On the structure of generalized effect algebras and separation algebras

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Abstract. *Separation algebras* are models of separation logic and *effect algebras* are models of unsharp quantum logics. We investigate these closely related classes of partial algebras as well as their noncommutative versions and the subclasses of (generalized) (pseudo-)orthoalgebras. We present an orderly algorithm for constructing all nonisomorphic generalized pseudoeffect algebras with n elements and use it to compute these algebras with up to 10 elements.

1 Introduction

Separation algebras were introduced by Calcagno, O'Hearn and Yang [3] as semantics for separation logic, and *effect algebras* were defined by Foulis and Bennett [6] as an abstraction of unsharp measurements in quantum mechanics. Detailed definitions are recalled in the next section, but we note here that they are both cancellative commutative partial monoids and that every effect algebra is a separation algebra. Hence results about separation algebras automatically apply to effect algebras, and the in-depth study of effect algebras over the past two decades provides insight into this particular subclass of separation algebras. Lattice effect algebras, MV-effect algebras, orthoalgebras, orthomodular posets, orthomodular lattices and Boolean effect algebras are all well known subclasses of effect algebras, so positioning effect algebras as a subclass of separation algebras provides many interesting algebraic models for separation logic.

In this paper, we are primarily interested in finite partial algebras since they can be computed for small cardinalities, and browsing models with up to a dozen elements is useful for investigating the structure of these finite algebras. To this end, we develop an algorithm for computing finite effect algebras and some of their noncommutative generalizations.

One of the aims of this paper is to increase awareness of the model theory of partial algebras since it has been developed quite extensively, but is not necessarily widely known. In the next section, we recall the basic notions of weak/full/closed homomorphisms, subalgebras and congruences for partial algebras. Examples from the classes of separation algebras and effect algebras are discussed in Section 3. Many of the results about these algebras do not depend on the commutativity of the partial binary operation $+$, hence we mostly consider noncommutative versions and often write the operation as $x \cdot y$ or xy rather than $x + y$. Effect algebras without the assumption of commutativity were introduced by Dvurecenski and Vetterlein [5] under the name *pseudoeffect algebras* and have been studied extensively since then.

Every partial algebra $\mathbf{A}$ can be easily *lifted* to a total algebra $\hat{A} = A \cup \{\mathbf{u}\}$, where the element $\mathbf{u}$ denotes undefined, and an operation on A produces $\mathbf{u}$ as output whenever the operation is undefined. In particular, if any of the inputs to the lifted operation are $\mathbf{u}$, then the output is $\mathbf{u}$. This map from partial to total algebras is a functor between the respective categories, but it does not preserve direct products of algebras, which means that the universal algebraic theory of partial algebras is not subsumed by total algebras. Given a partial monoid $\mathbf{A} = (A, \cdot, e)$, its lifted version is a well known total algebra called a *monoid with zero* $\hat{\mathbf{A}} = (A, \cdot, e, 0)$. These algebras occur, for example, as reducts of rings when $+$ and $-$ are removed from the signature. A monoid with zero is *cancellative* if all nonzero elements can be cancelled on the left and right of the multiplication operation. For partial algebras, a binary operation is *left-cancellative* if whenever $xy = xz$ are defined, then $y = z$. *Right-cancellativity* is defined analogously. Hence a partial monoid is cancellative if and only if the corresponding lifted total monoid with zero is cancellative.

Note that in the finite case these cancellative partial monoids are quite close to groups. For example, given any element x in a finite cancellative **total** monoid without a zero, the sequence $x, x^2, x^3, \dots, x^n, \dots$ must contain a duplicate when n exceeds the cardinality of the monoid. Hence, $x^i = x^j$ for some $i > j$ and by cancellativity $xx^{i-j-1} = e$, so x has an inverse. This well known argument shows that the class of finite cancellative monoids coincides with the class of finite groups. The significance of group theory in mathematics and its numerous fundamental applications in the sciences are well established, and allowing partiality of the binary operation leads to the class of finite cancellative partial monoids that properly contains all finite groups, thus making it an important class to study.

Complex algebras of separation algebras provide models of Boolean bunched implication logic and, in the noncommutative case, models of Boolean residuated lattices, also called residuated monoids in [8–10]. This indicates that separation algebras are functional Kripke structures, and in the past decade the field of modal logics and their Kripke semantics has been recognized as a branch of coalgebra. This meshes well with recent approaches to separation algebras [4] and effect algebroids [12]. We also highlight a method of [13] (Prop. 20) that converts a generalized pseudoeffect algebra to a total residuated partially ordered monoid by adding two elements $\bot, \top$, and we note that this totalization method preserves the property of being involutive.

In Section 2 we give basic definitions of partial algebras, homomorphisms, subalgebras, congruences and related concepts. Section 3 contains definitions and results about generalized separation algebras and (generalized pseudo-)effect algebras, and we map out some of the subclasses and implications between various axioms. Section 4 covers the results leading to the orderly algorithm for constructing all finite generalized pseudoeffect algebras up to isomorphism. In the subsequent section we prove new structural results about certain effect algebras that were suggested by the output of our enumeration program, and Section 6 concludes with some remarks and open problems.

2 Background on partial algebras

To facilitate our discussion of separation algebras and effect algebras, we begin with a brief summary of partial algebras. More details can be found in [1,2]. A *partial operation* g of arity n on a set A is a function from a subset $D(g)$ of A^n to A. The set $D(g)$ is the *domain* of g, and $(a_1,\dots,a_n)\in D(g)$ is also written as $g(a_1,\dots,a_n)\neq \mathbf{u}$ (but this is just convenient notation; $\mathbf{u}$ is not an element of A). Two partial operations g,h on A are equal if $D(g)=D(h)$ and $g(a_1,\dots,a_n)=h(a_1,\dots,a_n)$ for all $(a_1,\dots,a_n)\in D(g)$. The notation $g: A^n \multimap A$ is used to indicate that g is an n-ary partial function on A. If $D(g)=A^n$, then g is a *total operation*, or simply an *operation*. If $n=0$ then g is a *constant operation*, which we always assume to be total. A *signature* is a function $\sigma:\mathcal{F}\to\mathbb{N}$ where $\mathcal{F}$ is a set. The members of $\mathcal{F}$ are called (*partial*) *operation symbols*.

A *partial algebra of type* τ is a pair $\mathbf{A}=(A,\mathcal{F}^{\mathbf{A}})$ where A is a set and $\mathcal{F}^{\mathbf{A}}=\{f^{\mathbf{A}}: A^{\sigma(f)} \multimap A \mid f\in\mathcal{F}\}$ is a set of partial operations on A. If every partial operation in $\mathcal{F}^{\mathbf{A}}$ is in fact total, then $\mathbf{A}$ is a *total algebra*. Examples of partial algebras abound since any subset B of a total algebra $\mathbf{A}$ is the universe of an induced partial algebra $\mathbf{B}$, with *partial* operations $f^{\mathbf{B}}$ given by $f^{\mathbf{A}}$ restricted to B, so for $b_1,\dots,b_n\in B$, $f^{\mathbf{B}}(b_1,\dots,b_n)$ is undefined if and only if $f^{\mathbf{A}}(b_1,\dots,b_n)\notin B$. $\mathbf{B}$ is called a *relative subalgebra* of $\mathbf{A}$, and $\mathbf{A}$ is a *total extension* of $\mathbf{B}$. More natural examples are given by any field, such as the rational, real or complex numbers, with a signature that includes $^{-1}$ or division $/$, since $0^{-1}=\mathbf{u}=x/0$.

Terms, equations (= atomic formulas) and first-order formulas over a set of variables $X=\{x_1,x_2,\dots\}$ are defined inductively as for total algebras, but for a term t we also write $t=\mathbf{u}$ or $t\neq\mathbf{u}$ depending on whether t is undefined or defined. For a partial algebra $\mathbf{A}$ and an assignment $\mathbf{a}: X\to A$, the semantic interpretation of a term t as a *term function* $t^{\mathbf{A}}: A^n\to A$ is defined inductively by $t^{\mathbf{A}}(\mathbf{a})=\mathbf{a}(t)$ if $t\in X$, and for $t=f(t_1,\dots,t_n)$,

$$t^{\mathbf{A}}(\mathbf{a})=\begin{cases} f^{\mathbf{A}}(t_1^{\mathbf{A}}(\mathbf{a}),\dots,t_n^{\mathbf{A}}(\mathbf{a})) & \text{if } t_i^{\mathbf{A}}(\mathbf{a})\neq\mathbf{u} \text{ for all } i=1,\dots,n\\ \mathbf{u} & \text{otherwise.}\end{cases}$$

Hence if any subterm is undefined under the assignment, then the whole term is undefined. An *identity* (i.e. universally quantified equation with no free variables) $s=t$ is satisfied by an algebra $\mathbf{A}$, written $\mathbf{A}\models s=t$ if $s^{\mathbf{A}}=t^{\mathbf{A}}$, i.e., if the term functions are equal. Note that this means both sides have to be defined or both sides have to be undefined for any given input tuple. This interpretation of an equation in partial algebras is called a *strong identity* or *Kleene identity*. An even stronger form of satisfaction is given by *existence identities*: $\mathbf{A}\models s\overset{e}{=}t$ if $s^{\mathbf{A}}=t^{\mathbf{A}}$ and $D(s^{\mathbf{A}})=A$. Note that for an identity of the form $x=t$ the concept of strong identity and existence identity coincide since $x^{\mathbf{A}}$ is always defined for a variable x.

A quasi-identity is a formula $s_1 = t_1 \,\&\ldots\&\, s_m = t_m \implies s = t$, and is satisfied in $\mathbf{A}$ if any assignment to the variables that satisfies $s_1 = t_1, \ldots, s_m = t_m$ (both sides defined) also satisfies $s = t$ (both sides defined). Under this interpretation a quasi-identity with no premises ($m = 0$) is equivalent to an existence identity. A (*strong/existence/quasi*)*equational class* $\mathcal{K}$ of partial algebras is a class of algebras of the same signature such that for some set $\mathcal{I}$ of (strong/quasi/existence) identities we have $\mathcal{K} = \{\mathbf{A} : \mathbf{A} \models \varepsilon \text{ for all } \varepsilon \in \mathcal{I}\}$.

Direct products $\prod_{i\in I} \mathbf{A}_i$ are defined for partial algebras in exactly the same way as for total algebras, with pointwise fundamental operations $f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = (\ldots, f^{\mathbf{A}_i}(x_{1i}, \ldots, x_{ni}), \ldots)$ that are defined iff $f^{\mathbf{A}_i}(x_{1i}, \ldots, x_{ni})$ is defined for all $i \in I$. For a class $\mathcal{K}$ of partial algebras the class of products of members of $\mathcal{K}$ is denoted by $\mathbf{P}\mathcal{K}$.

There are three notions of homomorphism, with the weakest one being standard relational homomorphism. A function $h : \mathbf{A} \to \mathbf{B}$ is

- a (*weak*) *homomorphism* if for all $f \in \mathcal{F}$,

$$(a_1, \ldots, a_n) \in D(f^{\mathbf{A}}) \text{ implies } h(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)),$$

- *full* if for all $f \in \mathcal{F}$, $f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = h(a_0)$ implies there exists $(a'_1, \ldots, a'_n) \in D(f^{\mathbf{A}})$ such that $h(a_i) = h(a'_i)$ for $i = 0, \ldots n$,
- *closed* if for all $f \in \mathcal{F}$,

$$(h(a_1), \ldots, h(a_n)) \in D(f^{\mathbf{B}}) \text{ implies } (a_1, \ldots, a_n) \in D(f^{\mathbf{A}}).$$

Note that if h is a closed homomorphism, then it is a full homomorphism. The *category of partial algebras* with signature σ has morphisms given by the first (weak) notion of homomorphism. For a class $\mathcal{K}$ of partial algebras the class of homomorphic images is

$$\mathbf{H}\mathcal{K} = \{\mathbf{B} \mid h : \mathbf{A} \to \mathbf{B} \text{ is a surjective homomorphism for some } \mathbf{A} \in \mathcal{K}\}.$$

The class of full or closed homomorphic images of $\mathcal{K}$ are denoted by $\mathbf{H}_f\mathcal{K}$ and $\mathbf{H}_c\mathcal{K}$, respectively.

There are also three notions of a subalgebra $\mathbf{A}$ of $\mathbf{B}$. Assuming $A \subseteq B$, a partial algebra $\mathbf{A}$ is

- a *weak subalgebra* if for all $f \in \mathcal{F}$,

$$(a_1, \ldots, a_n) \in D(f^{\mathbf{A}}) \text{ implies } f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(a_1, \ldots, a_n),$$

- a *relative subalgebra* if for all $f \in \mathcal{F}$, $f^{\mathbf{A}} = f^{\mathbf{B}}\restriction_{A^n}$, and
- a (*closed*) *subalgebra* if for all $f \in \mathcal{F}$,

$$(a_1, \ldots, a_n) \in A^n \cap D(f^{\mathbf{B}}) \text{ implies } f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(a_1, \ldots, a_n).$$

These notions correspond to the injection map $i : A \to B$ being a weak/full/closed homomorphism. The class of weak, relative or closed subalgebras of a class $\mathcal{K}$

of partial algebras are denoted $\mathbf{S}_w\mathcal{K}$, $\mathbf{S}_r\mathcal{K}$ and $\mathbf{S}\mathcal{K}$ respectively. As the notation indicates, closed subalgebras are the standard concept for partial algebras.

A *congruence* θ on a partial algebra $\mathbf{A}$ is an equivalence relation such that for all $f \in \mathcal{F}$, $a_1\theta b_1, \ldots, a_n\theta b_n$ and $(a_1,\ldots,a_n),(b_1,\ldots,b_n) \in D(f^{\mathbf{A}})$ imply $f^{\mathbf{A}}(a_1,\ldots,a_n)\theta f^{\mathbf{A}}(b_1,\ldots,b_n)$. A congruence is *closed* if $(a_1,\ldots,a_n) \in D(f^{\mathbf{A}})$ implies $(b_1,\ldots,b_n) \in D(f^{\mathbf{A}})$. The *quotient algebra* $\mathbf{A}/\theta$ is defined on the set $A/\theta = \{[a]_\theta \mid a \in A\}$ of equivalence classes by

$$f^{\mathbf{A}/\theta}([a_1]_\theta,\ldots,[a_n]_\theta) = [f^{\mathbf{A}}(a'_1,\ldots,a'_n)]_\theta$$

if $(a'_1,\ldots,a'_n) \in D(f^{\mathbf{A}})$ and $a_1\theta a'_1,\ldots,a_n\theta a'_n$ for some $a'_1,\ldots,a'_n \in A$, and $f^{\mathbf{A}/\theta}$ is undefined otherwise. The *canonical map* $\gamma : \mathbf{A} \to \mathbf{A}/\theta$ given by $\gamma(a) = [a]_\theta$ is a full homomorphism and, if θ is closed, then γ is a closed homomorphism. We often write $[a]$ rather than $[a]_\theta$ when the confusion is unlikely.

For a set I, a *filter* F on I is a collection of subsets of I that is closed under finite intersection and if $X \in F$ and $X \subseteq Y$ then $Y \in F$. On a product $\mathbf{A} = \prod_{i\in I}\mathbf{A}_i$, a filter F on I induces a congruence θ_F by $a\theta_F b$ if and only if $\{i \in I \mid a_i = b_i\} \in F$. The resulting quotient algebra $\mathbf{A}/\theta_F$ is called a *reduced product*. For a class $\mathcal{K}$ of partial algebras the class of all reduced products is denoted by $\mathbf{P}_r\mathcal{K}$.

For total algebras Birkhoff's variety theorem says that $\mathcal{V}$ is an equational class if and only if $\mathcal{V}$ is a *variety*, i.e., $\mathcal{V} = \mathbf{HSP}\mathcal{K}$ for some class $\mathcal{K}$. Generalizations of this result to partial algebras are summarized below.

Theorem 1 ([2]). *Let $\mathcal{V}$ be a class of partial algebras.*

1. *$\mathcal{V}$ is an existence equational class if and only if $\mathcal{V} = \mathbf{HSP}\mathcal{K}$ for some $\mathcal{K}$.*
2. *If $\mathcal{V}$ is a strong equational class then $\mathcal{V} = \mathbf{H}_c\mathbf{SP}_r\mathcal{K}$ for some class $\mathcal{K}$.*
3. *$\mathcal{V}$ is a quasiequational class if and only if $\mathcal{V} = \mathbf{SP}_r\mathcal{K}$ for some class $\mathcal{K}$.*

A characterization for strong equational classes can be found in [14]. Since an equational class is a variety, the three classes above are also referred to as existence varieties, strong varieties and quasivarieties of partial algebras.

3 Generalized separation algebras and some subclasses

A *partial semigroup* $(S,\cdot)$ is a partial algebra with a binary operation that is *associative*, i.e., the identity $(xy)z = x(yz)$ holds. A *partial monoid* $(M,\cdot,e)$ is a partial semigroup with an identity element e such that $xe = x = ex$ holds. In fact, every variety of total algebras gives rise to a strong variety of partial algebras, simply by reinterpreting the same defining identities. However, some equational axioms only have total algebras as models. For example, the class of groups can be axiomatized as monoids that satisfy $xx^{-1} = e$, where $^{-1}$ is a unary operation symbol. Then $x = xe = x(yy^{-1}) = (xy)y^{-1}$ is defined for all values of x, y, hence the subterm xy is always defined. Consequently, the class of all partial algebras that satisfy these group axioms is simply the class of all (total) groups.

A *generalized separation algebra*, or GS-algebra, is a partial monoid that is *cancellative* and *conjugative*, i.e., satisfies the axioms

left cancellativity $xy = xz \implies y = z$
right cancellativity $xz = yz \implies x = y$
conjugation $\exists v(vx = y) \iff \exists w(xw = y)$

A *separation algebra* [3] is a commutative GS-algebra, i.e., the identity $xy = yx$ holds, making the conjugation axiom redundant. The category of generalized separation algebras with partial algebra homomorphisms has the category of (total) groups as a full subcategory, and the same is true for the category of total cancellative conjugative monoids. This includes all free commutative monoids such as the natural numbers with addition, but does not include any noncommutative free monoid.

Theorem 2. *The conjugation axiom is preserved by reduced products, but not by subalgebras, even in the presence of cancellative monoid axioms. Therefore the class of GS-algebras is not a quasivariety.*

Proof. Let $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ be a product of GS-algebras, F a filter on I, and assume $[c][a] = [b]$ for some $a, b, c \in A$, where $[x] = [x]_{\theta_F}$. Therefore $c_i a_i = b_i$is true for all i in some set $X \in F$. Since A_i is conjugative, $a_i d_i = b_i$ for some $b_i \in A_i$ and all $i \in X$. Let $d_j = e$ for $j \in I \setminus X$ and define b' by $b'_i = b_i$ if $i \in X$, and $b'_i = a_i$ otherwise. Then $[b'] = [b]$ and $ad = b'$, hence $[a][d] = [b'] = [b]$. The reverse implication is similar, so the reduced product $\mathbf{A}/\theta_F$ satisfies the conjugation axiom.

Let $A = \{e, a, b, c, d\}$ and define $\cdot$ on A by $ex = x = xe$ for $x \in A$, $ab = bc = ca = d$ and in all other cases xy is undefined. It is easy to check that $\mathbf{A} = (A, \cdot, e)$ is the smallest noncommutative generalized separation algebra. It has a closed subalgebra given by $B = \{e, a, b, d\}$ in which $ab = d$, but there is not element $x \in B$ such that $xa = d$, hence conjugation fails.

By the characterization theorem of quasivarieties for partial algebras, stated in Theorem 1, the class of GS-algebras is not a quasivariety. □

Note that the class of separation algebras **is** a quasivariety of partial algebras.

The following example demonstrates that conjugation is not preserved by weak homomorphisms. Consider the following GE-algebra **G** and partial algebra **A**,

G

$\cdot_G$	e	a	b
e	e	a	b
a	a	$-$	$-$
b	b	$-$	$-$

A

$\cdot_A$	e	a	b
e	e	a	b
a	a	$-$	$-$
b	b	e	$-$

The mapping that sends e, a and b in **G** to e, a and b in **A**, respectively, is a weak homomorphism from **G** to **A** that does not preserve conjugation.

A binary relation $\le$ is defined by $x \le y \iff \exists v(vx = y)$, and the conjugation axiom ensures that this binary relation could have also been defined by $\exists w(xw = y)$. An equivalent form of this axiom is $xy = z \implies \exists v, w(vx = yw = z)$. Reflexivity of $\le$ follows from $ex = x$, and $x \le y$, $y \le z$ imply $vx = y$, $wy = z$ for some v, w and therefore $wvx = z$, which proves transitivity. Hence $\le$ is a preorder, and its *symmetrization* is defined by $x \equiv y \iff x \le y$ and $y \le x$. As usual, the equivalence classes $[x]$ of $\equiv$ are partially ordered by $[x] \le [y] \iff x \le y$. An element v is *invertible* if there exists w such that $vw = e = wv$, and the set of invertible elements of a GS-algebra $\mathbf{A}$ is denoted by A^*. The inverse of v, if it exists, is unique and is denoted by v^{-1}.

Lemma 3. *Let* $\mathbf{A}$ *be a generalized separation algebra. Then*

1. A^* *is the bottom equivalence class* $[e]$ *of the poset* $A/{\equiv} = (\{[x] : x \in A\}, \le)$,
2. $\mathbf{A}^* = (A^*, \cdot, e, ^{-1})$ *is a (total) group and is a closed subalgebra of* $\mathbf{A}$,
3. $x \equiv y$ *holds if and only if* $x \in yA^*$, *and*
4. $\equiv$ *is the identity relation if and only if* e *is the only invertible element.*

Proof. 1. If $x \equiv e$ then $vx = e$ for some $v \in A$, so $(vx)v = ev = ve$. By associativity, $v(xv) = ve$, and from cancellativity we conclude that x is invertible. Conversely, we always have $e \le x$, and if x is invertible then $x^{-1}x = e$, hence $x \le e$, which proves that $[e] = A^*$ is the bottom element of $A/{\equiv}$. 2. It suffices to show that $\cdot$ restricted to A^* is a total operation. Given $v, w \in A^*$ there exists $u \in A$ such that $uv = e$. Thus $(uv)w = w$, and by associativity we get $u(vw) = w$, which implies that vw is defined. 3. From $x \equiv y$ we have $xv = y$ and $yw = x$ for some v, w. Therefore $ywv = y$ and $xvw = x$. By cancellativity it follows that $wv = e = vw$, so $w \in A^*$. Now $yw = x$ implies $x \in \{yz : z \in A^*\} = yA^*$. Conversely, assume $x \in yA^*$, whence $x = yv$ for some invertible element v. Then $xv^{-1} = y$, so $x \equiv y$. 4. Note that $\equiv$ is not the identity relation if and only if $x \equiv y$ for some $x \ne y$. By 2. this is equivalent to $|A^*| > 1$. □

A *generalized pseudoeffect algebra*, or GPE-algebra, is a GS-algebra that is *positive*, i.e., $xy = e \implies x = e$, in which case $y = e$ follows from $ey = y$. Equivalently, a GPE-algebra is a GS-algebra in which $\le$ is antisymmetric and hence a partial order. A commutative GPE-algebra is called a *generalized effect algebra*, or GE-algebra. As mentioned before, the conjugation axiom always holds in commutative partial algebras, hence separation algebras and GE-algebras are quasivarieties.

The following theorem shows that there is a close relationship between generalized separation algebras and generalized pseudoeffect algebras. In particular, the result shows that every separation algebra can be collapsed in a unique way to a largest generalized effect algebra. Hence a substantial part of the structure theory of separation algebras is covered by results about generalized effect algebras.

Theorem 4. *For a GS-algebra* $\mathbf{A}$,

1. *the relation* $\equiv$ *is a closed congruence,*

2. $\mathbf{A}/\!\equiv$ *is a GPE-algebra,*
3. the congruence classes of $\equiv$ *all have the same cardinality, and*
4. if $h : \mathbf{A} \to \mathbf{B}$ *is a homomorphism and* $\mathbf{B}$ *is a GPE-algebra then there exists a unique homomorphism* $g : \mathbf{A}/\!\equiv \to \mathbf{B}$ *such that* $g \circ \gamma = h$ *(where* $\gamma : \mathbf{A} \to \mathbf{A}/\!\equiv$ *is the canonical homomorphism* $\gamma(x) = [x]$*).*

Proof. Let $x \equiv y$, $z \equiv w$ and assume yw is defined. We want to show that xz is defined and $xz \equiv yw$. Using the assumptions and conjugation, we obtain $ux = y$ and $zv = w$ for some u, v, and, since yw is defined, $(ux)(zv) = yw$. By associativity it follows that $(u(xz))v = yw$, hence xz is defined. Using conjugation again, there exists r such that $r(u(xz)) = yw$, hence $xz \leq yw$. Given that xz is now known to be defined, a similar argument shows $yw \leq xz$, so $xz \equiv yw$.

The quotient algebra is positive since if $[x][y] = [e]$, then $xy \equiv e$. This gives $xyv = e$ for some v and therefore $x \leq e$, from which $[x] = [e]$ follows. Therefore $A/\!\equiv$ is a GPE-algebra. For $x \in A$, $x = x + 0$ and the map $x \mapsto x + v$ for $v \in \mathbf{A}^*$ is a bijection between $[x]$ and $[0]$. Hence the congruence classes have the same cardinality.

Now assume $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism and $\mathbf{B}$ is a GPE-algebra. Define $g : \mathbf{A}/\!\equiv \to \mathbf{B}$ by $g([a]) = h(a)$. To prove that g is well defined, assume $[a'] = [a]$, or equivalently $a \equiv a'$. This means $va = a'$, so $h(v)h(a) = h(va) = h(a')$, whence $h(a) \leq h(a')$. Similarly $h(a') \leq h(a)$, and since $\leq$ is a partial order in any GPE-algebra, $h(a) = h(a')$ follows.

Now suppose g' is a homomorphism that also satisfies $g' \circ \gamma = h$. This means $g'([a]) = h(a)$ for all $a \in A$, so $g' = g$. □

Theorem 5. *Let* $\mathbf{G}$ *be a group and* $\mathbf{B}$ *a GPE-algebra. Then* $\mathbf{A} = \mathbf{G} \times \mathbf{B}$ *is a GS-algebra with* $\mathbf{A}^* = \mathbf{G} \times \{e\}$*.*

Proof. The product of GS-algebras is again a GS-algebra since by, Theorem 2, this class of algebras is closed under reduced products. The element $(g, e) \in A$ has inverse (g^{-1}, e), and there are no other inverses by Lemma 3.4. □

Several of the prominent subclasses of GPE-algebras extend the signature of these algebras with a constant 1 and unary operations $^{\sim}, ^{-}$ or $'$. In the case of commutativity it is also traditional to replace $\cdot, e$ with $+, 0$ (or $\oplus, 0$).

Starting from GPE-algebras using the $+, 0$ signature, seven subclasses are obtained by adding combinations of the following three independent axioms:

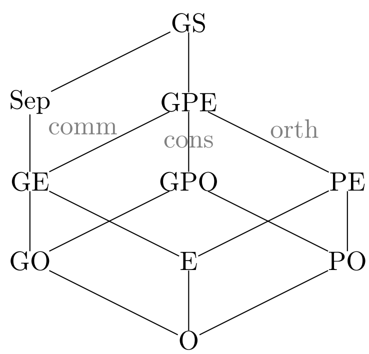
(comm) $x + y = y + x$ (*commutativity*)
(orth) $x + y = 1 \iff y = x^{\sim} \iff x = y^{-}$ (*orthocomplementation*)
(cons) $x + x \neq \mathbf{u} \implies x = 0$ (*consistency*)

In particular, adding these different combinations of the above axioms to a GPE-algebra produces the following subclasses:

Axioms	Name	Abbrev.
(comm)	*generalized effect algebra*	GE
(cons)	*generalized pseudo-orthoalgebra*	GPO
(orth)	*pseudoeffect algebra*	PE
(comm), (cons)	*generalized orthoalgebra*	GO
(comm), (orth)	*effect algebra*	E
(cons), (orth)	*pseudo-orthoaglebra*	PO
(comm), (cons), (orth)	*orthoaglebra*	O

In a pseudoeffect algebra, $x^\sim$ and x^- are called the *right* and *left complement* of x, and 1 is the top element. In fact any GPE-algebra with a top element, denoted by 1, can be extended with these two unary operations such that (orth) holds, and it is easy to check that they are total operations. For commutative subclasses such as effect algebras, we always have $x^\sim = x^-$ and in this case we write x' for the complement of x. From (orth) it follows that $x^{\sim-} = x = x^{-\sim}$, so for effect algebras and orthoalgebras this is written as $x'' = x$.

Below is a diagram that depicts the containment between these subclasses of GPE-algebras. The initial addition of the three independent axioms is shown as well as the larger classes of (generalized) separation algebras.



The two most studied subclasses of GPE-algebras are *effect algebras* (EA) which satisfy (comm) and (orth), and *orthoalgebras* (OA) which satisfy (comm), (orth) and (cons). The signature for effect algebras and orthoalgebras is $+, ', 0, 1$. Some examples of effect algebras are given below:

1. *One-element effect algebra* $(\{0\}, +, ', 0, 0)$
2. *Two-element effect algebra* $(\{0,1\}, +, ', 0, 1)$ where $0+x = x = x+0$, $1+1 =$ undefined, $0' = 1$, $1' = 0$.
3. The *standard MV-effect algebra* $[0,1]_E = ([0,1], +, ', 0, 1)$ where $x+y$ is addition, but *undefined* if the result is bigger than 1, $x' = 1 - x$.
4. For any MV-algebra $(A, \oplus, \neg, 0)$ define $x+y = \begin{cases} x \oplus y & \text{if } x \le \neg y \\ \text{undefined} & \text{otherwise} \end{cases}$ Then $(A, +, \neg, 0, \neg 0)$ is called an *MV-effect algebra*.

5. Let $(G, \cdot, ^{-1}, e, \leq)$ be a *partially ordered group* and $u \in G$ such that $u \geq e$. Then $([0,u], \cdot, ^{\sim}, ^{-}, e, u)$ is an *interval pseudoeffect algebra* where $\cdot$ is undefined if the result is outside of $[0,u]$, $x^{\sim} = x^{-1}u$ and $x^{-} = ux^{-1}$. If G is abelian this construction produces an *interval effect algebra*.
6. Let $(L, \vee, \wedge, ', 0, 1)$ be an *orthomodular lattice*, i.e., a lattice $(L, \vee, \wedge)$ that satisfies $x \wedge x' = 0$, $x \vee x' = 1$, $x'' = x$, $(x \wedge y)' = x' \vee y'$ and $x \leq y \implies x \vee (x' \wedge y) = y$. Define $x + y = \begin{cases} x \vee y & \text{if } x \leq y' \\ \text{undefined} & \text{otherwise} \end{cases}$. Then $(L, +, ', 0, 1)$ is an orthoalgebra since it is *consistent*: if $x + x$ is defined then $x = 0$.

Examples of generalized separation algebras (that are not GPE-algebras) can be constructed using Theorem 5.

All classes defined here are closed under *products*, but some of them are also closed under certain amalgamated disjoint unions. The *horizontal sum* $\mathbf{A} + \mathbf{B}$ of PE-algebras $\mathbf{A}$ and $\mathbf{B}$ is the disjoint union of $A - \{0,1\}$ and $B - \{0,1\}$ with new bottom and top added. The new operations $\cdot, '$ agree with $\cdot, '$ on $\mathbf{A}$ and $\mathbf{B}$, and, for $a \in A - \{0,1\}, b \in B - \{0,1\}$, the value of ab is undefined. The result is again a PE-algebra, and, if $\mathbf{A}, \mathbf{B}$ are effect algebras or orthoalgebras, the same is true for $\mathbf{A} + \mathbf{B}$. Clearly horizontal sums can also be defined for arbitrary families of PE-algebras. For the class of GPE-algebras or its subclasses one can define a *bottom sum* that takes the disjoint union of the factors and identifies all the bottom elements. The result is again a GPE-algebra, or a GE-algebra if all summands are commutative. In Figure 1 we give some diagrams of finite effect algebras and GPE-algebras to indicate the range of possible examples. A black dot is used for elements that are equal to their complements, and other elements are represented by open circles.

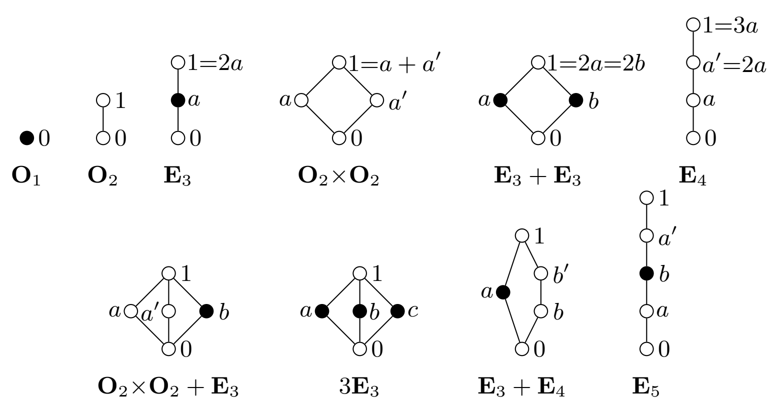


Fig. 1. Effect algebras with up to 5 elements

For a (pseudo)effect algebra $\mathbf{A}$, let $\bar{\mathbf{A}}$ denote the $'$-free reduct. Applying this to the 2-element orthoalgebra we obtain $\bar{\mathbf{O}}_2 = (\{0,1\}, +, 0)$, the 2-element GE-algebra.

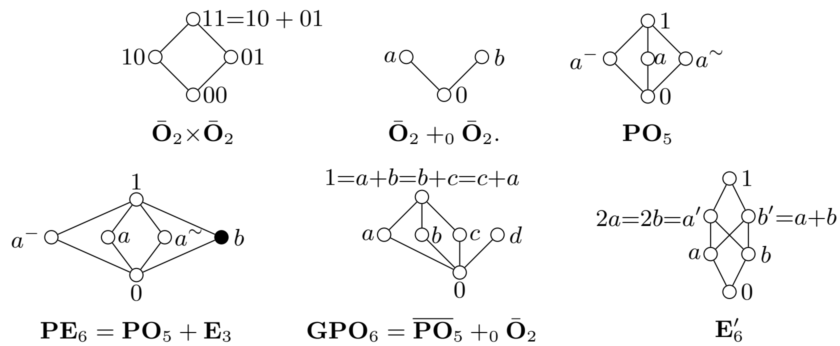


Fig. 2. Examples of GO-, GPO-, E-, GE- and GPE-algebras

4 An orderly algorithm for constructing generalized pseudoeffect algebras

Examples of GPE-algebras can provide insight into their structure that may not be apparent simply from studying their axioms. General purpose model generators such as Mace4 [11] can be used to find all models of cardinality n of a finitely axiomatized first-order theory. However, if a class has many models even for small cardinalities, as is the case with GPE-algebras, this approach becomes computationally unfeasible for $n > 8$. For this reason, it is helpful to have a more efficient algorithm that can construct all GPE-algebras of a given size. An orderly algorithm constructs nonisomorphic models of cardinality $n+1$ from models of cardinality n without checking for possible isomorphisms with all other models of the same size. This reduces space and time requirements of such algorithms and makes it possible to parallelize the model search.

A GPE-algebra of cardinality n is represented on the set $A = \{0, 1, \ldots, n-1\}$ by the $n \times n$ table for its partial binary operation, where undefined entries are marked with a special value not in A. The results below show how a GPE-table of size $n \times n$ is constructed by adding a new maximal element n to an already existing $(n-1) \times (n-1)$ table associated with a GPE-algebra of size $n-1$. A subset B of a poset is a *downset* if $x \le y \in B$ implies $x \in B$.

Lemma 6. *Let $\mathbf{A} = (A, +, 0)$ be a GS-algebra and B a nonempty downset with respect to the preorder $\le$. Then $\mathbf{B} = (B, +\restriction_B, 0)$ is a GS-algebra. If $\mathbf{A}$ is a GPE-algebra, the same holds for $\mathbf{B}$.*

Proof. Let B be a downset of A. For simplicity of notation, we will denote $+\restriction_B$, the restriction of $+$ onto B, as $+_B$. We need to check that $\mathbf{B}$ satisfies the axioms for GS-algebras.

Identity element: Since 0 is the bottom element of $\mathbf{A}$ it follows that 0 is in the downset B, hence it is also the identity element of $\mathbf{B}$ under the restriction of $+$ to B.

Cancellativity: Assume $x +_B y = x +_B z$ is defined for some $x, y, z \in B$. Then $x + y = x + z$ is defined in $\mathbf{A}$ and $x = y$ by cancellativity in $\mathbf{A}$. Similarly, if $x +_B y = z +_B y$, then $x + y = z + y$ in $\mathbf{A}$ and $x = z$ by cancellativity in $\mathbf{A}$.

Associativity: Assume $x +_B y$ and $(x +_B y) +_B z$ are defined in $\mathbf{B}$. Then $x + y$ and $(x + y) + z$ are defined in $\mathbf{A}$. By associativity in $\mathbf{A}$, it follows that $y + z$ and $x + (y + z)$ are defined in $\mathbf{A}$, and $(x + y) + z = x + (y + z)$. By substitution, $x + (y + z) = (x +_B y) +_B z$. It remains to show that $y +_B z$ is defined in $\mathbf{B}$. By conjugation in $\mathbf{A}$, there exists a $w \in A$ such that $x + (y + z) = (y + z) + w = (x +_B y) +_B z$. Therefore, $y + z \leq (x +_B y) +_B z \in B$, so $y +_B z \in B$ since $\mathbf{B}$ is a downset, and we can conclude that $(x +_B y) +_B z = x +_B (y +_B z)$.

Conjugation: Assume $x +_B y$ is defined in $\mathbf{B}$. Then $x + y$ is defined in $\mathbf{A}$ and by conjugation in $\mathbf{A}$, there exist $u, v \in A$ such that $x + y = u + x = y + v$. Hence $u \leq x + y = x +_B y \in B$, so $u \in B$ and $u +_B x = x +_B y$ in $\mathbf{B}$. Similarly, there exists $w \in A$ such that $y + v = v + w = x + y$ by conjugation in $\mathbf{A}$. Thus, $v \leq x + y = x +_B y \in B$, so $v \in B$ and $y +_B v = x +_B y$ in $\mathbf{B}$. □

The next result shows what needs to be checked to extend a GPE-algebra with a new maximal element n. The forward direction of the proof follows from the assumption that $\mathbf{A}$ is a relative subalgebra of $\mathbf{A}'$, and for the reverse direction it suffices to check that the GPE-axioms hold in $\mathbf{A}'$. The final statement of the theorem follows from Lemma 6

Theorem 7. *Let $\mathbf{A} = (A, \oplus, 0)$ be a GPE-algebra and let $A' = A \cup \{n\}$ for $n \notin A$. Then $\mathbf{A}' = (A', +, 0)$ is a GPE-algebra with $\mathbf{A}$ as a relative subalgebra and n as maximal element if and only if the following conditions hold for all $x, y, z \in A$*

1. *$x + y \in A$ if and only if $x \oplus y$ is defined, in which case $x + y = x \oplus y$,*
2. *$n + 0 = n = 0 + n$,*
3. *$x \neq 0 \implies n + x$ and $x + n$ are undefined, and $n + n$ is undefined,*
4. *$x + y = n = x + z \implies y = z$ and $x + y = n = z + y \implies x = z$,*
5. *$x + y = n \implies u + x = n = y + v$ for some $u, v \in A$, and*
6. *$(x + y) + z = n \iff x + (y + z) = n$.*

Furthermore, every GPE-algebra of cardinality $n + 1$ has a relative subalgebra of cardinality n.

Our algorithm uses the preceding result to construct all GPE-algebras of cardinality n starting with the one-element GPE-algebra. This is done by a backtracking search, ensuring that all possible one-point extensions of each algebra are computed. To remove isomorphic copies efficiently, the binary operation is coded as a directed graph and a canonical labeling algorithm is used to map to a unique fixed representative of the isomorphism class of the directed graph. Some optimizations are used for the cancellativity, conjugation and associativity checks. The algorithm was implemented in Python and uses a canonical labeling algorithm from the Sage computer algebra systems [15]. The number of algebras computed up to isomorphism in each subclass of GPE-algebras are summarized in Table 1 and the partial algebras can be downloaded from `https://github.com/jipsen/Effect-algebras`. For the class Sep of separation algebras and for the class of GS-algebras the Mace4 model finder [11] was used.

n	O	PO	GO	GPO	E	PE	GE	Sep	GPE	GS
2	1	1	1	1	1	1	1	2	1	2
3	0	0	1	1	1	1	2	3	2	3
4	1	1	2	2	3	3	5	8	5	8
5	0	1	2	3	4	5	12	13	13	14
6	1	2	4	7	10	12	35	39	42	48
7	0	2	8	19	14	19	119	120	171	172
8	2	5	18	68	40	52	496	507	1020	
9	0	4	42	466	60	84	2699		11742	
10	2	10	156	8740	172	240	21888		322918	
11	0	9	834		282		292496			

Table 1. Number of partial algebras in each class

As indicated by Theorem 4, there are only a small number of GS-algebras that are not GPE-algebras since the structure of a generalized separation algebra is highly restricted by its group of invertible elements and the GPE-quotient determined by this group.

5 Further results about GPE-algebras

The *height* of an element a in a finite GPE-algebra is the length of a maximal path from 0 to a in the Hasse diagram of the partial order. A set of elements of the same height make up a *level*. The *atoms* of a GPE-algebra are the elements in level 1, i.e, they only have the bottom element 0 below them.

Lemma 8. *Associativity holds automatically for naturally ordered partial algebras that have two levels or less.*

Proof. Level 1: If $(x + y) + z$ is defined in a partial algebra with 1 level, then $x + y = 0$ or $z = 0$. If $x + y = 0$, then $x, y = 0$ by positivity and so $(0 + 0) + z = z = 0 + (0 + z)$. If $z = 0$, then $(x + y) + 0 = x + y = x + (y + 0)$. In either case, associativity holds.

Level 2: Now assume $(x + y) + z$ is defined in a partial algebra with level 2. If $(x + y) + z$ has height 1, then it satisfies associativity by the same reasoning as the first part of this proof. If $(x + y) + z$ has height 2, then there are three possibilities:

(i) $x+y$ has height 2 and z has height 0, in which case $(x+y)+0 = x+y = x + (y + 0)$. (ii) $x + y$ has height 0 and z has height 2, in which case $x, y = 0$ and so $(0 + 0) + z = z = 0 + (0 + z)$. (iii) $x + y$ and z both have height 1. $x + y$ of height 1 implies that either $x = 0$ or $y = 0$. This means we either have $(0 + y) + z = y + z = 0 + (y + z)$ or $(x + 0) + z = x + z = x + (0 + z)$. □

Lemma 9. *A GPE-algebra is a GE-algebra if and only if it has a generating set in which all elements commute.*

Proof. Since GE-algebras are by definition commutative, the elements of any generating set will commute trivially. Thus we only need to prove the reverse implication, which we do by induction on the level n.

Let $\mathbf{A}$ be a GPE-algebra with a set of generators X such that for all $x, y \in X$ either $x + y, y + x$ are both undefined or $x + y = y + x$. $P(n)$: All levels up to and including n are commutative.

$P(2)$: Let $x, y \in A$ and w.l.o.g., let x have height 2 and y have any height. Then there exist atoms $a, b \in X$ such that $a + b = x$ and elements $c, d \in X \cup \{0\}$ such that $c + d = y$. If $x + y$ is defined, then by commutativity and associativity of $X \cup \{0\}$, we get that $x + y = (a + b) + (c + d) = (c + d) + (a + b) = y + x$.

Now assume $P(k)$ holds for all $2 \le n \le k$.

$P(k + 1)$: Let $x, y \in A$ and w.l.o.g., let x have height $k + 1$ and y have any height. Then there exist $a, b, c, d \in A$ with heights less than $k + 1$ such that $a + b = x$ and $c + d = y$. If $x + y$ is defined, then by the inductive hypothesis and associativity we have $(a + b) + (c + d) = (c + d) + (a + b)$ and thus $x + y = y + x$ for any x on level $k + 1$ with $x + y$ defined. □

It is an elementary result in group theory that every 1-generated group is commutative. For GPE-algebras a similar result holds for 1- and 2-generated algebras.

Theorem 10. *Every 2-generated GPE-algebra is commutative.*

Proof. Let $\mathbf{A}$ be a 2-generated GPE-algebra with atoms $a \neq b$. By symmetry, it suffices to show that if $a + b$ is defined, then $b + a$ is defined and the two values are equal. So assume $a + b$ is defined. Then, by conjugation, there exists a $w \in A$ such that $a + b = w + a$. It follows from the last equation that $w \le a + b$, which means w is either $a + b, a, b$ or 0, since a, b are atoms. By cancellativity, w cannot be 0, a or $a + b$, hence $w = b$ and we have that $a + b = b + a$. Since the atoms commute, the previous lemma implies that $\mathbf{A}$ is commutative. □

Let $L(n_1, n_2, ..., n_k)$ denote the number of GPE-algebras (up to isomorphism) with level structure $(n_1, n_2, ..., n_k)$ and $n = 1 + \sum_{i=1}^{k} n_i$ number of elements. The number of *integer partitions* for a positive integer n, also called the *partition* *function*, is the number of ways positive integers can sum to n, ignoring order, and is denoted by $p(n)$.

We now show that the number of GPE-algebras of height ≤ 2 with cardinality n is given by the sum of the partition function from 1 to n. We first observe that a partial operation $+$ can be viewed as a coalgebra $\alpha : A \to \mathcal{P}(A^2)$ where $\alpha(x) = \{(y, z) \in A^2 \mid x = y + z\}$.

Lemma 11. *For a GPE-algebra* $\mathbf{A}$ *and* $x \in A$*, the binary relation* $\alpha(x)$ *is a permutation of its domain, hence in the finite case the domain is partitioned into disjoint finite cycles.*

Proof. The relation $\alpha(x)$ is a function on its domain since $\cdot$ is left-cancellative, and injective since $\cdot$ is right-cancellative. By conjugation $\alpha(x)$ is surjective, hence it is a permutation. Letting α act on the domain gives the partition into cycles. □

Lemma 12. *For any GPE-algebra of size* $n \geq 3$, $L(n-2, 1) = L(n-3, 1) + p(n-2)$.

Proof. Consider an algebra $\mathbf{A} = (A, +, 0)$ of size $n-1$ with the level structure given by $(n-3, 1)$. Define a new algebra $\mathbf{A}' = (A', \oplus, 0)$ of size n by $A' = A +_0 \bar{O}_2$ (see Fig. 2 for examples of the bottom sum $+_0$). Then A' is a GPE-algebra of size n with a level structure given by $(n-2, 1)$. This means that at the very least there are $L(n-3, 1)$ GPE-algebras of size n with level structure $(n-2, 1)$.

Now let $\mathbf{A}$ be a pseudoeffect algebras with level structure $(n-2, 1)$, and let x be the top element. For every element y in the first level, there exists y' such that $y + y' = x$, hence the domain of α is $\mathbf{A}$. By Lemma 11, A is partitioned into disjoint cycles, with one of the cycles being $\{0, x\}$. Therefore the remaining $n-2$ elements in level 1 are partitioned into cycles, and there are $p(n-2)$ different possible partitions up to isomorphism. □

Theorem 13. *The number of GPE-algebra of cardinality* n *with level structure* $(n-2, 1)$ *is* $\sum_{k=1}^{n} p(k)$.

Recall that the partial order on a GPE-algebra is given by $a \leq b \iff \exists z\,(a + z = b) \iff \exists w\,(w + a = b)$. By cancellativity, z, w are unique, so we denote $z = a\backslash b$ and $w = b/a$. Rump and Yang [13] define a *two-point extension* for a GPE-algebra $\mathbf{A}$ that produces a total algebra $\mathbf{A}_\bot^\top = (A \cup \{\bot, \top\}, \leq, \cdot, e, \backslash, /, \bot, \top)$ such that $\leq$ is the natural order, extended with $\bot \leq x \leq \top$, let $e = 0$,

$$a \cdot b = \begin{cases} a + b & \text{if } a + b \text{ is defined} \\ \bot & \text{if } a = \bot \text{ or } b = \bot \\ \top & \text{otherwise} \end{cases}$$

$\bot\backslash x = x/\bot = x\backslash\top = \top/x = \top$ and if $a \not\leq b$ then define $a\backslash b = \bot = b/a$.

A *residuated partially ordered monoid* $(A, \leq, \cdot, e, \backslash, /)$ is a poset $(A, \leq)$ and a monoid $(A, \cdot, e)$, and for all $x, y, z \in A$, $xy \leq z \iff y \leq x\backslash z \iff y \leq z/x$.

Theorem 14 **([13]).** *Let* $\mathbf{A}$ *be a GPE-algebra. Then* $\mathbf{A}_\bot^\top$ *is a residuated po-monoid, i.e.,* $(A, \leq)$ *is a poset,* $(A, \cdot, e)$ *is a monoid and* $xy \leq z \iff y \leq x\backslash z \iff y \leq z/x$.

The preceeding result shows that every GPE-algebra is an interval in a total residuated po-monoid that has a unit e as its unique atom.

A residuated po-monoid is *involutive* [7] if there exists an element d such that the terms ${\sim}x = x\backslash d$ and $-x = d/x$ satisfy $-{\sim}x = x = {\sim}{-}x$. Then $-d = e = {\sim}d$ and $x\backslash y = {\sim}((-y)x)$, $x/y = -(y({\sim}x))$. Equivalently, $(A, \leq, \cdot, e, d, \sim, -)$ is an involutive residuated po-monoid if $-{\sim}x = x = {\sim}{-}x$ and $xy \leq d \iff x \leq -y$.

Theorem 15. *The two-point totalization of PE/PO-algebras, effect algebras and orthoalgebras produces involutive residuated po-monoids.*

Recall that a groupoid is a (small) category in which every morphism is an isomorphism. While groups capture the symmetries of individual mathematical objects, groupoids model symmetries of systems of related objects. For example, the fundamental groupoid of a topological space captures more information about the space than the fundamental group determined by a choice of base point.

We end this section with a recent generalization of effect algebras that is similar to modifications of separation algebras in [4] that allow several local identity elements.

A pseudoeffect algebra is *symmetric* if $x^\sim = x^-$. Roumen [12] has taken the important step of generalizing symmetric pseudoeffect algebras to effect algebroids. Here the concept is reformulated for a unisorted partial algebra.

An *effect algebroid* is a partial algebra $(A, +, ')$ such that

(asso) $(x + y) + z = x + (y + z)$
(idenL) $(x + x')' + x = x$
(orthL) $x + y$ defined and $x + y = x + x'$ implies $y = x'$
(orthR) $x + y$ defined and $x + y = y' + y$ implies $x = y'$
(dbl) $x'' = x$
(0-1) if $x + (x' + x)$ is defined then $x = (x' + x)'$.

For comparison with effect algebras and pseudoeffect algebras, we computed the number of effect algebroids of cardinality n.

$n =$	1	2	3	4	5	6	7	8
Effect algebroids	1	2	3	7	12	27	49	114

An effect algebroid is a symmetric pseudoeffect algebra if and only if it satisfies $x + x' = y + y'$. It is an effect algebra if, in addition, it satisfies $x + y = y + x$.

6 Conclusion

Partial algebras are considerably more general than total algebras. The class of generalized separation algebras and its subclass of generalized pseudoeffect algebras are closely related, but so far have been studied separately since they arose in the unrelated areas of separation logic and quantum logic. We proved that there is a canonical map from separation algebras to GPE-algebras and computed finite GPE-algebras up to 10 elements (up to 11 elements for GE-algebras). Insight from these finite models was used to prove that all 2-generated GPE-algebras are commutative and to describe all GPE-algebras with a single element on the second level.

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