ denote the subgraph of G consisting of the union of all P_3's with ends a and b *and* with middle vertex of degree 2 in G. If $D_{a,b}$ is nonempty we call it the *diamond with ends a and b* and say that two diamonds are *adjacent* if they have a common end, but nothing else in common. We usually write $V(D_{a,b}) - \{a, b\}$ as $\{c_1, c_2, \ldots, c_k\}$ and say that $D_{a,b}$ is *a trivial diamond* if $k = 1$, and *nontrivial* otherwise; we call k the *width* of $D_{a,b}$, and refer to $D_{a,b}$ as a *k-diamond.* Note that if $a \sim b$, the edge ab is not included in $D_{a,b}$. To distinguish the two possibilities, we say that the diamond $D_{a,b}$ is *braced* if $a \sim b$ and *unbraced* otherwise.

Note that two diamonds can share an edge, but only if they are determined by a P_4 with internal vertices of degree 2 in G or by a C_3 with two vertices of degree 2 in G. In either case both diamonds are trivial, and the C_3 case is characterized by the fact that the overlapping diamonds are $D_{a,b}$ and $D_{a,d}$ for distinct vertices a, b and d.

Suppose that $D_{a,b}$ is a nontrivial diamond with vertices labelled as above. For $1 \le i < j \le k$, the P_3's ac_ib are called *diamond paths* while the pair of P_3's c_iac_j and c_ibc_j is called the *diamond pair* associated with ac_ib and ac_jb. A P_3 of the form cad where $\deg(c) = \deg(d) = 2$ that is not one of a diamond pair (thus c and d are in different diamonds) is called a *diamond connector.* Two diamonds are therefore adjacent if and only if they each contain one (but not the same) edge of the same diamond connector.

Our fourth type of swap is now as follows.

(4) Suppose c_iac_j and c_ibc_j are a diamond pair. Since $N(c_iac_j) = N(c_ibc_j) = \{ac_ib, ac_jb\}$, swapping c_iac_j and c_ibc_j gives a P_3-isomorphism, which we call a *diamond pair swap,* or *D-swap* for short. It is induced if $G \cong C_4$ and not induced otherwise. For example, in Figure 1(a) swapping g_1ag_2 and g_1dg_2 is a D-swap.

Note that diamond pair swaps provide a multitude of counterexamples to Li's conjecture [7] that all P_3-isomorphisms between graphs with minimum degree at least 2 are induced.

In characterizing P_3-isomorphisms, we need only do this up to T-relation, which takes T-swaps (and more) into account. We also need to take the other three kinds of swap into account. Suppose τ_1 and τ_2 are P_3-isomorphisms from G to H. We say τ_1 and τ_2 are *B-related* if $\tau_2^{-1} \circ \tau_1$ is the identity or a composition of B-swaps; *S-related* and *D-related* are defined similarly. We use joins of these four equivalence relations: for example, two P_3-isomorphisms are *TBSD-related* if we can get from one to the other by a chain of zero or more T-, B-, S- and/or D-relations. Other joins will be denoted by analogous notation.

It is natural at this point to ask if all P_3-isomorphisms are TBSD-related to an induced one. Unfortunately, the answer is no. We have not even accounted for the examples of Broersma and Hoede [2] of nonisomorphic connected graphs with isomorphic P_3-graphs. If there is a P_3-isomorphism from G to H which is TBSD-related to an induced one, then G and H have the same nonstar components, which is not true for the Broersma and Hoede examples.

Before proceeding to examine P_3-isomorphisms that are not TBSD-related to induced ones, we deal with the case of disconnected graphs. Star components produce only isolated vertices in a P_3-graph. For the nonstar components, we have the following lemma, whose proof is straightforward.

Lemma [Component]. *Let α and β be two P_3's in a graph G, neither of which is a 2-thorn, and whose middle vertices are connected by a path in G. Then α and β are connected by a path in $P_3(G)$.*

Thus, there is a one-to-one correspondence between the nonstar components of G and the nontrivial components of $P_3(G)$, and a P_3-isomorphism τ from G to H induces a one-to-one correspondence between the nonstar components of G and H. Suppose G_i is a nonstar component of G, with counterpart H_i in H. Then τ restricts to a bijection τ_i from $\Pi_3(G_i) - T_2(G_i)$ to $\Pi_3(H_i) - T_2(H_i)$. If necessary, add one or more stars (copies of $K_{1,2} = P_3$ always work) to one but not both of G_i or H_i to obtain graphs G_i^* and H_i^* with the same number of 2-thorns. By putting the 2-thorns of G_i^* and H_i^* into correspondence in an arbitrary way, we extend τ_i to a P_3-isomorphism τ_i^* from G_i^* to H_i^*. Understanding each τ_i^* is the key to understanding τ, and so we now make the following assumption.

Standing Assumption. Throughout the rest of the paper we let τ be a P_3-isomorphism from G to H where at least one of G or H is connected and neither is a star. Thus, each has one nonstar component (which we call G_0 and H_0, respectively), while all other components, if any, are stars. For $\alpha \in \Pi_3(G)$, we let α' denote $\tau(\alpha)$.

3. $K_{3,3}$ AND RELATED GRAPHS

Besides the relations discussed in Section 2, there are three families of graphs which provide noninduced P_3-isomorphisms. One is a family of small graphs related to $K_{3,3}$, the second is an infinite family derived from Whitney's examples of noninduced edge isomorphisms, and the third is obtained from bipartite graphs. In this section we discuss the first family, and show that they correspond to one particular type of noninduced P_3-isomorphism.

The following construction shows that even P_3-isomorphisms between graphs of minimum degree 3 may not be induced.

Construction on $K_{3,3}$. Let vertex sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ be a bipartition of $K_{3,3}$ and define $\tau_0 : \Pi_3(K_{3,3}) \to \Pi_3(K_{3,3})$ by $\tau_0(u_i v_i u_j) = u_k v_k u_j, \tau_0(v_i u_i v_j) = v_k u_k v_j, \tau_0(u_i v_j u_k) = u_i v_j u_k$ and $\tau_0(v_i u_j v_k) = v_i u_j v_k$ for each i, j and k with $\{i, j, k\} = \{1, 2, 3\}$. It is not difficult to check that τ_0 is a noninduced P_3-isomorphism of $K_{3,3}$ to itself. Since $K_{3,3}$ has no degree 1 or 2 vertices, τ_0 is TBSD-related only to itself, so it is not TBSD-related to an induced P_3-isomorphism, either.

The important property of $\tau = \tau_0$ is the following.

Situation 1. Either τ or τ^{-1} disperses some bundle with base of degree 3 or more.

For example, $\tau_0(u_1 v_1 u_2)$ and $\tau_0(u_1 v_1 u_3)$ have different middle vertices. For minimum degree at least 3, it turns out that Situation 1 characterizes τ_0. Before verifying this, we give two lemmas with some basic properties that will be useful here and later.

Lemma [End]. *If α and β are P_3's in G with a common end (possibly both ends in common) but no common terminal edge, then α' and β' have a common end.*

Proof. Two P_3's have a common end but no common terminal edge if and only if there is a third P_3 adjacent to both. Now τ preserves the latter property and hence the former. ∎

Lemma [Cycle]. *Let τ be a P_3-isomorphism from G to H with at least one of G or H connected.*

(i) *If $\alpha\beta\gamma\alpha$ is a 3-cycle in $P_3(G)$, then G contains a 3-cycle C with $\Pi_3(C) = \{\alpha, \beta, \gamma\}$.*
(ii) *If $\alpha\beta\gamma\delta\alpha$ is a 4-cycle in $P_3(G)$, then either G contains a 4-cycle C with $\Pi_3(C) = \{\alpha, \beta, \gamma, \delta\}$, or else G contains an edge ab with $\alpha, \gamma \in a \vdash b$ and $\beta, \delta \in b \vdash a$. Loosely, $\alpha\beta\gamma\delta\alpha$ comes either from a 4-cycle in G or from pairs of P_3's at opposite ends of an edge. Note that the first possibility is distinguished from the second in that $m(\alpha) \neq m(\gamma)$ in the first.*

Proof. Obvious. ∎

Theorem $[K_{3,3}]$. *Let G and H be connected graphs of minimum degree at least 2 and let τ be a P_3-isomorphism from G to H. If τ disperses $a \vdash b$ where deg$(b) \geq 3$, then $G \cong H \cong K_{3,3}$ and τ is equivalent to τ_0 as in the Construction on $K_{3,3}$.*

Proof. Suppose that $m(\alpha'_1) \neq m(\alpha'_2)$ where $\alpha_1 = cab$ and $\alpha_2 = dab$ with $c \neq d$. Let $\beta_1 = abe$ and $\beta_2 = abf$ with $e \neq f$. Note that $m(\alpha'_1) = m(\alpha'_2)$ if and only if $m(\beta'_1) = m(\beta'_2)$ by (ii) of the Cycle Lemma, so anything we prove about α_1 and α_2 is also true for β_1 and β_2. By (ii) of the Cycle Lemma, there is a 4-cycle $u_1 v_1 u_2 v_2 u_1$ in H with $\alpha'_i = v_1 u_i v_2$ and $\beta'_i = u_1 v_i u_2$ for $i = 1, 2$.

Next we note that none of $\alpha_i, \alpha'_i, \beta_i$, or $\beta'_i, i = 1, 2$, is in a 3-cycle in $P_3(G)$ or $P_3(H)$. For, 3-cycles in $P_3(G)$ correspond to 3-cycles in $P_3(H)$ and so, by symmetry, we only need to consider

α_1. If α_1 is in a 3-cycle, then we may choose β_1 to be the other member of that 3-cycle that also contains the edge ab. Thus, by (i) of the Cycle Lemma, α_1' and β_1' are in a common 3-cycle in $P_3(H)$, a contradiction of the above. Consequently, a, b, c, d, e and f are all distinct while $u_1 \not\sim u_2$ and $v_1 \not\sim v_2$. Letting $\alpha_3 = cad$ and $\beta_3 = ebf$ and using symmetry and the End Lemma, we can assume that there are vertices $u_3, v_3, x, y \in V(H)$ with $\alpha_3' = v_1u_3x, \beta_3' = u_1v_3y$ and $u_3, v_3 \notin \{u_1, u_2, v_1, v_2\}$.

Suppose that $\alpha_4, \beta_4 \in \Pi_3(G)$ with $\alpha_4' = v_1u_1v_3$ and $\beta_4' = u_1v_1u_3$. Now $\alpha_4' \sim \beta_1', \beta_3'$, so $m(\alpha_4) = e \neq m(\alpha_1)$. Hence, since $\alpha_1', \alpha_4' \in u_1 \vdash v_3$, we conclude, as above, that $u_3 \neq v_3$, that $c \sim e$, and that $\alpha_4 = bec$ and $\beta_4 = ace$. If $\alpha_5, \beta_5 \in \Pi_3(G)$ with $\alpha_5' = v_2u_1v_3$ and $\beta_5' = u_2v_1u_3$, we have $\beta_5 = adg$ and $\alpha_5 = bfh$, where g and h need not be new vertices. Since β_5 has at least three neighbors, so does β_5'. Hence, $\deg(u_2) \geq 3$ or $\deg(u_3) \geq 3$. In the first case, we must have a new member of $\Pi_3(H)$ that is adjacent to both β_1' and β_5', while the second case requires one adjacent to both β_4' and β_5'; in either case, we conclude that $g = e$ in G. And, by symmetry, we get $h = c$.

Now, let $\alpha_6 = ced, \alpha_7 = bed, \beta_6 = ecf$ and $\beta_7 = acf$. The adjacencies of α_7 and β_7 with the previously found P_3's require that $y = u_2$ with $\alpha_7' = v_1u_2v_3$ and $x = v_2$ with $\beta_7' = u_1v_2u_3$, while the combined adjacencies of α_6 and β_6 require that $u_3 \sim v_3$ with $\alpha_6' = v_1u_3v_3$ and $\beta_6' = u_1v_3u_3$. Similarly, supposing that $\alpha_8, \alpha_9, \beta_8, \beta_9 \in \Pi_3(G)$ with $\alpha_8' = v_2u_2v_3, \alpha_9' = v_2u_3v_3, \beta_8' = u_2v_2u_3$ and $\beta_9' = u_2v_3u_3$ requires $d \sim f$ with $\alpha_8 = bfd, \alpha_9 = cfd, \beta_8 = adf$ and $\beta_9 = edf$.

To finish the proof, we note that $\deg(a) = 3$; otherwise, we would have a third P_3 in $P_3(H)$ that is adjacent to both β_1' and β_2', an impossibility. So, by the total symmetry in G and H at this stage, we conclude that $G \cong H \cong K_{3,3}$. Moreover, if we let σ be the isomorphism from G to H given by $\sigma(a) = u_3, \sigma(b) = v_3, \sigma(c) = v_2, \sigma(d) = v_1, \sigma(e) = u_2$ and $\sigma(f) = u_1$, then one easily verifies that $\tau = \tau_0 \circ \sigma^*$. ∎

Construction of the Generalized $K_{3,3}$ Pairs. If we relax the minimum degree restriction of the above theorem, and allow terminal vertices in G and H, then we obtain six more P_3-isomorphisms τ that disperse bundles. All seven are listed below, using the following notation. We write $(c, d)ab(e, f) \mapsto uvwxu$ if G contains edges ab, ac, ad, be, bf, H contains the C_4 $uvwxu$, and τ maps $cab \mapsto xuv, dab \mapsto vwx, abe \mapsto uvw$ and $abf \mapsto wxu$. We also write $abc(d, e) \mapsto uvwxy$ if G contains edges ab, bc, cd, ce, H contains the P_5 $uvwxy$, and τ maps $abc \mapsto vwx, bcd \mapsto uvw$ and $bce \mapsto wxy$. This notation will be reversed (e.g., $abcda \mapsto (w, x)uv(y, z)$) as needed. Each description below contains one or more such items which together completely specify all edges of G and H except those in star components, and all P_3's except 2-thorns. Some edges and P_3's occur in more than one item. The vertex names mostly follow those in the above proof.

(i) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1$, and cad and ebf map to P_3 components of H.

(ii) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1, kebfh \mapsto yv_3u_1(v_1, v_2)$, and cad maps to a P_3 component.

(iii) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1, (k, l)eb(a, f) \mapsto u_1v_1u_2v_3u_1, (h, i)fb(a, e) \mapsto u_1v_2u_2v_3u_1$, and cad, kel and hfi map to P_3 components.

(iv) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1, ecadg \mapsto xu_3v_1(u_1, u_2)$, and $cebfh \mapsto yv_3u_1(v_1, v_2)$. Note that G and H are connected and isomorphic.

(v) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1, ebfhe \mapsto (v_1, v_2)u_1v_3(y, z)$, and cad maps to yv_3z. Again G and H are connected and isomorphic.

(vi) $(c, d)ab(e, f) \mapsto u_1v_1u_2v_2u_1, (c, d)eb(a, f) \mapsto u_1v_1u_2v_3u_1, (h, i)fb(a, e) \mapsto u_1v_2u_2v_3u_1, aceda \mapsto (w, x)u_3v_1(u_1, u_2)$, and hfi maps to wu_3x. Again G and H are connected and isomorphic.

(vii) The Construction on $K_{3,3}$: $G \cong H \cong K_{3,3}$.

Either τ or τ^{-1} as in cases (i) through (vii) above, or any equivalent P_3-isomorphism, is said to be of *generalized $K_{3,3}$ type.*

Corollary [Generalized $K_{3,3}$ Type]. *Situation* 1 *occurs if and only if τ is T-related to a P_3-isomorphism of generalized $K_{3,3}$-type.*

Proof. The 'if' part is trivial, so consider the 'only if' part. Under our standing assumuption, suppose τ disperses $a \vdash b$, where $\deg(b) \geq 3$. As in the proof of the $K_{3,3}$ Theorem, we know that (i) occurs inside G and H. The remaining cases come about by the addition of more edges. We only sketch this since the details are just as in the proof of the $K_{3,3}$ Theorem. As there, we have that a, b, c, d, e, f are all distinct, $u_1 \not\sim u_2$ and $v_1 \not\sim v_2$. Likewise, any vertices of degree at least 3 are of degree exactly 3.

So we can assume by symmetry that there is an edge v_3u_1 in H. Constructing the P_3's in G corresponding to $v_1u_1v_3$ and $v_2u_1v_3$ forces us to (ii).

There are now four nonsymmetric ways of adding further edges to H in (ii): at u_2 which gives (iii), at v_1 which gives (iv), at v_3 which gives (v), and at y which once again gives (iii). Moreover, following any one of those three additions by another gives (vi). Finally, adding anything to G or H in (vi) gives (vii). ∎

Note that in the statement of the above corollary we need T-relation to take 2-thorns into account in cases (i), (ii) and (iii). There are no split 1-thorn pairs, so we do not need S-relation. We do not need B- or D-relation, despite the presence of diamond pairs and 1-thorns in the same bundle in cases (i), (ii), (iii), (v) and (vi), because each diamond pair swap in G corresponds to an *induced* bundle 1-thorn swap in H, and vice versa. This means that composing with a diamond pair swap or a bundle 1-thorn swap just takes each P_3-isomorphism in these cases to an equivalent one.

4. EXAMPLES FROM WHITNEY'S EDGE ISOMORPHISMS

In this section we construct an infinite family of noninduced P_3-isomorphisms from Whitney's examples of noninduced edge isomorphisms. We begin with a rather general idea which will be used here and in the next section.

Diamond Inflation. Suppose F is a graph. A *diamond inflation* of F is a graph obtained by replacing each $ab \in E(F)$ by an unbraced s_{ab}-diamond $D_{a,b}$ ($s_{ab} \geq 1$), and adding t_a terminal edges incident with each $a \in V(F)$ ($t_a \geq 0$). For example, in Figure 1(a) we have a diamond inflation of a $K_{1,3}$ with vertices a, b, c, d where $s_{ab} = 2, s_{ac} = 1, s_{ad} = 3, t_a = 0, t_b = 1, t_c = 2$ and $t_d = 3$. Figure 1(b) is also a diamond inflation, of a K_3 with vertices u, v, w.

Now suppose φ is an edge isomorphism between graphs F and F'. Suppose I and I' are diamond inflations of F and F' respectively, with the following property: for every $ab \in E(F)$, if $\varphi(ab) = uv$ then (i) $s_{uv} = s_{ab}$ and (ii) $t_u + t_v = t_a + t_b$. Obtain G and H from I and I' respectively by adding star components to one of them (if necessary) to make the number of 2-thorns equal. Then we can define a P_3-isomorphism τ from G to H, as follows. Suppose $ab \in E(F)$ and $\varphi(ab) = uv$. Let $\tau|\Pi_3(D_{a,b})$ be induced by any isomorphism from $D_{a,b}$ to $D_{u,v}$: the two diamonds are the same size by (i). For any diamond path α of $D_{a,b}$, the $t_a + t_b$ terminal 1-thorns adjacent to α can be mapped arbitrarily to the $t_u + t_v$ terminal 1-thorns adjacent to $\tau(\alpha)$, since the numbers are equal by (ii). The 2-thorns of G can be mapped arbitrarily to the

2-thorns of H. This only leaves the diamond connectors: the image of each diamond connector is uniquely determined by the images of the two diamond paths which are its neighbors. We say τ is a *diamond inflation* of φ. Note that anything TBS-related to τ is also a diamond inflation of φ.

It turns out there are two situations in which diamond inflation yields P_3-isomorphisms not TBSD-related to induced ones. The first situation comes from Whitney's exceptional edge isomorphisms.

Theorem [Whitney [8], also see [3]**].** *Suppose that φ is an edge isomorphism from G to H where G and H are both connected. If φ is not induced, then $i = |E(G)| = |E(H)| \in \{3,4,5,6\}$, G and H are isomorphic to W_i and W_i' in some order, and φ is equivalent to φ_i or φ_i^{-1}, where*

(i) *$W_6 \cong W_6' \cong K_4$, with $V(W_6) = \{a,b,c,d\}, V(W_6') = \{u,v,w,x\}$, and φ_6 maps $ab \mapsto uv, ac \mapsto uw, ad \mapsto vw, bc \mapsto ux, bd \mapsto vx$ and $cd \mapsto wx$;*
(ii) *$W_5 = W_6 - cd, W_5' = W_6' - wx$ and $\varphi_5 = \varphi_6|E(W_5)$;*
(iii) *$W_4 = W_6 - \{bd, cd\}, W_4' = W_6' - \{vx, wx\}$ and $\varphi_4 = \varphi_6|E(W_4)$; and*
(iv) *$W_3 = W_6 - \{bc, bd, cd\} \cong K_{1,3}, W_3' = W_6' - x \cong K_3$, and $\varphi_3 = \varphi_6|E(W_3)$.*

Construction on the Whitney graphs. A P_3-isomorphism τ is said to be of *Whitney type i* if τ or τ^{-1} is equivalent to a diamond inflation of φ_i as above, $i = 3,4,5,6$. As an example, the two graphs of Figure 1 are related by a Whitney type 3 P_3-isomorphism, with vertices labelled exactly as above.

For Whitney type P_3-isomorphisms, condition (i) of Diamond Inflation just requires that both of a pair of corresponding edges, for example ad and vw in all four types, are replaced by diamonds of the same width. Letting t_z denote the number of terminal edges incident with z for z in $\{a,b,c,d\}$ or $\{u,v,w,x\}$, we see that condition (ii) of Diamond Inflation gives one equation from each pair of corresponding edges of the original Whitney graphs: for example, in Whitney type 4, 5 or 6 the corresponding edges bc and ux give that $t_b + t_c = t_u + t_x$. Solving for t_u, t_v, t_w, t_x in terms of t_a, t_b, t_c, t_d we find the same solutions for all four types:

$$t_u = \frac{1}{2}(t_a + t_b + t_c - t_d), \qquad t_w = \frac{1}{2}(t_a - t_b + t_c + t_d),$$
$$t_v = \frac{1}{2}(t_a + t_b - t_c + t_d), \qquad t_x = \frac{1}{2}(-t_a + t_b + t_c + t_d) \text{ (except for type 3)}.$$

These equations impose obvious conditions on t_a, t_b, t_c, t_d to make t_u, t_v, t_w and (except for type 3) t_x integral and nonnegative.

Note that the case where no terminal edges are added ($t_a = t_b = t_c = t_d = 0$) includes an infinite family of pairs of P_3-isomorphic but not isomorphic graphs of minimum degree at least 2. Thus, even the weak form of Li's conjecture for minimum degree 2 is false.

The Connected Case for the Whitney Types. If we want both G and H to be connected, then both must have the same number of 2-thorns in their nonstar components. For a Whitney pair of type 3, this requires that we have

$$\binom{t_a}{2} + \binom{t_b}{2} + \binom{t_c}{2} + \binom{t_d}{2} = \binom{t_u}{2} + \binom{t_v}{2} + \binom{t_w}{2}.$$

After some computations that use the equations above, we can reduce this to a quadratic equation whose solutions are $t_a = t_b + t_c + t_d$ and $t_a = t_b + t_c + t_d - 2$.

Somewhat surprisingly, for Whitney types 4, 5 and 6 the graphs turn out to be already connected, since the conditions given in their construction can be shown to imply that

$$\binom{t_a}{2} + \binom{t_b}{2} + \binom{t_c}{2} + \binom{t_d}{2} = \binom{t_u}{2} + \binom{t_v}{2} + \binom{t_w}{2} + \binom{t_x}{2}.$$

The Special Whitney Type. There is one special example of a noninduced P_3-isomorphism which is closely related to the Whitney types, and so we also introduce it here. Let SW be the graph obtained by subdividing each edge of $K_{1,3}$ exactly once; then $P_3(SW) \cong C_6$. Rotation of this C_6 by one step is a noninduced P_3-isomorphism from SW to itself; we say this or any equivalent P_3-isomorphism is of *special Whitney type.*

The important property of the Whitney types (including the special Whitney type) is the following.

Situation 2. Situation 1 does not occur, but either τ or τ^{-1} disperses two P_3's in the same bundle, neither of which is a thorn or one of a diamond pair.

Below we show that this characterizes the Whitney types; first we need some basic properties of diamonds.

Lemma [Diamond]. *Suppose Situation* 1 *does not occur.*

(i) *α, β is a diamond pair if and only if α', β' is, in which case $N(\alpha) = N(\beta)$ and consists solely of the two diamond paths associated with α and β.*

(ii) *If $D_{a,b}$ is a nontrivial diamond in G, then there exist a unique and nontrivial diamond $D_{u,v}$ in H and an isomorphism φ from $D_{a,b}$ to $D_{u,v}$ such that $\varphi(a) = u, \varphi(b) = v$ and φ^* is D-related to $\tau|\Pi_3(D_{a,b})$. Moreover, $D_{a,b}$ is braced if and only if $D_{u,v}$ is. We call $D_{u,v}$ the image, under τ, of $D_{a,b}$.*

Proof. (i) If α, β is a diamond pair but α', β' is not, then, by (ii) of the Cycle Lemma, Situation 1 occurs for α' and β'.

(ii) The first claim follows from (i); for if $\delta_1 = ac_1b$ is a diamond path in $D_{a,b}$, then δ is a diamond path in $D_{a,b}$ if and only if there is a diamond pair α, β with $N(\alpha) = \{\delta_1, \delta\}$. Thus $\Pi_3(D_{a,b})$ maps into $\Pi_3(D_{u,v})$ for some nontrivial diamond $D_{u,v}$. Applying the same argument to τ^{-1} and $D_{u,v}$ we see that τ maps $\Pi_3(D_{a,b})$ onto $\Pi_3(D_{u,v})$. The second claim is immediate from (i) of the Cycle Lemma. ∎

Note that the simplest pairs (that is, using 1-diamonds and no terminal edges in the construction) of Whitney types 3, 4, and 5 show the necessity of requiring a nontrivial diamond in part (ii) of the Diamond Lemma.

Theorem [Whitney Type]. *Situation* 2 *occurs if and only if τ is of special Whitney type or D-related to a P_3-isomorphism of Whitney type.*

Proof. The 'if' part is trivial, so consider the 'only if' part. We suppose that $\alpha_1 = c_1ac_2, \alpha_2 = c_2ac_3, \alpha_3 = c_3ac_1$ with neither α_1 nor α_2 a thorn or one of a diamond pair and $m(\alpha_1') \neq m(\alpha_2')$. By symmetry we assume that $m(\alpha_1') \neq m(\alpha_3')$. Since $\deg(c_i) \geq 2$ for each $i = 1, 2, 3$, we have P_3's $\beta_i = ac_ib_i, i = 1, 2, 3$. In fact, $\deg(c_1) = \deg(c_2) = 2$ since Situation 2 implies that Situation 1 does not occur, and so β_1 and β_2 are unique and α_1 is a diamond connector. Also,

a, c_1 and c_2 cannot form a triangle, or the uniqueness of β_1 and β_2 would force $m(\alpha_1') = m(\alpha_2')$. Hence, b_1, b_2, c_1, c_2 are all distinct. If α_1' is a thorn, a short argument using the fact that Situation 1 does not occur shows that we have the special Whitney type.

Consequently, we can assume that α_1' is not a thorn. By (i) of the Diamond Lemma, α_1' is not a one of a diamond pair, but deg$(\alpha_1') = $ deg$(\alpha_1) = 2$, so α_1' is a diamond connector with diamond paths β_1' and β_2' as neighbors. Thus $m(\alpha_2') \neq m(\alpha_3')$ since $m(\alpha_2')$ (resp. $m(\alpha_3')$) and $m(\alpha_1')$ are the ends of β_2' (resp. β_1'). Since Situation 1 does not occur, deg$(c_3) = 2$. And as with α_1', we see that α_2' and α_3' are diamond connectors in H; that is, deg$(m(\beta_i')) = 2, i = 1, 2, 3$ and so the middle vertices of $\alpha_1', \beta_2', \alpha_2', \beta_3', \alpha_3', \beta_1', \alpha_1'$ form a 6-cycle in H. Hence β_1', β_2' and β_3' are all in different diamonds and b_1, b_2 and b_3 are pairwise distinct.

So we have distinct diamonds D_{a,b_1}, D_{a,b_2} and D_{a,b_3}, pairwise adjacent at a, whose images in H form a triangle of diamonds, that is, we can choose the notation so that their images are of the form D_{u_1,u_2}, D_{u_2,u_3} and D_{u_3,u_1}, respectively, with $u_i = m(\alpha_i'), i = 1, 2, 3$. Note that these diamonds are all unbraced; in fact, all edges at a other than terminal ones are accounted for; otherwise, we would have another $\alpha \in a \vdash c$, and so just as in the preceding, we would get a fourth diamond at a whose image, along with those of two of the previous ones, would form yet another triangle of diamonds in H, an impossibility.

If this accounts for all of G and H, other than for some terminal edges, then τ is D-related to a P_3-isomorphism of Whitney type 3. Otherwise, we can assume that there is an unaccounted-for vertex of degree at least 2 that is adjacent to u_1. Applying the same reasoning as above, but for τ^{-1} at u_1, we see that there is a third unbraced diamond D_{u_1,u_4} at u_1 that is the image of D_{b_1,b_2} and that there can be no more nonterminal edges at u_1. If this is all of G and H, except for some terminal edges, then τ is D-related to a P_3-isomorphism of Whitney type 4. If this still doesn't account for all of G, then continuing in the same way, we see that τ must be D-related to a P_3-isomorphism of Whitney type 5 or 6, and we cannot continue beyond this. ∎

5. EXAMPLES FROM BIPARTITE GRAPHS

So far we have seen two families of noninduced P_3-isomorphisms. However, we still have not captured Broersma and Hoede's examples! In this section we construct noninduced P_3-isomorphisms starting from an arbitrary bipartite graph, using the idea of diamond inflation introduced in the previous section. This construction accounts for the Broersma and Hoede examples, and produces many others.

Construction on a Bipartite Graph. Start with a positive integer k and an arbitrary bipartite graph F with at least one edge and with bipartition (A, B). Let I and I' be different diamond inflations of F, where each edge e is inflated to a diamond of the same width s_e both times, but in producing I each vertex v has t_v terminal edges added, while in producing I' it has t_v' terminal edges added, where

$$t_v' = \begin{cases} t_v - k & \text{if } v \in A, \\ t_v + k & \text{if } v \in B. \end{cases}$$

(Thus, we need $t_v \geq k$ for all $v \in A$.) Let φ be the identity edge isomorphism from F to itself. Clearly φ, I and I' satisfy condition (i) of Diamond Inflation, and condition (ii) is satisfied because each edge of F has the form ab with $a \in A$ and $b \in B$, so that $t_a' + t_b' = (t_a - k) + (t_b + k) = t_a + t_b$. We can therefore obtain a P_3-isomorphism τ by diamond inflation; τ is in general not induced. We say τ and τ^{-1}, or any equivalent P_3-isomorphisms, are of *bipartite type.*

Some Examples. We note that the Broersma and Hoede examples [2, Figs. 2 and 3] are fairly simple, albeit quite illustrative, cases of this construction. With a slight change in labelling, the graphs of their Figure 2 appear in our Figure 2. They may be obtained by starting with a tree F of diameter 3, whose vertices are the solid vertices of the figure. To obtain $I = T_{m,l}$ from F, replace all edges with unbraced 1-diamonds (that is, subdivide the edges of F), add a single terminal edge at each vertex of $A = \{x, r_1, r_2, \ldots, r_l\}$, and add nothing at each vertex of $B = \{y, u_1, u_2, \ldots, u_m\}$. To obtain $I' = T'_{m,l}$, we take $k = 1$ in our construction. The added terminal edges for I and I' are emboldened in the figure.

The graphs of Broersma and Hoede's Figure 3 are equally simple, differing only in that the starting graph is $K_{1,n+1}$ and one of the replacing diamonds is a 2-diamond, again an unbraced one. If we do the same thing to P_3, but using only unbraced 1-diamonds, then we get the other example they noted, namely, P_7 and the graph that results from deleting a terminal vertex from a subdivision of $K_{1,3}$. Both of these graphs have P_5 as their P_3-graphs; on the other hand, it is not hard to show that if G is connected and $P_3(G) \cong P_n, n \neq 5$, then $G \cong P_{n+2}$.

The simplest pairs obtained by the Construction on a Bipartite Graph are the ones that start by subdividing a K_2 with the resulting pair of graphs having one nonthorn P_3 with an arbitrary, but same total number of terminal edges attached at each end; the smallest nonisomorphic pair obtained this way is $P_5 \cup P_3$ and the graph Y that results from deleting two terminal vertices from a subdivision of $K_{1,3}$. Note that the relationship between P_5 and Y occurs inside several of the

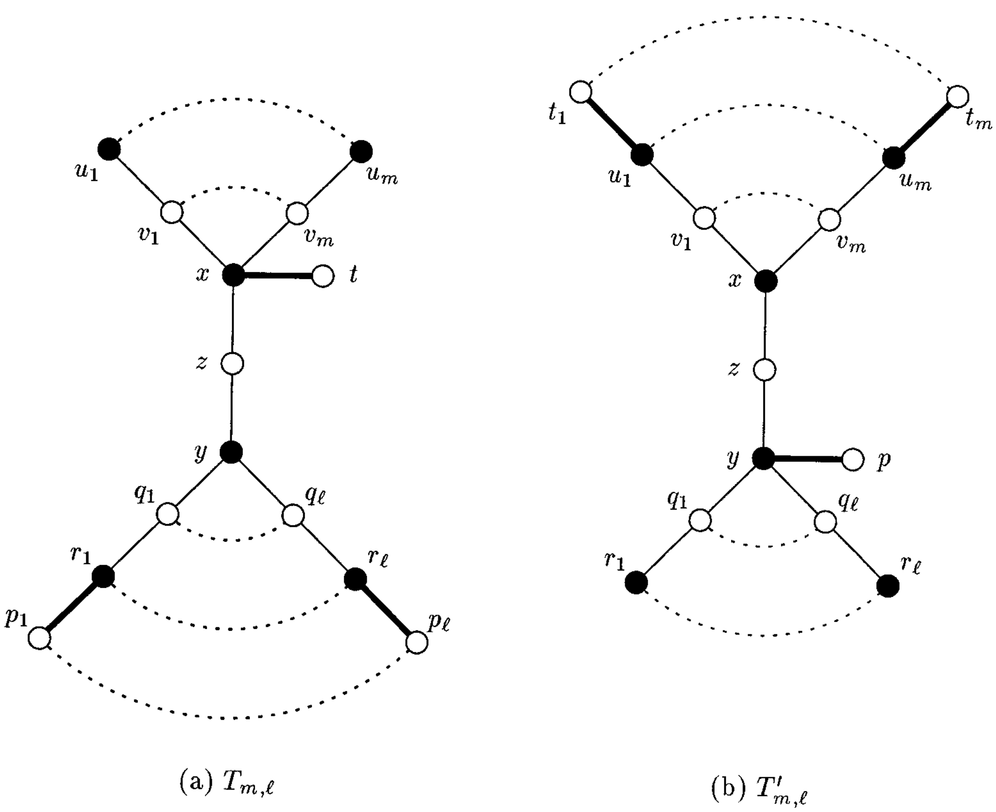

FIGURE 2. Broersma and Hoede's $T_{m,l}$ and $T'_{m,l}$ $(m > l \geq 0)$.

Generalized $K_{3,3}$ examples, and was used in describing them, but we do not consider it one of them because Situation 1 does not occur.

The Connected Case for the Construction on a Bipartite Graph. Using t_z for the number of terminal edges incident with z in G and requiring that both G and H be connected implies that

$$\sum_{a\in A}\binom{t_a}{2}+\sum_{b\in B}\binom{t_b}{2}=\sum_{a\in A}\binom{t_a-k}{2}+\sum_{b\in B}\binom{t_b+k}{2}$$

which reduces to

$$\sum_{a\in A}(2t_a-k-1)=\sum_{b\in B}(2t_b+k-1).$$

As with the Generalized $K_{3,3}$ and Whitney types, we will characterize the P_3-isomorphisms of bipartite type in terms of things that are not preserved. First we must investigate the consequences of the following assumption.

Additional Standing Assumption. For the rest of this section we assume that Situations 1 and 2 do not occur.

Under this assumption, we know that in many cases two P_3's with the same middle vertex will not be dispersed by τ, so that their images will have the same middle vertex. This allows us to 'bind' certain vertices of G to corresponding vertices in H. A P_3 is said to be *nonbinding* if it is one of a diamond pair, a terminal 1-thorn, or a 2-thorn; otherwise it is *binding.* Let $B(a)$ denote the set of binding P_3's in $S(a)$. A vertex a is *strongly bound* if $B(a)$ is nonempty. It is not difficult to see that there are only three (overlapping) types of vertices that are not strongly bound: first, terminal vertices; second, the central vertex of a star component; and third, a vertex a all of whose nonterminal neighbors belong to a single unbraced diamond $D_{a,b}$. In the third case, if there is some diamond path acb with both b and c strongly bound, we say that a is *weakly bound.* A vertex is *bound* if it is either strongly or weakly bound, and *unbound* otherwise. Since all diamond paths ending at a weakly bound vertex a are similar, all are binding; thus, every neighbor of a is either an unbound terminal vertex or a strongly bound vertex of degree 2, and the unique vertex b at distance 2 from a is strongly bound. Note that a weakly bound vertex may be a terminal vertex.

Lemma [Strong Binding].

(i) *A P_3-isomorphism maps binding P_3's to binding P_3's, and nonbinding P_3's to nonbinding P_3's.*

(ii) *For every strongly bound vertex a of G there is a unique strongly bound vertex of H, which we denote a', such that $\tau(B(a)) = B(a')$. Thus, if $\alpha \in \Pi_3(G)$ is binding with middle vertex a, then α' has middle vertex a'.*

(iii) *If a is strongly bound, deg$(b) \geq 2$, and $a \sim b$, then there exists a binding P_3 in $a \vdash b$.*

(iv) *If a is strongly bound and $\alpha \in \Pi_3(G)$ has an end at a, then α' has an end at a'.*

Proof. (i) This follows from (i) of the Diamond Lemma and the fact that α is a 2-thorn or terminal 1-thorn if and only if deg$(\alpha) \leq 1$.

(ii) Suppose $\alpha_1, \alpha_2 \in B(a)$. Let $a' = m(\alpha_1')$. First, suppose that one of α_1 or α_2, say α_1, has an end c of degree 3 or more. If c is also an end of α_2 then $m(\alpha_2') = a'$ since Situation 1 does not occur. If c is not an end of α_2', then α_2' has an end d of degree 2 or more. Let $\gamma = cad$. Then $m(\alpha_1') = m(\gamma')$ (since Situation 1 does not occur) and $m(\gamma') = m(\alpha_2')$ (since Situation

1 does not occur, if $\deg(d) \geq 3$, or since Situation 2 does not occur, if $\deg(d) = 2$). Therefore $m(\alpha_2') = a'$.

Second, suppose all ends of both α_1 and α_2 have degree 2. Write $\alpha_1 = c_1ac_2$ and $\alpha_2 = d_1ad_2$. Since α_1 is not one of a diamond pair, at most one of $\gamma_1 = c_1ad_1$ and $\gamma_2 = c_2ad_1$ is one of a diamond pair: suppose γ_1 is not one of a diamond pair. Then since Situation 2 does not occur, $m(\alpha_2') = m(\gamma_1') = m(\alpha_1') = a'$.

So $\tau(B(a)) \subseteq S(a')$, and by (i) above, $\tau(B(a)) \subseteq B(a')$. Applying the same reasoning to τ^{-1} yields $\tau^{-1}(B(a')) \subseteq B(a)$, so $\tau(B(a)) = B(a')$.

(iii) If $\deg(b) \geq 3$, then any P_3 in $a \vdash b$ is binding, so suppose $\deg(b) = 2$. Since a is strongly bound, there is some neighbour c of a which is either of degree 3 or more, or in a different diamond from b, and then bac is binding.

(iv) Suppose $\alpha = abc$. By (iii) there is some binding dab. Now $\alpha' = (abc)' \sim (dab)' = ua'v$ (for some u, v), so a' is an end of α'. ∎

Lemma [Weak Binding].

(i) *The ends of a binding diamond path are either both unbound, both strongly bound, or one strongly bound and one weakly bound.*

(ii) *For every weakly bound vertex a of G there is a unique weakly bound vertex of H, which we denote a', such that for every binding P_3 α with an end at a, α' has an end at a'.*

Proof. Part (i) follows from the definition of weakly bound. For (ii), let $D_{a,b}$ be the unique (necessarily unbraced) diamond in G with end a, having diamond paths $\alpha_i = ac_ib, i = 1, 2, \ldots, k$: these are the binding P_3's with an end at a. For each i, α_i' is binding and b', c_i' are strongly bound, so $\alpha_i' = xc_i'b'$ for some x by (ii) and (iv) of the Strong Binding Lemma. By (ii) of the Diamond Lemma, if $k \geq 2$ then x is the same for all i. Since b' is strongly bound and α_i' is binding, x is bound by (i). If x is strongly bound, then a is also strongly bound by (iv) of the Strong Binding Lemma, and so x is weakly bound. Now x has all the required properties, so we let $a' = x$. ∎

We may summarize the important implications of the Strong and Weak Binding Lemmas as follows.

Theorem [Binding]. *Let σ be the map $a \mapsto a'$ from the bound vertices of G to the bound vertices of H. Let a, b be bound vertices of G.*

(i) *The map σ is a bijection which maps strongly bound vertices to strongly bound vertices, and weakly bound vertices to weakly bound vertices.*

(ii) *For every P_3 α with an end at a, α' has an end at a'.*

(iii) *For every binding P_3 α with middle vertex a, α' has middle vertex a'.*

(iv) *$a \sim b$ if and only if $a' \sim b'$.*

Proof. Parts (i), (ii) and (iii) follow directly from the Strong and Weak Binding Lemmas applied to τ and τ^{-1}—note that (iii) is vacuously true when a is weakly bound. For (iv), if $a \sim b$ then one of a or b is strongly bound, and by the definition of weakly bound or by (iii) of the Strong Binding Lemma there is a binding P_3 containing the edge ab to which (ii) and (iii) can be applied. ∎

Theorem [Trivial Bipartite Type]. *Suppose Situations 1 and 2 do not occur, and there is an unbound nonterminal vertex in G_0 or H_0. Then τ is either D-related to a P_3-isomorphism of bipartite type starting from $F \cong K_2$ or is TBSD-related to an induced P_3-isomorphism.*

Proof. Suppose a is a nonterminal unbound vertex in G_0. Every edge incident with a is terminal or an edge of a unique unbraced diamond $D = D_{a,b}$, of width k. Consider a diamond path acb: if it is one of a diamond pair, then $G \cong H \cong C_4$; if it is a terminal 1-thorn then $G \cong H \cong P_4$. In either case τ is induced, so we may suppose acb is binding. Then b is unbound by (i) of the Weak Binding Lemma. Now G_0 consists only of D and possibly terminal edges incident with a and/or b. It follows (immediately when $k = 1$, and by (ii) of the Diamond Lemma if $k \geq 2$) that H_0 consists of a diamond $D_{u,v}$ of width k and possibly terminal edges incident with u and/or v. If $\{\deg(a), \deg(b)\} = \{\deg(u), \deg(v)\}$ then τ is TBSD-related to an induced P_3-isomorphism, and otherwise τ is D-related to some τ' of bipartite type starting from $F \cong K_2$. ∎

After eliminating the possibility of unbound nonterminal vertices, $\sigma : a \mapsto a'$ is very close to inducing τ. In the following lemma, we summarize what we can say about the images of P_3's of various types.

Lemma [Image]. *Suppose every nonterminal vertex is bound. Let $abc \in \Pi_3(G)$ with deg$(a) \geq$ deg(c).*

(i) *If abc is a 2-thorn, $(abc)'$ is a 2-thorn.*
(ii) *If abc is a terminal 1-thorn, with neighbor dab, then d is bound and $(abc)'$ is either $a'b'x$ or $a'd'x$ for some unbound x.*
(iii) *If abc is one of a diamond pair, the other being adc, then $(abc)'$ is either $a'b'c'$ or $a'd'c'$.*
(iv) *If abc is binding with all of a, b, c bound, then $(abc)' = a'b'c'$.*
(v) *If abc is binding with not all of a, b, c bound, then only c is unbound, and $(abc)' = a'b'x$ for some unbound x.*
(vi) *If abc is a diamond path that is not a 2-thorn or terminal 1-thorn, then it is binding, and a, b, c are all bound.*

Proof. Part (i) is obvious. Part (iv) follows from (ii) and (iii) of the Binding Theorem. For (v), only c is unbound since a and b are nonterminal, and $(abc)' = a'b'x$ by (ii) and (iii) of the Binding Theorem, where x is unbound since otherwise we could apply (iv) to τ^{-1}.

For (vi), a diamond path cannot also be one of a diamond pair, or else $G_0 \cong C_4$ which has unbound nonterminal vertices, so abc is binding and b is strongly bound. Also, deg$(a) \geq 2$ so a is bound, and hence c is bound by (i) of the Weak Binding Lemma.

For (iii), parts (vi) and (iv) apply to the diamond paths bad and bcd, so $N((abc)') = \{b'a'd', b'c'd'\}$, and (iii) follows.

For (ii), dab cannot be a terminal 1-thorn, otherwise $G_0 \cong P_4$ which has unbound nonterminal vertices, so (vi) and (iv) apply to dab. Now $(abc)' \sim (dab)' = d'a'b'$, so $(abc)' = a'b'x$ or $a'd'x$ for some x. If x is bound, then c is bound by (iii) of the Binding Theorem, but c is not strongly bound since it is terminal, and not weakly bound since there is no binding P_3 ending at c. Thus, x is unbound. ∎

Theorem [Nontrivial Bipartite Type]. *Suppose that Situations 1 and 2 do not occur and every nonterminal vertex is bound. Then there is a bound vertex a with deg$(a) \neq$ deg(a') if and only if τ is D-related to a P_3-isomorphism of bipartite type starting from $F \not\cong K_2$.*

Proof. The ‘if’part is not difficult, so consider the ‘only if’part. Without loss of generality, suppose that a is bound and $\deg(a') = \deg(a) + k$ for some $k > 0$. If a has a neighbour c of degree 3 or more, then a cannot be weakly bound, so a is strongly bound. By (ii) and (iii) of the Binding Theorem, applied to both τ and τ^{-1}, there is a bijection between $a \vdash c$ and $a' \vdash c'$, contradicting $\deg(a) \neq \deg(a')$.

Thus, all edges incident with a are terminal or edges of diamonds with an end at a. The same applies to a'. Let b be a vertex at distance 2 from a, which must be an end of a diamond path $\alpha = acb$. By (iii) of the Binding Theorem, $\alpha' = a'wx$ for some w, x. Now $\deg(a)+\deg(b)-2 = \deg(\alpha) = \deg(\alpha') = \deg(a') + \deg(x) - 2$, so $\deg(b) = \deg(x) + k \geq 2$, and b is bound. Thus, $x = b'$ and $\deg(b') = \deg(b) - k$. By (ii) of the Diamond Lemma, $D_{a,b} \cong D_{a',b'}$.

Now we can apply the same reasoning, replacing a with b'. Continuing in this way, we can decompose G_0 and H_0 into terminal edges with unbound ends and corresponding diamonds, each of which has $\deg(a') - \deg(a) = k$ for one end, a, and $\deg(b') - \deg(b) = -k$ for the other end, b. Thus, G_0 and H_0 are isomorphic to diamond inflations of some bipartite graph $F \not\cong K_2$, and from (i)–(v) of the Image Lemma, τ is D-related to τ' of bipartite type. ∎

6. CONCLUSION

Finally, we show that we have now identified all P_3-isomorphisms.

Main Theorem. *Let τ be a P_3-isomorphism from G to H with at least one of G or H connected. Then τ is one of the following*:

(i) *T-related to a P_3-isomorphism of generalized $K_{3,3}$-type*;
(ii) *of special Whitney type*;
(iii) *D-related to a P_3-isomorphism of Whitney type* 3, 4, 5 *or* 6;
(iv) *D-related to a P_3-isomorphism of bipartite type*; *or*
(v) *TBSD-related to an induced P_3-isomorphism.*

Proof. We may suppose that Situations 1 and 2 do not occur, every nonterminal vertex is bound, and $\deg(a') = \deg(a)$ for every bound a, as we have seen that (i)–(v) cover all other situations. By (iv) of the Binding Theorem, the subgraphs G_1 and H_1 induced by the bound vertices of G and H respectively are isomorphic, with isomorphism $\sigma : a \mapsto a'$. So G_0 and H_0 differ only in the placement of some terminal edges with unbound ends. But since $\deg(a) = \deg(a')$ for all bound vertices a, σ extends to an isomorphism from G_0 to H_0, which we still call σ. Applying (i)–(v) of the Image Lemma shows that τ is TBSD-related to σ^*. ∎

One special case is easily stated.

Corollary. *If τ is a P_3-isomorphism from G to H, where G has minimum degree at least* 3, *then $G \cong H$. Moreover*, *τ is induced unless τ is equivalent to τ_0 as in the Construction on $K_{3,3}$.*

It is also not difficult to extract a result for when G and H both have minimum degree at least 2; we omit the details.

We remark that although our results are stated for finite graphs, our reasoning appears to hold without change for infinite graphs that are locally finite (every vertex has finite degree). In this case, the situation for each infinite component must be as in (iv) or (v) of our Main Theorem. If we allow vertices of infinite degree, however, it becomes easier to construct P_3-isomorphisms by diamond inflation, including some that are of neither Whitney nor bipartite type. As the simplest example, consider two diamond inflations I and I' of a K_3 with vertices a, b, c, where $s_e = s'_e$ for each edge e, and each set $\{t_a, t_b, t_c\}$ and $\{t'_a, t'_b, t'_c\}$ has two countably infinite elements while the third element is finite or countably infinite. For each edge $e = uv$ we have $t_u + t_v = t'_u + t'_v = \aleph_0$, and so there is a P_3-isomorphism τ from I to I' that is a diamond inflation of the identity edge isomorphism of the K_3. In general τ is of a type not mentioned in our Main Theorem.

We conclude with some obvious questions about extensions of our work. Can we characterize P_k-isomorphisms for $k \geq 4$? What can we say about P_3-isomorphisms if we allow multiple edges? Or if we restrict ourselves to *induced* P_3's? Or if we allow P_3's to be adjacent whenever they share any edge? We know the answer to only the last of these questions. The graph with vertex set $\Pi_3(G)$, with two P_3's adjacent when they share an edge, is just $L(L(G))$, and it is a relatively simple exercise to characterize isomorphisms between $L(L(G))$ and $L(L(H))$ using Whitney's characterization of edge isomorphisms.

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