Periodic lattice-ordered pregroups are distributive

Nikolaos Galatos and Peter Jipsen

Abstract. It is proved that any lattice-ordered pregroup that satisfies an identity of the form $x^{ll\cdots l} = x^{rr\cdots r}$ (for the same number of ^{l,r}-operations on each side) has a lattice reduct that is distributive. It follows that every such ℓ-pregroup is embedded in an ℓ-pregroup of residuated and dually residuated maps on a chain.

Lambek [9] defined *pregroups* as partially ordered monoids $(A, \cdot, 1, \leq)$ with two additional unary operations ^{l}, ^{r} that satisfy the inequations

$$x^l x \leq 1 \leq x x^l \qquad \text{and} \qquad x x^r \leq 1 \leq x^r x.$$

These algebras were introduced to model some aspects of grammars, and have been studied from algebraic and proof-theoretic points of view in several papers by W. Buskowski [2, 3, 4, 5].

A *lattice-ordered pregroup*, or *ℓ-pregroup*, is of the form $(L, \wedge, \vee, \cdot, 1, {}^l, {}^r)$ where $(L, \wedge, \vee)$ is a lattice and $(L, \cdot, 1, {}^l, {}^r, \leq)$ is a pregroup with respect to the lattice order. Alternatively, an ℓ-pregroup is a residuated lattice that satisfies the identities $x^{lr} = x = x^{rl}$ and $(xy)^l = y^l x^l$ where $x^l = 1/x$ and $x^r = x\backslash 1$. Another equivalent definition of ℓ-pregroups is that they coincide with involutive FL-algebras in which $x \cdot y = x + y$ and $0 = 1$. In particular, the following (quasi-)identities are easy to derive for (ℓ-)pregroups:

$$\begin{array}{rl}
x^{lr} = x = x^{rl} & 1^l = 1 = 1^r \\
(xy)^l = y^l x^l & (xy)^r = y^r x^r \\
x x^l x = x & x x^r x = x \\
x^l x x^l = x^l & x^r x x^r = x^r \\
x(y \vee z)w = xyw \vee xzw & x(y \wedge z)w = xyw \wedge xzw \\
(x \vee y)^l = x^l \wedge y^l & (x \vee y)^r = x^r \wedge y^r \\
(x \wedge y)^l = x^l \vee y^l & (x \wedge y)^r = x^r \vee y^r
\end{array}$$

$$x^l = x^r \iff x^l x = 1 = x x^l \iff x x^r = 1 = x^r x$$

Lattice-ordered groups are a special case of ℓ-pregroups where the identity $x^l = x^r$ holds, which is equivalent to x^l (or x^r) being the inverse of x. It is well-known that ℓ-groups have distributive lattice reducts. Other examples of ℓ-pregroups occur as subalgebras of the set of finite-to-one order-preserving functions on $\mathbb{Z}$ (where *finite-to-one* means the preimage of any element is a finite set). These functions clearly form a lattice-ordered monoid, and if a is such a function then $a^l(y) = \bigwedge\{x \in \mathbb{Z} | a(x) \geq y\}$ and $a^r(y) = \bigvee\{x \in \mathbb{Z} | a(x) \leq y\}$.

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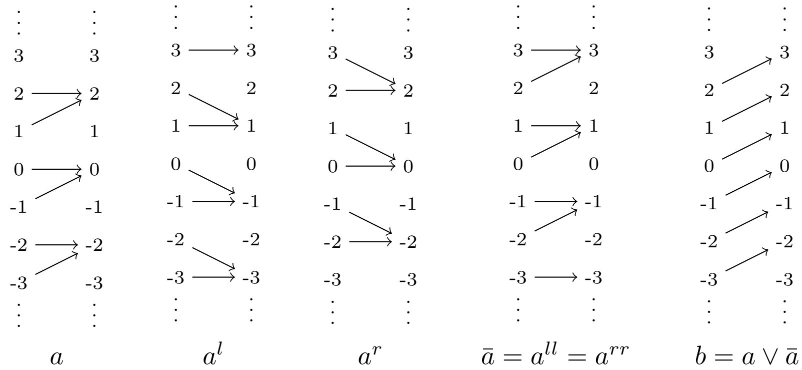

$$
\begin{aligned}
&A = \{ab^n, b^n, \bar{a}b^n : n \in \mathbb{Z}\}\\
&b^l = b^r = b^{-1}, \quad b^m b^n = b^{m+n}\\
&aa = a, \quad \bar{a}\bar{a} = \bar{a}\\
&a\bar{a} = ab, \quad \bar{a}a = \bar{a}b\\
&b^n a = \begin{cases} ab^n & \text{if } n \text{ is even} \\ \bar{a}b^n & \text{if } n \text{ is odd} \end{cases}\\
&b^n \bar{a} = \begin{cases} \bar{a}b^n & \text{if } n \text{ is even} \\ ab^n & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

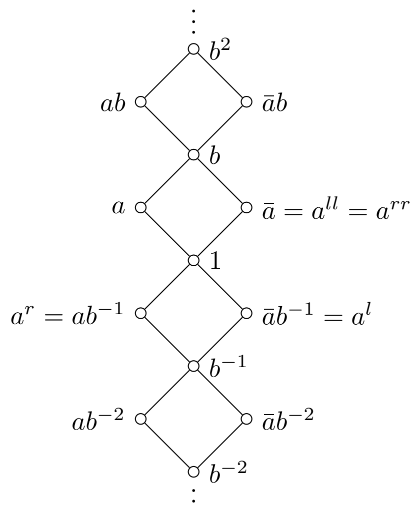

FIGURE 1. The ℓ-pregroup of period 2

The notation x^{l^n} is defined by $x^{l^0} = x$ and $x^{l^{n+1}} = (x^{l^n})^l$ for $n \geq 0$, and similarly for x^{r^n}. We say that an ℓ-pregroup is *periodic* if it satisfies the identity $x^{l^n} = x^{r^n}$ for some positive integer n. The aim of this note is to prove that if an ℓ-pregroup is periodic then the lattice reduct must also be distributive. For $n = 1$ this identity defines ℓ-groups, but for $n = 2$ it defines a strictly bigger subvariety of ℓ-pregroups since it contains the ℓ-pregroup generated by the function $a : \mathbb{Z} \to \mathbb{Z}$ defined by $a(2m) = a(2m-1) = 2m$ for $m \in \mathbb{Z}$. A diagram of this algebra is given in Figure 1. Note that all functions in this algebra have period 2. Similarly the function $a_n : \mathbb{Z} \to \mathbb{Z}$ defined by $a_n(nm) = a_n(nm-1) = \cdots = a_n(nm-(m-1)) = nm$ generates an ℓ-pregroup that satisfies $x^{l^n} = x^{r^n}$, and all functions in it have period n.

The proof of Theorem 5 below was initially found with the help of the WaldmeisterII equational theorem prover [11] and contained 274 lemmas (about

1900 equational steps). Lemmas 2-4 below were extracted by hand from the automated proof.

The first lemma is true for any binary operation that distributes over $\wedge$,$\vee$ and has an identity element.

Lemma 1. $(1 \vee x)(1 \wedge x) = x = (1 \wedge x)(1 \vee x)$

Proof. $(1{\vee}x)(1{\wedge}x) = (1{\wedge}x){\vee}(x{\wedge}xx) \leq x \leq (1{\vee}x){\wedge}(x{\vee}xx) = (1{\vee}x)(1{\wedge}x)$. $\square$

The next few lemmas are true for ℓ-pregroups in general.

Lemma 2.

(i) $x(1 \wedge x^l y) = xx^l(x \wedge y)$
(ii) $x(1 \vee x^r y) = xx^r(x \vee y)$
(iii) $1 \vee x^l y = 1 \vee x^l(x \vee y)$
(iv) $1 \wedge x^r y = 1 \wedge x^r(x \wedge y)$

Proof. (i) $x(1 \wedge x^l y) = x \wedge xx^l y = xx^l x \wedge xx^l y = xx^l(x \wedge y)$, and (ii) is similar.

(iii) $1 \vee x^l y = 1 \vee x^l x \vee x^l y = 1 \vee x^l(x \vee y)$, and again (iv) is similar. $\square$

Lemma 3. *If $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$ then $x^l y = x^l z$, $x^r y = x^r z$, $yx^l = zx^l$ and $yx^r = zx^r$.*

Proof. Assume $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$.

By Lemma 2 (i) we have $x(1 \wedge x^l y) = xx^l(x \wedge y) = xx^l(x \wedge z) = x(1 \wedge x^l z)$, and similarly from (ii)-(iv) we get $x(1 \vee x^r y) = x(1 \vee x^r z)$, $1 \vee x^l y = 1 \vee x^l z$ and $1 \wedge x^r y = 1 \wedge x^r z$.

Using Lemma 1 $xx^l y = x(1 \wedge x^l y)(1 \vee x^l y) = x(1 \wedge x^l z)(1 \vee x^l z) = xx^l z$, hence $x^l y = x^l xx^l y = x^l xx^l z = x^l z$

Similarly $x^r y = x^r z$, $yx^l = zx^l$ and $yx^r = zx^r$. $\square$

Lemma 4. *If $x^{ll} = x^{rr}$ then $x^l \vee x^r$ and $x^l \wedge x^r$ are invertible.*

Proof. If $x^{ll} = x^{rr}$ then $(x^l \vee x^r)^l = x^{ll} \wedge x = x^{rr} \wedge x = (x^l \vee x^r)^r$, hence $(x^l \vee x^r)^l(x^l \vee x^r) = 1$, i.e. $x^l \vee x^r$ is invertible, and similarly for $x^l \wedge x^r$. $\square$

Theorem 5. *If the identity $x^{ll} = x^{rr}$ holds in an ℓ-pregroup then the lattice reduct is distributive.*

Proof. It is well-known that a lattice is distributive if every element has a unique relative complement. Hence we assume $a, b, c \in L$ satisfy $a \wedge b = a \wedge c$, $a \vee b = a \vee c$ and we have to prove that $b = c$.

By Lemma 3 we have $a^l b = a^l c$ and $a^r b = a^r c$, so $(a^l \vee a^r)b = a^l b \vee a^r b = a^l c \vee a^r c = (a^l \vee a^r)c$. By Lemma 4 it follows that $b = c$. $\square$

Note that the converse of Lemma 4 also holds, since if $x^l \vee x^r$ and $x^l \wedge x^r$ are invertible then $x^{ll} \wedge x = (x^l \vee x^r)^l = (x^l \vee x^r)^r = x^{rr} \wedge x$ and $x^{ll} \vee x = x^{rr} \vee x$, so as in the proof of Theorem 5 $x^{ll} = x^{rr}$.

To extend the proof to subvarieties of ℓ-pregroups defined by $x^{l^n} = x^{r^n}$ we first prove a few more lemmas.

Lemma 6. $x \vee (x^r \wedge 1) = x \vee 1$

Proof. It suffices to show that $x \vee (x^r \wedge 1) \geq 1$. We have $1 \leq (x \vee 1)^r (x \vee 1) = (x \vee 1)^r x \vee (x \vee 1)^r \leq x \vee (x^r \wedge 1)$ since $(x \vee 1)^r \leq 1$. □

Lemma 7. *$x \vee (yx^r \wedge 1)y = x \vee y$ and $x \wedge (x^r y \vee 1)y = x \wedge y$*

Proof. $x \vee (yx^r \wedge 1)y = x \vee xy^l y \vee ((xy^l)^r \wedge 1)y = x \vee (xy^l \vee ((xy^l)^r \wedge 1))y = x \vee (xy^l \vee 1)y$ by the preceding lemma. The last expression equals $x \vee xy^l y \vee y$, and since $y^l y \leq 1$ we have shown that $x \vee (yx^r \wedge 1)y = x \vee y$.

The second identity follows from the first by substituting x^l, y^l for x, y and then applying $(\ )^r$ to both sides. □

Note that if $(L, \wedge, \vee, \cdot, 1, {}^l, {}^r)$ is an ℓ-pregroup, then so is the 'opposite' algebra $(L, \wedge, \vee, \odot, 1, {}^r, {}^l)$, where $x \odot y = y \cdot x$.

Lemma 8. *If $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$ then $yx^l y \wedge y = zx^l z \wedge z$, $yx^l y \vee y = zx^l z \vee z$, $x^{ll} \vee y = x^{ll} \vee z$ and $x^{ll} \wedge y = x^{ll} \wedge z$. The 'opposite' identities with ^{l} replaced by ^{r} also hold.*

Proof. Assume $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$.

By Lemma 3 $yx^l y \wedge y = zx^l y \wedge y \leq zx^l y \wedge zz^l y = z(x^l \wedge z^l)y = z(x^l \wedge y^l)y = zx^l y \wedge zy^l y \leq zx^l z \wedge z$, and the reverse inequality is proved by interchanging y, z. The second equation has a dual proof.

From these two equations and Lemma 7 we obtain $x^{ll} \vee y = x^{ll} \vee (yx^{llr} \wedge 1)y = x^{ll} \vee (yx^l y \wedge y) = x^{ll} \vee (zx^l z \wedge z) = x^{ll} \vee (zx^{llr} \wedge 1)z = x^{ll} \vee z$, and the fourth equation is proved dually. □

Using the preceding lemma repeatedly, it follows that if $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$ then $x^{l^{2m}} \vee y = x^{l^{2m}} \vee z$ and $x^{l^{2m}} \wedge y = x^{l^{2m}} \wedge z$ for all $m \in \omega$. As in Lemma 3, it follows that $x^{l^{2m+1}} y = x^{l^{2m+1}} z$ and $x^{r^{2m+1}} y = x^{r^{2m+1}} z$ for all $m \in \omega$. Now in the presence of the identity $x^{l^n} = x^{r^n}$, the term $t(x) = x^l \vee x^{lll} \vee \cdots \vee x^{l^{2n-1}}$ produces invertible elements only. Indeed, $x^{l^n} = x^{r^n}$ entails $x^{l^{2n}} = (x^{l^n})^{l^n} = (x^{r^n})^{l^n} = x$, and therefore $t(x)^l = t(x)^r$. As in the proof of Theorem 5, if we assume $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ then we have $t(a)b = t(a)c$, hence $b = c$. Thus we obtain the following result.

Theorem 9. *If the identity $x^{l^n} = x^{r^n}$ holds in an ℓ-pregroup then the lattice reduct is distributive.*

However, it is not known whether the lattice reducts of all ℓ-pregroups are distributive. It is currently also not known if the identity $(x \vee 1) \wedge (x^l \vee 1) = 1$ holds in every ℓ-pregroup (it is implied by distributivity). Recently M. Kinyon [8] has shown with the help of Prover9 that if an ℓ-pregroup is modular then it is distributive. The following result has been proved in [1] and [10].

Theorem 10. *An ℓ-monoid can be embedded in the endomorphism ℓ-monoid of a chain if and only if $\cdot, \vee$ distribute over $\wedge$.*

Recall that a map f from a poset $\mathbf{P}$ to a poset $\mathbf{Q}$ is called *residuated* if there is a map $f^*: Q \to P$ such that $f(p) \leq q \Leftrightarrow p \leq f^*(q)$, for all $p \in P$ and $q \in Q$. Then f^* is unique and is called the *residual* of f, while f is called the *dual residual* of f^*. It is well-known that residuated maps between posets are necessarily order-preserving. The map $(f^*)^*$, if it exists, is called the *second-order residual* of f, and likewise we define higher-order residuals and dual residuals of f. In [7] (page 206) it is mentioned, using different terminology, that the set $RDR^\infty(\mathbf{C})$ of all maps on a chain $\mathbf{C}$ that have residuals and dual residuals of all orders forms a (distributive) ℓ-pregroup, under pointwise order and functional composition. Hence we obtain our final result, which was first noted in [6].

Corollary 11. *Every periodic ℓ-pregroup can be embedded in $RDR^\infty(\mathbf{C})$, for some chain $\mathbf{C}$.*

Proof. Let $\mathbf{A}$ be a periodic ℓ-pregroup. By the two preceding theorems there is a chain $\mathbf{C}$ and an ℓ-monoid embedding $h: \mathbf{A} \to \mathrm{End}(\mathbf{C})$. Since $\mathbf{A}$ satisfies $xx^r \leq 1 \leq x^r x$ we have

$$h(x) \circ h(x^r) \leq \mathrm{id}_C \leq h(x^r) \circ h(x). \qquad (*)$$

The functions $h(x)$ and $h(x^r)$ are order-preserving, so $h(x^r)$ is the residual of $h(x)$. Therefore $h(x^r) = h(x)^*$, and by substitution $h(x^{rr}) = h(x^r)^* = h(x)^{**}$, etc., which shows that residuals $h(x)^{*\cdots*}$ of all orders exists in $h[A]$. Similarly $h(x^l)$ is the dual residual of $h(x)$ and dual residuals of all orders exist. Hence $h(x) \in RDR^\infty(\mathbf{C})$. From $(*)$ above it also follows that $h(x^r) = h(x)^r$, and an analogous argument shows $h(x^l) = h(x)^l$, thus $h: \mathbf{A} \to RDR^\infty(\mathbf{C})$ is an ℓ-pregroup embedding. $\square$

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