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ULTRAPRODUCTS OF ATOMIC BOOLEAN ALGEBRAS

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0. Introduction

It follows from Tarski's characterization of elementary equivalent Boolean algebras (see [1] Theorem 5.5.10 p. 300), that every two infinite atomic Boolean algebras are elementary equivalent, and therefore the theory of infinite atomic Boolean algebras is complete. Thus if there exists a saturated infinite atomic Boolean algebra of cardinality α, then it is unique up to isomorphism.

In Section 3 we "identify" the saturated atomic Boolean algebra of regular power $\alpha>\omega$ (we denote it by B_α). If α is a successor cardinal then B_α is shown to be isomorphic to an ultraproduct of finite Boolean algebras. For α inaccessible B_α is the union of the elementary chain $\{B_\gamma : \gamma \text{ is a successor cardinal} <\alpha\}$. It is also shown that B_α is ω_1-incomplete for any regular power $\alpha>\omega$.

In Section 4 it is shown that if B is any infinite atomic Boolean algebra and α is a regular cardinal such that $|B|<\alpha$ ($|B|\leq\alpha$ if α is successor) then B is elementary embeddable in B_α.

Finally, all the results are proven under the assumption of the generalized continuum hypothesis (CCH).

1. Preliminaries

1.1. Notation and terminology

For a Boolean algebra B we denote by 0 and 1 the bottom and top element of B respectively. If $a, b\in B$ then we let $a+b$ and ab be the join and meet of a and b.

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We refer the reader to [1] and [5] for the following concepts concerning Boolean algebras and Model Theory:

(i) Boolean algebras: atomic, α–complete (α–incomplete).

(ii) Model Theory: ultraproducts, good ultrafilters, elementary chains, saturated and special models.

1.2. Required results

In this paper we need the following wellknown results about the structure of atomic Boolean algebras:

THEOREM 1.2.1. *Let A be the set of atoms of an atomic Boolean algebra B. Then*

(i) *B is complete if and only if $B \cong \mathcal{P}(A)$.*

(ii) *every element of B is the (possibly infinite) join of all the atoms less than it.*

We also require the following results from model theory. For more details the reader is referred to [1] as well as the papers indicated.

THEOREM 1.2.2 (Morley and Vaught [4]). *Two elementary equivalent saturated models of the same cardinality are isomorphic.*

THEOREM 1.2.3 (Keisler [2]). *Let $|I|=\beta$ and let $E \subseteq \mathcal{P}(A)$ such that $|E| \leq \beta$, every element of E has power β and E is closed under finite intersections. Then E can be extended to a β^+–good ultrafilter.*

THEOREM 1.2.4 (Keisler [3]). *Let α be an infinite cardinal and let $\mathcal{D}$ be a countably incomplete α–good ultrafilter over a set I. Suppose the power of the language $\mathcal{L}$ is less than α. Then for any family A_i ($i \in I$) of models of $\mathcal{L}$, the ultraproduct $\Pi_{\mathcal{D}} A_i$ is α–saturated.*

2. The ultraproduct

2.1. The ultrafilter

Let B be an infinite atomic Boolean algebra, and consider the set

$$I = \{i \in B : i \text{ is the join of a finite set of atoms of } B\}.$$

Notice that Theorem 1.2.1 implies that I is a sublattice of B, in fact I is the sublattice of compact elements of B. Also the structure of I depends only on the cardinality of the set of atoms of B. For $i \in I$ consider the set

$$I_i = \{j \in I : j \geq i\}.$$

Then the collection

$$E = \{I_i : i \in I\}$$

has the finite intersection property. In fact, for $i_i, \ldots, i_n \in I$ and $x = i_i + \ldots + i_n$ we have

$$\bigcap_{k=1}^{n} I_{i_k} = I_x.$$

We denote by $\mathcal{D}$ an ultrafilter over I with basis E. Since for any $i \in I$ we have $|I_i| = |I| = |E| = \beta$ say, we may assume that $\mathcal{D}$ is a β^+-good ultrafilter (see Theorem 1.2.3).

2.2. The ultraproduct $\Pi_{\mathcal{D}}(i]$

Let B and I be as in 2.1. For $i \in I$ consider the principal ideal

$$(i] = \{j \in B : j \leq i\}.$$

Clearly $(i]$ is isomorphic to a finite Boolean algebra, and $(i] \subseteq I$ for all $i \in I$ (see Theorem 1.2.1). Consider the product.

$$\prod_{i \in I} (i].$$

A function $f : I \to I$ is a member of this product if and only if it satisfies $f(i) \leq i$ for all $i \in I$. We denote by

$$\prod_{\mathcal{D}} (i]$$

an ultraproduct, where $\mathcal{D}$ is any ultrafilter from 2.1.

2.3. The cardinality of the ultraproduct

Let B be any infinite atomic Boolean algebra with $I, \mathcal{D}$ and $\Pi_{\mathcal{D}}(i]$ as in 2.2. For $a \in B$ we consider

$$f_a / \mathcal{D} \in \prod_{\mathcal{D}} (i]$$

where $f_a \in \prod_{i \in I} (i]$ is given by $f_a(i) = ai$ for all $i \in I$.

Lemma 2.3.1. *If $a, b \in B$ and $a \neq b$ then $f_a / \mathcal{D} \neq f_b / \mathcal{D}$.*

Proof. Suppose $a \nleq b$ then for some $i \in I$ we have $i \leq a$ and $i \nleq b$. Then for any $j \in I_i$, $i \leq f_a(j)$ and $i \nleq f_b(j)$.

Hence $\{j\in I : f_a(j) \neq f_b(j)\} \supseteq I_i \in \mathcal{D}$ so the result follows.

COROLLARY 2.3.2. *Suppose that I has cardinality β and let C be any infinite atomic Boolean algebra with $|C| \leq 2^\beta$. Then*

$$|\Pi_{\mathcal{D}}(i]| = |\Pi_{\mathcal{D}}C| = 2^\beta.$$

Proof. Here we assume that B is complete, with set of atoms A so $|A| = |I| = \beta$ and $|B| = |\mathcal{P}(A)| = 2^\beta$ (from Theorem 1.2.1). By the preceeding lemma $2^\beta = |B| \leq |\Pi_{\mathcal{D}}(i]|$, but we also have

$|\Pi_{\mathcal{D}}(i]| \leq |\Pi_{\mathcal{D}}C| \leq |C^I| \leq (2^\beta)^\beta = 2^\beta$ so the result follows.

3. Saturated atomic Boolean algebras of regular power

As in the introduction we denote the unique saturated atomic Boolean algebra of power $\alpha \geq \omega$ by B_α (if it exists). For α a successor cardinal, $\alpha = \beta^+$, denote by $\mathcal{D}_\alpha$ an α-good ultrafilter as defined in 2.1. ($\mathcal{D}_\alpha$ can be constructed on the set I of compact elements of any infinite atomic Boolean algebra for which $|I| = \beta$). The next theorem identifies B_α for regular α.

THEOREM 3.1. *Let α be an uncountable regular cardinal.*

(i) *If α is a successor cardinal then*

$$B_\alpha \cong \prod_{\mathcal{D}_\alpha} (i] \cong \prod_{\mathcal{D}_\alpha} C$$

for any infinite atomic Boolean algebra C with $|C| \leq \alpha$.

(ii) *If a is a limit regular (inaccessible) cardinal then*

$$B_\alpha \cong U\{B_\gamma \mid \gamma \textit{ is a successor cardinal} < \alpha\}.$$

Proof. (i) Suppose $\alpha = \beta^+$ then by Theorem 1.2.4 both $\prod_{\mathcal{D}_\alpha}(i]$ and $\prod_{\mathcal{D}_\alpha} C$ are β^+-saturated and by Corollary 2.3.2 they are both of cardinality 2^β. Hence by GCH they are saturated. It now follows from Theorem 1.2.2 and the completeness of the theory of infinite atomic Boolean algebras that they are isomorphic.

(ii) Let $\gamma < \delta$ be two successor cardinals then by (i) above it follows that $B_\delta \cong \prod_{\mathcal{D}_\delta} B_\gamma$ so B_γ can be regarded as an elementary subalgebra of B_δ. Thus the collection $\varphi = \{B_\gamma \mid \gamma$ is a successor cardinal $< \alpha\}$ forms a specializing chain for the special model $\cup\varphi$ which is of cardinality

α. But special models of regular limit power are saturated (see [1] p. 217. Prop. 5. 16(iv)) so $\bigcup\varphi\cong B_\alpha$.

In the remainder of this section we show that none of the Boolean algebras in Theorem 3. 1 are complete. Let B, I and E be as in 2. 1 and suppose $\mathcal{D}$ is any ultrafilter having base E. We construct acountable strictly increasing sequence of elements of $\Pi_{\mathcal{D}}(i]$ which has no least upper bound. Let A be the set of all atoms of B and let $A=\{a_\gamma : \gamma<\beta\}$ be an enumeration A, $|A|=\beta$. It follows from Theorem 1. 2. 1 that each $i\in I$ is the join of a unique finite subset of A which we will denote by $\bar{i}$. Clearly $i=\sum\bar{i}, \bar{0}=\phi$ and $\bar{i}=\{a\in A : a\leq i\}=\{a_{\gamma_1}, a_{\gamma_2}, \ldots, a_{\gamma_m}\}$ for some $\gamma_1<\gamma_2<\ldots<\gamma_m<\beta, m=|\bar{i}|$, (i. e. we assume that the elements of i are listed in strictly increasing order according to the well-ordering of A). For $n<\omega$ set $i_0=0$, $i_n=a_1+\ldots+a_n$. We now consider the sequence $S=\{f_{i_n}/\mathcal{D} : n<\omega\}\leq\Pi_{\mathcal{D}}(i]$ (for definition of $f_a/\mathcal{D}$ see 2. 3). S is a strictly increasing sequence since $\{i\in I : f_{i_n}(i)<f_{i_{n+1}}(i)\}\supseteq I_{i_{n+1}}\in\mathcal{D}$ for each $n<\omega$. We now define a map $\mu : \Pi_{\mathcal{D}}(i]\to\Pi_{\mathcal{D}}(i]$ by $\mu(g/\mathcal{D})=\hat{g}/\mathcal{D}$ where

$$\hat{g}(i)=\begin{cases}\sum\{a_{\gamma_1},\ldots, a_{\gamma_{m-1}}\} & \text{if } \overline{g(i)}=\{a_{\gamma_1},\ldots,a_{\gamma_m}\},\ \gamma_1<\gamma_2\ \ldots<\gamma_m \\ 0 \quad \text{if} \quad g(i)=0.\end{cases}$$

(i. e. $g(i)$ is the join of all but the last element of $\overline{g(i)}$.)

LEMMA 3. 2.

(i) *μ is well defined and order preserving.*
(ii) *if $g/\mathcal{D}\neq 0/\mathcal{D}$ then $\mu(g/\mathcal{D})<g/\mathcal{D}$ (μ is strictly decreasing).*
(iii) *for $f_{i_n}/\mathcal{D}\in S$ defined above and $n\geq 1$, $\mu(f_{i_n}/\mathcal{D})=f_{i_{n-1}}/\mathcal{D}$.*

Proof. (i) $g/\mathcal{D}=h/\mathcal{D}$ implies $\hat{g}/\mathcal{D}=\hat{h}/\mathcal{D}$ since $\{i\in I : g(i)=h(i)\}\subseteq\{i\in I : \hat{g}(i)=\hat{h}(i)\}$. Hence μ is well defined. Now let $g/\mathcal{D}\leq h/\mathcal{D}$ then $\{i\in I : g(i)\leq h(i)\}\in\mathcal{D}$. If $g(i)=0$ then $\hat{g}(i)=0\leq\hat{h}(i)$ so suppose $\overline{g(i)}=\{a_{\gamma_1},\ldots,a_{\gamma_m}\}$, $\overline{h(i)}=\{a_{\delta_1},\ldots,a_{\delta_n}\}$. $g(i)\leq h(i)$ implies $\overline{g(i)}\subseteq\overline{h(i)}$ so $a_{\gamma_m}=a_{\delta_k}$ forsome $k\leq n$. We are assuming $\gamma_1<\ldots<\gamma_m$, $\delta_1<\ldots<\delta_n$ hence $\overline{\hat{g}(i)}=\{a_{\gamma_1},\ldots,a_{\gamma_{m-1}}\}\subseteq\{a_{\delta_1},\ldots,a_{\delta_{k-1}}\}\subseteq\overline{\hat{h}(i)}$ so $\hat{g}(i)\leq\hat{h}(i)$. It follows that $\{i\in I : \hat{g}(i)\leq\hat{h}(i)\}\supseteq\{i\in I : g(i)\leq h(i)\}\in\mathcal{D}$ so $\hat{g}/\mathcal{D}\leq\hat{h}/\mathcal{D}$.

(ii) If $g/\mathcal{D}\neq 0/\mathcal{D}$ then $\{i\in I : g(i)\neq 0\}\in\mathcal{D}$. But this is precisely the set of i for which $\hat{g}(i)<g(i)$ so $\mu(g/\mathcal{D})=\hat{g}/\mathcal{D}<g/\mathcal{D}$.

(iii) $\{i\in I : \hat{f}_{i_n}(i)=f_{i_{n-1}}(i)\}\supseteq I_{i_n}\in\mathcal{D}$ so the result follows.

The next lemma is valid for join-semilattices but we need it only for Boolean algebras.

LEMMA. 3. 3. *Let B be a Boolean algebra on which we can define an order preserving function $\mu : B \to B$ such that $\mu(b) < b$ for all $b \neq 0$. Suppose also that we can find a sequence $S = b_n : n \in \omega\} \subseteq B$ with the property $\mu(b_n) = b_{n-1}$ for all $n \geq 1$. Then*

(i) *S has no join in B.*

(ii) *the canonical image of S in any ultrapower of B has no join.*

Proof. (i) Suppose c is a least upper bound for S. Then $c \geq b_n$ for all $n \in \omega$ and $c \neq 0$. Since the function μ is order preserving $\mu(c) \geq \mu(b_n) = b_{n-1}$ for all $n \geq 1$. Hence we have found a strictly smaller upper bound for S.

(ii) Let $d : B \to \Pi_{\mathcal{D}} B$ be the canonical embedding $d(b) = \langle b : i \in I \rangle / \mathcal{D}$ and define the map $\mu_{\mathcal{D}}$ on the ultrapower in the obvious way: for $h/\mathcal{D} \in \Pi_{\mathcal{D}} B$ put $\mu_{\mathcal{D}}(h/\mathcal{D}) = \langle \mu(h(i)) : i \in I \rangle / \mathcal{D}$. This definition is well defined and one easily checks that $\mu_{\mathcal{D}}$ has the same properties with respect to $\Pi_{\mathcal{D}} B$ and $d(S) = \{d(b_n) : n \in \omega\}$ as μ has with respect to B and S. If follows from part (i) that $d(S)$ has no join in $\Pi_{\mathcal{D}} B$.

COROLLARY 3. 4. *For each regular cardinal $\alpha > \omega$ B_α is ω_1-incomplete.*

Proof. If α is a successor cardinal this follows from Theorem 3. 1 (i), Lemma 3. 2 and 3. 3 (i). So let α be inaccessible and suppose B_α is ω_1-complete. Let γ be a successor cardinal $< \alpha$ then by Lemma 3. 3 there exists a sequence $\{b_n : n \in \omega\} \subseteq B_\gamma$ which has no least upporbound in B_γ. But B_γ is embedded in $B_\alpha = \bigcup \{B_\gamma : \gamma$ is a successor cardinal $< \alpha\}$. Let b be the join of $\{b_n : n \in \omega\}$ in B_α then $\{b\} \cup \{b_n : n \in \omega\} \subseteq B_\delta$ for some $\gamma \leq \delta < \alpha$, δ successor and b is the join of $\{b_n\}$ in B_δ. But by Theorem 3. 1 (i) B_δ is isomorphic to an ultrapower of B_γ so by Lemma 3. 3 (ii) $\{b_n : n \in \omega\}$ has no join in B_δ.

(We identify B_γ with its isomorphic cannonical image in B_δ.)

4. The embedding theorem

In this section we reformulate our main result (Theorem 3. 1) in the following way.

THEOREM 4. 1. *Let C be an infinite atomic Boolean algebra of cardi-*

nality β.

(i) *For any successor cardinal* $\alpha \geq \beta$, C *is elementary embeddable in an ultraproduct of finite Boolean algebras and this ultraproduct has cardinality* α.

(ii) *For any inaccessible cardinal* $\alpha > \beta$, C *is elementary embeddable in* B_α *which is the union of an elementary chain of Boolean algebras each isomorphic to an ultraproduct of finite Boolean algebras.*

Proof. (i) Since α is a successor cardinal and $|C| \leq \alpha$ it follows from Theorem 3.1 (i) that $B_\alpha \cong \prod_{\mathcal{D}_\alpha} (i] \cong \prod_{\mathcal{D}_\alpha} C$ where $\mathcal{D}_\alpha$ is an α-good ultrafilter over I as in 2.1. For each $i \in I$ $(i]$ is a finite Boolean algebra and C is elementary embeddable in any ultrapower of itself, so the result follows.

(ii) Since $|C| < \alpha$ inaccessible, there exists successor cardinal $\gamma < \alpha$ such that $|C| \leq \gamma$. By part (i) B' is elementary embeddable in B_γ and from the construction of B_α (Theorem 3.1 (ii)) it follows that B_γ is an elementary subalgebra of B_α.

Remark. Unlike the theory of atomless Boolean algebras, the theory of infinite atomic Bloolean algebras is not model complete: In $\mathcal{P}(\omega)$ consider the subalgebra R generated by the set

$$X = \{\{2n,\ 2n+1\} : n \in \omega\}.$$

Then $a \in X$ iff a is an a is an atom (element of height 1) in B but a has height 2 in $\mathcal{P}(\omega)$. Since the height of an element is a first order property we have that B is not an elementary subalgebra of $\mathcal{P}(\omega)$.

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