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Small representations of the relation algebra $\mathscr{E}_{n+1}(1, 2, 3)$

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Abstract. Applying combinatorial methods, we prove that the symmetric relation algebra $\mathscr{E}_{n+1}(1, 2, 3)$ of $n + 1$ atoms is finitely representable for all $n \geq 1$, on at most $(2 + o(1))n^2$ elements as $n \to \infty$. We explicitly construct a representation of size $\leq 4.5n^2$, for every $n \geq 1$.

The finite relation algebra $\mathscr{E}_{n+1}(1, 2, 3)$ is defined as follows. The atoms of $\mathscr{E}_{n+1}(1, 2, 3)$ are 1', $a_1, a_2, \ldots, a_n$, and $\mathscr{E}_{n+1}(1, 2, 3)$ is symmetric, i.e., $\breve{x} = x$ for every x. If x and y are distinct atoms different from 1', then $x; y = 0'$ and $x; x = 1$. Thus, the product of any two elements is as big as possible.

R. D. Maddux proved in [4] that $\mathscr{E}_{n+1}(1, 2, 3)$ is representable, and asked whether $\mathscr{E}_{n+1}(1, 2, 3)$ is representable over a finite set. In [6] he remarks (p. 182) that the answer is "yes". In this note we give two proofs of this result: a constructive and a nonconstructive one, the former based on geometric ideas inspired by results of R. Lyndon in [3], and the latter one based on probabilistic techniques which originated with Erdös, and which were also used by Maddux to answer his own question. With regards to our constructive proof, we wish to point out that H. Andréka also has explicit constructions (which produce representations of $\mathscr{E}_{n+1}(1, 2, 3)$ on Cn^5 points), essentially different from ours (private communication, 1989) but they are more complicated and would make this note unreasonably long. For readers familiar with Lyndon's algebras $\mathscr{L}_1(q)$ constructed from a projective line with $q + 1$ points, we note that our proof arises from the observation that $\mathscr{E}_{n+1}(1, 2, 3)$ is a proper subalgebra of $\mathscr{L}_1(q)$ for $q + 1 \geq 2n$. As noted by Jónsson [2] (p. 277), every proper subalgebra of $\mathscr{L}_1(q)$ is (finitely) representable.

Finite representations of $\mathscr{E}_{n+1}(1, 2, 3)$ are useful because they can be taken as part of other constructions. Examples of those applications are given in [1], where finite representations of $\mathscr{E}_{n+1}(1, 2, 3)$ are used in the construction of a proper relation algebra which cannot be represented in such a way that each of its automorphisms comes from a permutation of the base set.

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Representations of the relation algebra $\mathscr{E}_{n+1}(1, 2, 3)$ arise from certain edge-colorings f of complete undirected graphs with n colors, in which, for any two distinct vertices u and v and any two colors i and j (which may be the same), there is a third vertex w such that $f(u, w) = i$ and $f(v, w) = j$. (See [5] and [7].)

THEOREM 1. (i) *For every* $n \geq 0$, $\mathscr{E}_{n+1}(1, 2, 3)$ *is finitely representable.*

(ii) *The smallest representation of* $\mathscr{E}_{n+1}(1, 2, 3)$ *has at least* $n^2 + n + 1$ *and at most* $4.5n^2$ *vertices. Moreover, the upper bound can be improved to* $(2 + o(1))n^2$ *as* $n \to \infty$.

(iii) *For every* $\epsilon > 0$ *there is an* $n(\epsilon)$ *such that for all* $n \geq n(\epsilon)$ *and all* $N \geq (6 + \epsilon)n^2 \log n$, *almost all edge colorings of the complete graph on N vertices with n (or at most n) colors are representations of* $\mathscr{E}_{n+1}(1, 2, 3)$.

Proof. Part (i) follows from either of (ii) or (iii).

(ii): We prove the lower bound by showing that every vertex u has degree at least $n + 1$ in each color i. Trivially, u is adjacent to some v in color i. Then for every $j \in \{1, \ldots, n\}$ there is a w with $f(u, w) = i$ and $f(v, w) = j$. For distinct j, those w must be distinct, providing n further edges of color i incident to u.

Coloring each pair of a 3-element set with color 1, the upper bound is trivial for $n = 1$. Hence, assume $n \geq 2$. Let q be the smallest prime power not smaller than $2n - 1$. Consider a finite affine plane, say the Galois plane $AG(q)$, of order q. This plane has $q + 1 \geq 2n$ parallel classes $\Lambda_1, \ldots, \Lambda_{q+1}$ of lines. Partition the lines into n groups $\Gamma_1, \ldots, \Gamma_n$, each of which is the union of two or more parallel classes. Recall that each pair (u, v) of points in $AG(q)$ is contained in precisely one line. Define $f(u, v) = i$ if and only if the line passing through u and v belongs to Γ_i.

We prove that f is a representation of $\mathscr{E}_{n+1}(1, 2, 3)$. Since every line of $AG(q)$ has $q \geq 2n - 1 \geq 3$ points, every edge is contained in a monochromatic triangle. Let L be the line containing the vertices u and v, and let i and j be two arbitrary colors. Take any two nonparallel lines $L_i \in \Gamma_i \setminus \{L\}$ and $L_j \in \Gamma_j \setminus \{L\}$ such that u is on L_i and v is on L_j. (This is always possible because each group contains at least two parallel classes.) Then the unique point $w \in L_i \cap L_j$ satisfies the requirements.

The upper bounds $4.5n^2$ and $(2 + o(1))n^2$ on the number of vertices follow from a stronger variant of Chebycheff's classical theorem on the distribution of primes. (Analysis of small cases shows that an interval $[2n - 1, 2.2n)$ always contains a prime power. For large n, there exists a prime p with $p - 2n = o(n)$.)

(iii): For a fixed $\epsilon > 0$ and a desired proportion of random graphs that are representations, the choice of $n(\epsilon)$ is delayed until the end of the proof. Let $n > n(\epsilon)$ and let $K = (V, E)$ be a complete graph on N vertices, where $N \geq (6 + \epsilon)n^2 \log n$.

Assign independent random variables $\xi(u, v)$ to the edges $(u, v) \in E$, such that for every (u, v) and every $i \in \{1, \ldots, n\}$ the probability $Pr(\xi(u, v) = i)$ is equal to $1/n$. We are going to prove that the probability of the event that the random coloring $\mathscr{C} = \{\xi(u, v) \mid (u, v) \in E\}$ is a representation of $\mathscr{E}_{n+1}(1, 2, 3)$ tends to 1 as $N \to \infty$. Note that in this model each particular coloring with $\leq n$ colors occurs with the same probability. (If colorings were assumed to have precisely n colors, then the probability of a particular coloring would be larger, but we would lose the advantage of having totally independent random variables on E.)

Let $u, v \in V$ be any two distinct vertices of K. Then for any two (not necessarily distinct) fixed colors $i, j \in \{1, \ldots, n\}$ and any $w \in V \backslash \{u, v\}$ we have $Pr(\xi(u, w) \neq i$ or $\xi(u, w) \neq j) = 1 - n^{-2}$. If u and v are fixed, then the independence of the random variables $\xi(u, v)$ implies that

$$Pr(\xi(u, w) \neq i \text{ or } \xi(v, w) \neq j \text{ for all } w \in V \backslash \{u, v\})$$
$$= (1 - n^{-2})^{N-2} = ((1 - n^{-2})^{n^2})^{(N-2)/n^2} < e^{-(N-2)/n^2}.$$

Since the edge (u, v) can be chosen in $\binom{N}{2} < N^2/2$ different ways, and there are n^2 possible choices for the ordered pair (i, j) of colors, we obtain

$$Pr(\exists i, j \in \{1, \ldots, n\}, \exists u, v \in V \text{ with } \xi(u, w) \neq i \text{ or } \xi(v, w) \neq j$$
$$\text{for all } w \in V \backslash \{u, v\}) < N^2 n^2 / (2e^{(N-2)/n^2}) = P(n, N).$$

Observe that for $N \geq (6 + \epsilon) n^2 \log n$ the function $P(n, -)$ is decreasing. Thus the proportion of the colorings which do not provide a representation of $\mathscr{E}_{n+1}(1, 2, 3)$ is less than $P(n, (6 + \epsilon) n^2 \log n) = O(n^{-\epsilon} \log^2 n)$, and by choosing $n(\epsilon)$ large enough this proportion can be made as small as desirable. □

It remains an open problem to determine the smallest constant c $(1 \leq c \leq 2)$ such that $\mathscr{E}_{n+1}(1, 2, 3)$ has a representation on $(c + o(1)) n^2$ elements as $n \to \infty$.

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