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Adjoining units to residuated Boolean algebras

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Dedicated to the memory of Alan Day

Abstract. We consider a variety $\mathscr{V}$ of r-algebras, – residuated Boolean algebras, – and ask under what conditions a member $\mathbf{A}$ of $\mathscr{V}$ can be embedded in a member $\mathbf{A}'$ having a unit element. The answer, although quite simple, is somewhat surprising for two reasons. First, to a large extent the answer is independent of the variety $\mathscr{V}$, as long as $\mathscr{V}$ is closed under canonical extensions. This is so because if any extension of $\mathbf{A}$ has a unit, then the canonical extension has a unit. The second surprise is that, for varieties $\mathscr{V}$ closed under canonical extensions, the members for which this extension has a unit form a subvariety with a very simple equational basis relatively to $\mathscr{V}$. Applied to the variety of all relation algebras, this latter result solves a problem of long standing due to A. Tarski. This problem was solved independently by H. Andréka and I. Németi.

1. Introduction

A binary operation $\circ$ on a Boolean algebra $\mathbf{A}_0 = (A, +, 0, 1, ^-)$ is said to be *residuated* if there exist binary operations $\backslash$ and $/$ on $\mathbf{A}_0$ such that, for all $x, y, z \in A$,

$$x \circ y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z/y.$$

Equivalently, $\circ$ is residuated iff there exist binary operations $\rhd$ and $\lhd$ on $\mathbf{A}_0$ such that, for all $x, y, z \in A$,

$$(x \circ y) \cdot z = 0 \quad \text{iff} \quad (x \rhd z) \cdot y = 0 \quad \text{iff} \quad (z \lhd y) \cdot x = 0.$$

These operations, if they exist, are unique, and they are connected by the formulas

$$x \rhd z = (x \backslash z^-)^-, \qquad x \backslash z = (x \rhd z^-)^-,$$

$$z \lhd y = (z^-/y)^-, \qquad z/y = (z^- \lhd y)^-.$$

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We refer to $\backslash$ and $/$ as the *right* and *left residuals* of $\circ$, and to $\rhd$ and $\lhd$ as the *right* and *left conjugates* of $\circ$. For technical reasons we prefer to treat $\rhd$ and $\lhd$ as basic operations, and we therefore refer to $\mathbf{A} = (\mathbf{A}_0, \circ, \rhd, \lhd)$ as a *residuated Boolean algebra*, or an *r-algebra*, if $\rhd$ and $\lhd$ are respectively, right and left conjugates of $\circ$. By a *unit* for $\mathbf{A}$ we mean a unit for the operation $\circ$, i.e., an element $e \in A$ with

$$e \circ x = x \circ e = x \qquad \text{for all } x \in A.$$

If e is a unit for $\mathbf{A}$, then $(\mathbf{A}_0, \circ, e, \rhd, \lhd)$ is said to be a *unital residuated Boolean algebra*, or a *ur-algebra*. If the operation $\circ$ is associative, then $\mathbf{A}$ is said to be associative. An associative *ur*-algebra is referred to as a *residuated Boolean monoid*, or an *rm-algebra*.

This note is concerned with the following question: Which r-algebras can be embedded in r-algebras having a unit or, more briefly, which r-algebras are subreducts of *ur*-algebras? Our main tool will be the canonical extension. A detailed discussion of this concept can be found e.g. in [2]. Most of the information needed here is contained in I–IV below. A term or an equation that does not involve the Boolean complementation is said to be *strictly positive* and an operation on $\mathbf{A}$ is said to be strictly positive if it is defined by a strictly positive term. The clone of all strictly positive operations on $\mathbf{A}$ is called the *strictly positive clone* of $\mathbf{A}$.

I. The canonical extension of an r-algebra $\mathbf{A} = (\mathbf{A}_0, \circ, \rhd, \lhd)$ is a complete and atomic r-algebra $\mathbf{A}^\sigma = (\mathbf{A}_0^\sigma, \circ, \rhd, \lhd)$ having $\mathbf{A}$ as a subalgebra, and if $\mathbf{A}$ has a unit e, then e is also a unit for $\mathbf{A}^\sigma$.
II. The map $F \mapsto \prod F$ is a bijection from the set of all ultrafilters F of $\mathbf{A}_0$ onto the set of all atoms of $\mathbf{A}_0^\sigma$.
III. The map $g \mapsto g^\sigma$ is an isomorphism from the strictly positive clone of $\mathbf{A}$ onto the strictly positive clone of $\mathbf{A}^\sigma$. In other words, for any strictly positive n-ary term t in the language of $\mathbf{A}$, $t^{\mathbf{A}^\sigma} = (t^{\mathbf{A}})^\sigma$. Hence, every strictly positive identity that holds in $\mathbf{A}$ holds also in $\mathbf{A}^\sigma$.
IV. If t is a strictly positive n-ary term in the language of $\mathbf{A}$, and if S is a down-directed subset of A^n, then

$$t^{\mathbf{A}^\sigma}(\prod S) = \prod \{t^{\mathbf{A}}(x) : x \in S\}.$$

We remind the reader that a residuated operation on a Boolean algebra preserves all joins. In particular, if the Boolean algebra is complete, then the operation is completely additive. Consequently, the operations $\circ$, $\rhd$ and $\lhd$ on the canonical extension of an r-algebra are completely additive in each argument.

Finally, we list some arithmetic properties of r-algebras that will be used later.

LEMMA 1.1. *Suppose* **A** *is an* r*-algebra and* $a, b, c \in A$. *The following inclusions hold*:

(i) $(a/b) \circ b \leq a$,

(ii) $a \circ (a \backslash b) \leq b$,

(iii) $(a \circ b) \cdot c \leq (a \cdot (c \lhd b)) \circ b$,

(iv) $(a \circ b) \cdot c \leq a \circ ((a \rhd c) \cdot b)$.

Proof. (i) and (ii) follow from the trivial inclusions $a/b \leq a/b$ and $a \backslash b \leq a \backslash b$. (iii) and (iv) are special cases of an observation about conjugate maps, i.e., maps f and g such that for all $x, y \in A$, $f(x) \cdot y = 0$ iff $x \cdot g(y) = 0$. The observation is that, for all $x, y \in A$, $f(x) \cdot y \leq f(x \cdot g(y))$. To prove this, let $z = f(x \cdot g(y))^{-}$. Then $f(x \cdot g(y)) \cdot z = 0$, $x \cdot g(y) \cdot g(z) = 0$, $x \cdot g(y \cdot z) = 0$ (because g is necessarily isotone), $f(x) \cdot y \cdot z = 0$, and finally $f(x) \cdot y \leq z^{-} = f(x \cdot g(y))$. To obtain (iii), take $f(x) = x \circ b$ and $g(y) = y \lhd b$, and to obtain (iv) take $f(x) = a \circ x$ and $g(y) = a \rhd y$. □

2. Adjoining one-sided units

We begin by considering embeddings of an r-algebra in an algebra with a one-sided unit, say with a right unit.

THEOREM 2.1. *Suppose* **A** *is an* r*-algebra. The following conditions are equivalent*:

(i) **A** *can be embedded in an* r*-algebra with a right unit.*

(ii) $\mathbf{A}^{\sigma}$ *has a right unit.*

(iii) *For all* $n \in \omega$ *and* $x, y_0, y_1, \ldots, y_n \in A$,

$$x \circ \prod \{y_i \backslash y_i : i \leq n\} \geq x.$$

Proof. Obviously (ii) implies (i), and if **A** can be embedded in an r-algebra $\mathbf{A}'$ with a right unit e, then for all $y \in A'$, $y \circ e \leq y$, whence $e \leq y \backslash y$. From this (iii) follows. Finally, assuming (iii), we want to show that the element $e = \prod \{y \backslash y : y \in A\}$ is a right unit for $\mathbf{A}^{\sigma}$. From IV it follows that the inclusion $x \leq x \circ e$ holds whenever x is the meet of elements of **A**. By II this inequality therefore holds whenever x is an atom of $\mathbf{A}^{\sigma}$, and using the fact that the operation $\circ$ is completely additive, we infer that it holds for every $x \in A^{\sigma}$. By Lemma 1.1(ii), $x \circ (x \backslash x) \leq x$, and hence $x \circ e \leq x$ for all x in A. Proceeding as with the opposite inclusion, we extend this first to elements of A^{σ} that are meets of subsets of A, and then to arbitrary elements of A^{σ}. Thus e is a right unit for $\mathbf{A}^{\sigma}$. □

3. Adjoining two-sided units

From Theorem 2.1 and the corresponding result for left units we obtain an infinite equational basis for the variety of all r-algebras that are subreducts of ur-algebras, but as we shall see, this variety is in fact finitely based.

LEMMA 3.1. *If* $\mathbf{A}$ *is an* r*-algebra satisfying the identities*

$$x \circ (y/y)(z/z) \geq x, \qquad x \circ (y \backslash y) \geq x, \tag{1}$$

then $\mathbf{A}^\sigma$ *has a right unit.*

Proof. Letting $E = \{u \in A : u \geq u/u\}$, we are going to show that the element $e = \prod E$ is a right unit in $\mathbf{A}^\sigma$. We first note that

$$u \in E \qquad \text{iff for all } x \in A \quad x \circ u \geq x. \tag{2}$$

The forward implication follows from the first formula in (1), and the backward implication is obtained by taking $x = u/u$, using the fact that by Lemma 1.1(i), $(u/u) \circ u \leq u$. Next, we note that

$$a \backslash a \in E \qquad \text{for all } a \in A. \tag{3}$$

Indeed, letting $u = a \backslash a$ we have $x \circ u \geq x$ for all $x \in A$ by the second formula in (1), and therefore $u \in E$ by (2). We now show that

$$E \text{ is closed under meets.} \tag{4}$$

Suppose $u, v \in E$. Then $u \geq u/u$ and $v \geq v/v$, and by Lemma 1.1(i), we always have $uv \geq (uv/uv) \circ uv$. Therefore, $uv \geq (uv/uv) \circ (u/u)(v/v) \geq uv/uv$ by the first formula in (1), so that $uv \in E$.

The remainder of the proof is essentially as the proof of Theorem 2.1. The set E is down-directed, and $x \circ u \geq x$ for all $u \in E$ and $x \in A$. Hence by IV, $x \circ e \geq x$ for all $x \in A$. The opposite inclusion also holds, because $x \backslash x \in E$ by (3), and $x \circ (x \backslash x) \leq x$ by Lemma 1.1(ii). Thus $x \circ e = x$ for all $x \in A$. By II and IV it follows that this equality holds whenever x is an atom of $\mathbf{A}^\sigma$, and we conclude by the complete additivity of $\circ$ that it holds for all $x \in A^\sigma$.

THEOREM 3.2. *For any* r*-algebra* $\mathbf{A}$, *the following conditions are equivalent*:

(i) $\mathbf{A}$ *is embeddable in an* r*-algebra with a unit.*

(ii) $\mathbf{A}^\sigma$ *has a unit.*

(iii) *For all* $x, y, z \in A$,

$$x \circ (y/y)(z/z) \geq x, \qquad x \circ (y\backslash y) \geq x,$$
$$(y\backslash y)(z\backslash z) \circ x \geq x, \qquad (y/y) \circ x \geq x.$$

Proof. By the preceding lemma, and by right-left symmetry. □

COROLLARY 3.3. *If* **B** *is a subalgebra of an r-algebra* **A**, *and if* $\mathbf{A}^\sigma$ *has a unit, then* $\mathbf{B}^\sigma$ *has a unit.*

COROLLARY 3.4. *If* **B** *is a finite subalgebra of an r-algebra with a unit, then* **B** *has a unit.*

Proof. If **B** is finite, then $\mathbf{B}^\sigma = \mathbf{B}$. □

We now consider *ur*-algebras rather than *r*-algebras with unit. The significance of this is that the language of *ur*-algebras has a constant denoting the unit element, and an equational basis for a variety of *ur*-algebras may contain identities that involve this constant.

THEOREM 3.5. *Suppose* $\mathscr{U}$ *is a variety of ur-algebras. Let* $\mathscr{K}$ *be the class of all r-algebras that are reducts of members of* $\mathscr{U}$, *and let* $\mathscr{V}$ *be the variety generated by* $\mathscr{K}$. *Then* $\mathscr{V} = \mathbb{S}(\mathscr{K})$. *If* $\mathscr{U}$ *is canonical, then* $\mathscr{V}$ *is canonical and, for every r-algebra* **A**,

$$\mathbf{A} \in \mathscr{V} \qquad \textit{iff } \mathbf{A}^\sigma \in \mathscr{K}.$$

Proof. The claim $\mathscr{V} = \mathbb{S}(\mathscr{K})$ is equivalent to the assertion that the class of all *r*-algebras that are subreducts of algebras in $\mathscr{U}$ is a variety. In fact, for any variety $\mathscr{W}$ of algebras, and for any subtype τ of the type of $\mathscr{W}$, the class $\mathscr{H}$ of all algebras of type τ that are subreducts of algebras in $\mathscr{W}$ is closed under $\mathbb{S}$ and $\mathbb{P}$. In the present case, $\mathscr{H} = \mathbb{S}(\mathscr{K})$ is also closed under $\mathbb{H}$. This follows from the following two observations. First, $\mathbb{S}(\mathscr{K})$ has the congruence extension property (CEP) and second, a member **A** of $\mathscr{U}$ and its reduct have the same congruence lattice. To clarify the first statement, an algebra **A** has CEP if every congruence relation on a subalgebra of **A** can be extended to a congruence relation on **A**, and a class of algebras has CEP if all of its members have CEP. The statement results from the fact that $\circ$, $\lhd$, and $\rhd$ are additive and normal (take the value 0 whenever one of the arguments is 0). Indeed, if $\mathbf{A} = (\mathbf{A}_0, f_i, i \in I)$, where $\mathbf{A}_0$ is a Boolean algebra and the operations f_i are additive and normal, then **A** has CEP. This follows from the facts that a congruence relation R on $\mathbf{A}$ is completely determined by the congruence ideal $0/R$, that a Boolean ideal is a congruence ideal if and only if it is closed under the operation f_i, and that if J is a congruence ideal of a subalgebra of $\mathbf{A}$, then the Boolean ideal generated by J is a congruence ideal. The second statement is trivial. The only difference between $\mathbf{A}$ and its reduct is that in the former the unit is treated as a distinguished element, but in the latter it is not.

Now suppose $\mathscr{U}$ is canonical. Then $\mathscr{K}$ is obviously closed under canonical extensions, and hence so is $\mathscr{V}$. Furthermore, every member of $\mathscr{K}$ has a unit, and every member of $\mathscr{V}$ therefore can be embedded in an r-algebra with a unit. From this it follows by Theorem 3.2 that, for every $\mathbf{A} \in \mathscr{V}$, the member $\mathbf{A}^\sigma$ of $\mathscr{V}$ has a unit, whence $\mathbf{A}^\sigma \in \mathscr{K}$. The converse is obvious: If the algebra $\mathbf{A}^\sigma$ is in $\mathscr{K}$, then the subalgebra $\mathbf{A}$ of $\mathbf{A}^\sigma$ is in $\mathscr{V}$. □

4. Specification algebras

Relation algebras, as defined by A. Tarski, are algebras $\mathbf{A} = (\mathbf{A}_0, \circ, e, {}^\cup)$ such that $\mathbf{A}_0 = (A, +, 0, \cdot, 1, {}^-)$ is a Boolean algebra, $(A, \circ, e)$ is a monoid, and $\circ$ is residuated, its right conjugate being $a \rhd x = a^\cup \circ x$ and its left conjugate $x \lhd a = x \circ a^\cup$. More compactly, $\mathbf{A}$ is a relation algebra iff the algebra $(\mathbf{A}_0, \circ, e, \rhd, \lhd)$ is an rm-algebra. Over twenty years ago, Tarski conjectured in a private conversation with some of his students that the set of identities holding in all relation algebras and not containing the constant denoting the unit had a finite basis. Using the results from the preceding sections, we shall verify this conjecture.

DEFINITION 4.1. By a *specification algebra* we mean an algebra $\mathbf{A} = (\mathbf{A}_0, \circ, {}^\cup)$ that satisfies all the identities that hold in every relation algebra and do not contain the constant denoting the unit.

The reason for this terminology is that it appears that these algebras are suitable for an abstract treatment of program specifications.

An axiomatic characterization of specification algebras is easily obtained by applying Theorem 3.2 to the r-algebra $(\mathbf{A}_0, \circ, \rhd, \lhd)$ with $\rhd$ and $\lhd$ defined as above. The operations $\backslash$ and $/$ are of course the left and right residuation,

$$a \backslash b = (a^\cup \circ b^-)^-, \qquad a/b = (a^- \circ b^\cup)^-.$$

It will be shown that in this case the four conditions in Theorem 3.2(iii) can be replaced by a special case of the first condition (the case $z = y^-$), provided we also postulate some simple properties of the operation ${}^\cup$, which of course hold in all relation algebras, and therefore in all specification algebras.

THEOREM 4.2. *For any algebra* $\mathbf{A} = (\mathbf{A}_0, \cdot, {}^\smile)$, *the following conditions are equivalent*:

(i) $\mathbf{A}$ *is a specification algebra.*

(ii) $\mathbf{A}$ *is a subreduct of a relation algebra.*

(iii) $\mathbf{A}^\sigma$ *is a reduct of a relation algebra.*

(iv) *The algebra* $(\mathbf{A}_0, \ , \triangleright, \triangleleft)$ *with* $a \triangleright b = a^\smile \cdot b$ *and* $a \triangleleft b = a \cdot b^\smile$ *for all* $a, b \in A$ *is an r-algebra, and the following conditions hold for all* $a, b, c \in A$.

(iv_1) $a \ \ (b \ \ c) = (a \ \ b) \ \ c$

(iv_2) $(a \ \ b)^\smile = b^\smile \ \ a^\smile$

(iv_3) $(a + b)^\smile = a^\smile + b^\smile$

(iv_4) $a^{\smile\smile} = a$

(iv_5) $a \ \ (b/b)(b^-/b \ \ ) \geq a.$

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv) are obvious. In order to prove that (iv) implies (iii), we are going to show that if (iv) holds, then the four inclusions in Theorem 3.2(iii) are also satisfied.

By (iv_3) and (iv_4), the map $^\smile$ is an isotone bijection from A to A, and it is therefore an automorphism of the Boolean algebra $\mathbf{A}_0$. In particular,

$$(a \cdot b)^\smile = a^\smile \cdot b^\smile, \qquad a^{-\smile} = a^{\smile -}. \tag{1}$$

Using this, we easily see that

$$b \ c = b^{-\smile}/c^{-\smile}, \qquad (b/c)^\smile = c^-/b^- = c^\smile \backslash b^\smile. \tag{2}$$

E.g., $b \backslash c = (b^\smile \ \ c^-)^-$ and $b/c = (b^- \ \ c^\smile)^-$, so that $b^{-\smile}/c^{-\smile} = (b^\smile \cdot c^-)^{\ } = b \ \ c$.

Using (1) and (2), we show that the first inclusion in Theorem 3.2(iii) implies the other three. Thus we assume that

$$a \cdot (b/b)(c/c) \geq a. \tag{3}$$

Applying $^\smile$ to both sides, we obtain

$$(b^\smile \ \ b^\smile)(c^\smile \ \ c^\smile) \cdot a^\smile \geq a^\smile$$

and hence, replacing a, b, and c by their converses,

$$(b \ \ b)(c \ \ c) \cdot a \geq a. \tag{4}$$

From (3) it follows that $a \cdot (b/b) \geq a$ and replacing b by $b^{-\smile}$ we infer by (2) that $a \cdot (b \ \ b) \geq a$. Similarly, (4) implies that $(b/b) \cdot a \geq a$.

Our problem has now been reduced to showing that (3) follows from (iv). Letting $u = (b/b)(b^-/b^-)$ and $v = (c/c)(c^-/c^-)$, we have

$$a \circ u \geq a \quad \text{and} \quad a \circ v \geq a \tag{5}$$

by (iv_5), and it suffices to show that

$$a \circ uv \geq a. \tag{6}$$

We begin by showing that u is an equivalence element, i.e., that $u \circ u \leq u = u^\cup$. By Lemma 1.1 (i), $u \circ b \leq b$, whence $u \circ u \circ b \leq b$ by (iv_1). Hence $u \circ u \leq b/b$. By symmetry, $u \circ u \leq b^-/b^-$, so that $u \circ u \leq u$. By (2), $u = (b/b)(b/b)^\cup$, whence $u^\cup = u$. Similarly, v is an equivalence element, and hence so is uv.

We next show that $1 \circ uv = 1$. By (5), $u \leq u \circ v$, whence

$$u = (u \circ v)u = (u \circ uv + u \circ u^- v)u.$$

But $(u \circ u^- v)u = 0$, because $(u^\cup \circ u)u^- v \leq uu^- v = 0$, and we therefore have $u = (u \circ uv)u$, or $u \leq u \circ uv$. Consequently, since (5) also implies $1 \circ u = 1$,

$$1 \circ uv \geq 1 \circ u \circ uv \geq 1 \circ u = 1.$$

To complete the proof, we show that, for any equivalence element w, $1 \circ w = 1$ implies $a \circ w \geq a$. Using Lemma 1.1(iii), (iv_1), and the fact that w is an equivalence element, we have

$$a = a \cdot (1 \circ w) \leq ((a \lhd w) \cdot 1) \circ w = a \circ w \circ w \leq a \circ w.$$

Taking $w = uv$, we obtain (6), and the proof is complete. □

This solves Tarski's problem. According to Maddux [4], Tarski had considered the formula (3) in the above proof, and also the formula $x \circ (y/y) \geq x$, and he believed that he had at one time shown that one of them (he was not sure which one) characterizes what is here called specification algebras.

A relation algebra can be reconstructed from the associated rm-algebra by using the fact that $a^\cup = a \rhd e$. Consequently, the variety of all relation algebras is definitionally equivalent to a variety of rm-algebras. In fact, by [3], Theorem 5.3, the variety of all relation algebras is definitionally equivalent to the variety of all ur-algebras that satisfy the equivalent identities $a \rhd (b \circ c) = (a \rhd b) \circ c$ and $a \circ (b \lhd c) = (a \circ b) \lhd c$.

5. A counterexample

In view of Theorem 3.2, one might expect that the infinite set of inclusions in Theorem 2.1 could be replaced by a finite set. It will be shown below that this is not the case. In other words, we are going to show that, for any positive integer n, there exists an r-algebra satisfying the inclusion

$$x \circ \prod \{y_i \backslash y_i : i \leq n\} \geq x \tag{1_n}$$

that cannot be embedded in an r-algebra with a right unit. The following lemma will be used.

LEMMA 5.1. *For any r-algebra* $\mathbf{A}$, *the following conditions are equivalent.*
(i) *For all $x \in A$, if $x \neq 0$, then $x \rhd x = 1$.*
(ii) *For all $x, y \in A$, if $y \neq 0$, then $x \leq x \circ y$.*

Proof. Assume (i). Then for all non-zero elements $x, y \in A$, $(x \rhd x) \cdot y \neq 0$, or equivalently, $(x \circ y) \cdot x \neq 0$. Letting $z = x \cdot (x \circ y)^-$, we infer that $(z \circ y) \cdot z \leq (x \circ y) \cdot (x \circ y)^- = 0$, hence $z = 0$, or equivalently, $x \leq x \circ y$.
Conversely, assume that (ii) holds. Then, for any non-zero elements $x, y \in A$,

$x \cdot (x \circ y) \neq 0$, or equivalently, $(x \rhd x) \cdot y \neq 0$.

Since $(x \rhd x) \cdot (x \rhd x)^- = 0$, it follows that $(x \rhd x)^- = 0$, and hence $x \rhd x = 1$.
□

THEOREM 5.2. *The variety of all r-algebras that can be embedded in r-algebras with a right unit is not finitely based.*

Proof. We are going to construct a sequence of r-algebras $\mathbf{A}_m$ such that each of the conditions (1_n) holds in $\mathbf{A}_m$ for sufficiently large m, but each of the algebras $\mathbf{A}_m$ violates (1_n) for sufficiently large n. These algebras will be finite; in fact, the set X of all atoms in $\mathbf{A}_m$ will be of order $(2^m)^2 - 2^m$. To complete the definition of $\mathbf{A}_m$, it suffices to describe the way one of the three operations $\circ$, $\rhd$, $\lhd$ acts on the atoms. We are going to consider the operation $\rhd$. Choose a subset U of X of order 2^m, let V be the set of all non-diagonal elements of $U \times U$, and noting that V and X have the same order, fix a bijection ϕ from V onto X. For $p, q \in X$, define

$$p \rhd q = \begin{cases} 1 & \text{if } p = q \\ \phi(p, q) & \text{if } (p, q) \in V \\ 0 & \text{otherwise.} \end{cases}$$

The algebra $\mathbf{A}_m$ satisfies the condition (i) in Lemma 5.1, and therefore also the condition (ii). It follows that (1_n) fails in $\mathbf{A}_m$ iff there exist elements $a_i \in A_m$, for $i < n$, such that the meet of the elements $a_i \backslash a_i$ is 0, or equivalently, such that

$$\sum \{a_i \rhd a_i^- : i < n\} = 1. \tag{1}$$

For $i < n$, let U_i be the set of all $p \in U$ with $p \leq a_i$. Then, for $(p, q) \in V$,

$$\phi(p, q) \leq a_i \rhd a_i^- \qquad \text{iff } p \in U_i \text{ and } q \notin U_i.$$

Consequently, (1) holds iff

$$V = \bigcup \{U_i \times (U - U_i) : i < n\}. \tag{2}$$

For $i < n$, let ψ_i be the map from U into $\{0, 1\}$ that takes each member of U_i into 0 and every member of $U - U_i$ into 1, and let ψ be the induced map from U into $P = \{0, 1\}^n$. Then (2) holds iff for all distinct $u, v \in U$, $\psi(u)$ and $\psi(v)$ are incomparable in the poset P, in other words ψ is an injection that maps U onto an antichain in P. If $n < m$, then U has more elements than P, and no such map ψ exists. In this case, $\mathbf{A}_m$ therefore satisfies (1_n). On the other hand, (1) holds for $n = 2^m$ if we take the a_i's to be the n distinct members of U, and (1_n) therefore fails in this case, and of course also for $n > 2^m$. This completes the proof. □

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