The Structure of Locally Integral Involutive Po-monoids and Semirings

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Abstract. We show that every locally integral involutive partially ordered monoid (ipo-monoid) $\mathbf{A} = (A, \leqslant, \cdot, 1, \sim, -)$, and in particular every locally integral involutive semiring, decomposes in a unique way into a family $\{\mathbf{A}_p : p \in A^+\}$ of integral ipo-monoids, which we call its *integral components*. In the semiring case, the integral components are semirings. Moreover, we show that there is a family of monoid homomorphisms $\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant q\}$, indexed on the positive cone $(A^+, \leqslant)$, so that the structure of $\mathbf{A}$ can be recovered as a glueing $\int_\Phi \mathbf{A}_p$ of its integral components along Φ. Reciprocally, we give necessary and sufficient conditions so that the Płonka sum of any family of integral ipo-monoids $\{\mathbf{A}_p : p \in D\}$, indexed on a lower-bounded join-semilattice $(D, \vee, 1)$, along a family of monoid homomorphisms Φ is an ipo-monoid.

Keywords: Residuated lattices · Involutive partially ordered monoids · Semirings · Płonka sums · Frobenius quantales.

1 Introduction

Idempotent semirings are algebras of the form $(A, \vee, \cdot, 1)$ where $(A, \vee)$ is a semilattice (with order $x \leqslant y \iff x \vee y = y$), $(A, \cdot, 1)$ is a monoid, and the monoid operation distributes over the join. They play an important role in mathematics, logic, and theoretical computer science, since they generalize distributive lattices and expand to Kleene algebras and residuated lattices. An *involutive semiring* is an idempotent semiring with operations $\sim$ and $-$ satisfying:

$$x \leqslant y \iff x \cdot {\sim}y \leqslant -1 \iff -y \cdot x \leqslant -1.$$

These algebras are term-equivalent to involutive residuated lattices and, in the case that the lattice is complete, to Frobenius quantales (see [5] and [4]). Furthermore, algebras of binary relations are involutive semirings under the operations of union, composition, and complement-converse. The structural characterization obtained in this paper is valid for more general partially ordered structures called *involutive po-monoids* where the semilattice $(A, \vee)$ is replaced by a poset $(A, \leqslant)$.

An ipo-monoid is *integral* when the monoid identity 1 is also the top element of the order, that is, the inequality $x \leqslant 1$ holds. This is a very important property for residuated lattices, since it is equivalent to the proof-theoretical rule called weakening. In this work we identify a much larger class, namely the class of locally integral ipo-monoids.

The main result in this paper is that every locally integral ipo-monoid $\mathbf{A}$ can be decomposed in a unique way into a family of integral ipo-monoids $\{\mathbf{A}_p : 1 \leqslant p\}$, which we call its *integral components*. Two locally integral ipo-monoids can have the same integral components, but may differ in the way these components are *glued* together. We find in the literature similar situations in which a number of structures are glued together to form a new one: for instance, in [6] it is described how chains can be attached to an odd Sugihara monoid in order to form a commutative idempotent residuated chain, and in [8] how Boolean algebras can be glued together to form commutative idempotent involutive residuated lattices.

In our present case, we associate to every locally integral ipo-monoid $\mathbf{A}$ a join-semilattice indexed family of monoid homomorphisms $\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : 1 \leqslant p \leqslant q\}$ between its integral components so that the structure of $\mathbf{A}$ can be completely recovered as an aggregate or *glueing* $\int_\Phi \mathbf{A}_p$ of these integral components along Φ in two stages: first, the monoid part of $\mathbf{A}$ turns out to be the Płonka sum of the family Φ, and the involutive negations can be defined componentwise. Then, we recover the order of $\mathbf{A}$ using the product, the negations, and the local identities.

As an application of our results, we can combine certain semantics for fuzzy logics with semantics for relevance logic using, for example, the well-understood MV-algebras as building blocks of a glueing.

We exploit this decomposition in order to prove that several properties of locally integral ipo-monoids are *local*, in that a locally integral ipo-monoid satisfies them if and only if all its integral components satisfy them. One of the most significant local properties established here is local finiteness.

Previous research into the structure of doubly-idempotent semirings can be found in [1, 2]. The structure of all finite commutative idempotent involutive residuated lattices is completely described in [8] in a step-by-step decomposition. In the current paper, this is significantly generalized to all locally integral ipo-monoids, without any restrictions regarding finiteness, commutativity, or full idempotence. A similar use of Płonka sums can be found in [7], where the structure of even and odd involutive commutative residuated chains is studied.

We set the terminology and notation in Section 2, and describe the fundamental properties of ipo-monoids needed in the rest of the paper. In Section 3, we introduce the class of locally integral ipo-monoids and show that every locally integral ipo-monoid is the glueing of its integral components. Finally, in Section 4, we solve the reverse problem, that is, we provide necessary and sufficient conditions so that the glueing of a system of integral ipo-monoids is an ipo-monoid.

2 Involutive Partially Ordered Monoids and Semirings

An *involutive partially ordered monoid*, or *ipo-monoid* for short, is a structure of the form $(A, \leqslant, \cdot, 1, \sim, -)$ such that $(A, \leqslant)$ is a poset (i.e., $\leqslant$ is a reflexive, antisymmetric, and transitive binary relation on A), $(A, \cdot, 1)$ is a monoid (i.e., $\cdot$

is an associative binary operation on A and 1 is its identity element) satisfying:

$$x \leqslant y \iff x \cdot {\sim} y \leqslant 0 \iff -y \cdot x \leqslant 0, \tag{ineg}$$

where, by definition, $0 = -1$.[1] The unary operations $\sim$ and $-$ are called *involutive negations*. If there is no danger of confusion, we will write xy instead of $x \cdot y$. Given an ipo-monoid $\mathbf{A}$, we say that $\mathbf{A}$ is *cyclic* if it satisfies ${\sim} x = -x$. An element x of $\mathbf{A}$ is *central* if $x \cdot y = y \cdot x$ for any other $y \in A$, and $\mathbf{A}$ is *commutative* if all its elements are central. An element x of $\mathbf{A}$ is *idempotent* if $x \cdot x = x$, and $\mathbf{A}$ is *idempotent* if all its elements are idempotent. We will be specially interested in ipo-monoids with a lattice order. These can be then presented as algebraic structures $(A, \wedge, \vee, \cdot, 1, \sim, -)$ called *$i\ell$-monoids* or *involutive semirings*.[2]

Lemma 1. *Every ipo-monoid satisfies the following properties:*

1. *double negation:* ${\sim}{-}x = x = -{\sim}x$ (dn)
2. *rotation:* $x \cdot y \leqslant z \iff y \cdot {\sim} z \leqslant {\sim} x \iff -z \cdot x \leqslant -y$ (rot)
3. *antitonicity:* $x \leqslant y \iff {\sim} y \leqslant {\sim} x \iff -y \leqslant -x$ (ant)
4. *residuation:* $xy \leqslant z \iff x \leqslant -(y \cdot {\sim} z) \iff y \leqslant {\sim}(-z \cdot x)$ (res)
5. *constants:* $0 = {\sim} 1$, ${\sim} 0 = 1$, *and* $-0 = 1$. (ct)

The properties of the previous lemma will often be used without mentioning them explicitly. Notice also that the multiplication is residuated, with *left* and *right residuals* $z/y = -(y \cdot {\sim} z)$ and $x \backslash z = {\sim}(-z \cdot x)$, respectively, as (res) can be rewritten as:

$$x \cdot y \leqslant z \iff x \leqslant z/y \iff y \leqslant x \backslash z.$$

The fact that $\cdot$ preserves arbitrary existing joins, and therefore is order-preserving, in both arguments follows easily from these observations. It can be also readily checked that the involutive negations can be expressed in terms of the residuals as follows: ${\sim} x = x \backslash 0$ and $-x = 0/x$. Since in any commutative ipo-monoid the equality $y/x = x \backslash y$ holds, every commutative ipo-monoid is cyclic.

Lemma 2. *Every ipo-monoid satisfies the following properties:*

1. $-({\sim} x \cdot {\sim} y) = {\sim}(-x \cdot -y)$,
2. ${\sim} x$ *is idempotent if and only if* $-x$ *is idempotent.*

Proof. 1. Using (res), (rot), (dn), and (res) again, we obtain $z \leqslant -({\sim} x \cdot {\sim} y) \iff z \cdot {\sim} x \leqslant y \iff -y \cdot z \leqslant -{\sim} x \iff -y \cdot z \leqslant x \iff z \leqslant {\sim}(-x \cdot -y)$. Since z is arbitrary, we deduce that $-({\sim} x \cdot {\sim} y) = {\sim}(-x \cdot -y)$.

2. Assume that ${\sim} x$ is idempotent. Then, by (dn) and the previous part, $-x \cdot -x = -{\sim}(-x \cdot -x) = --({\sim} x \cdot {\sim} x) = --{\sim} x = -x$. The rest is analogous. □

[1] Notice that the symmetry of all the properties of Lemma 1, and specially (ct), suggests that we would obtain the same results had we defined $0 = {\sim} 1$.

[2] This terminology is based on the observation that $(A, \vee, \cdot, 1)$ is an idempotent unital semiring since the residuation property of Lemma 1 implies that $x(y \vee z) = xy \vee xz$ and $(x \vee y)z = xz \vee yz$, and $\wedge$ is term definable by the De Morgan laws.

Lemma 3. *For every ipo-monoid* $\mathbf{A}$*, the following conditions are equivalent:*

1. *The identity* $-x \cdot x = x \cdot {\sim}x$ *holds in* $\mathbf{A}$,
2. *The identity* ${\sim}(-x \cdot x) = -(x \cdot {\sim}x)$*, that is,* $x\backslash x = x/x$*, holds in* $\mathbf{A}$.

Proof. Suppose that the equation $-x \cdot x = x \cdot {\sim}x$ holds in $\mathbf{A}$. In particular, we have that $-{\sim}x \cdot {\sim}x = {\sim}x \cdot {\sim}{\sim}x$, that is, $x \cdot {\sim}x = {\sim}x \cdot {\sim}{\sim}x$. Hence,

$${\sim}(-x \cdot x) = {\sim}(-x \cdot -{\sim}x) = -({\sim}x \cdot {\sim}{\sim}x) = -(x \cdot {\sim}x),$$

where the middle equality follows from Lemma 2(1). In order to prove the other implication, suppose that the equation ${\sim}(-x \cdot x) = -(x \cdot {\sim}x)$ holds in $\mathbf{A}$. In particular, we have that ${\sim}(x \cdot {\sim}x) = {\sim}(-{\sim}x \cdot {\sim}x) = -({\sim}x \cdot {\sim}{\sim}x) = {\sim}(-x \cdot -{\sim}x) = {\sim}(-x \cdot x)$, where again the last but one equality follows from Lemma 2(1). Using (dn), we deduce that $-x \cdot x = x \cdot {\sim}x$. □

Given an ipo-monoid $\mathbf{A}$, we call $A^+ = \{x \in A : 1 \leqslant x\}$ the *positive cone* of $\mathbf{A}$, and its elements the *positive* elements of $\mathbf{A}$, and ${\downarrow}0 = \{x \in A : x \leqslant 0\}$ the principal order-ideal generated by 0. We say that an ipo-monoid $\mathbf{A}$ is ${\downarrow}0$-*idempotent* if all the elements in ${\downarrow}0$ are idempotent. Thus, an involutive semiring is ${\downarrow}0$-idempotent if and only if the quasiequation $x \wedge 0 = x \implies x^2 = x$ holds in $\mathbf{A}$. Furthermore, this property can be expressed by the identity $(x \wedge 0)^2 = x \wedge 0$. Our next result characterizes ${\downarrow}0$-idempotence in ipo-monoids.

Lemma 4. *An ipo-monoid is* ${\downarrow}0$*-idempotent* $\iff$ *for all* $x, y \leqslant 0$, $x \cdot y = x \wedge y$.

Proof. If $\mathbf{A}$ is ${\downarrow}0$-idempotent, then $0 \cdot 0 \leqslant 0$, and applying (rot) we obtain $0 = 0 \cdot 1 = 0 \cdot {\sim}0 \leqslant {\sim}0 = 1$. Thus, if $x, y \leqslant 0$, in particular $x, y \leqslant 1$, and therefore $x \cdot y \leqslant x$ and $x \cdot y \leqslant y$. Also, if $z \leqslant x$ and $z \leqslant y$, then in particular $z \leqslant 0$ and so it is idempotent. Thus, $z = z \cdot z \leqslant x \cdot y$ and hence $x \cdot y = x \wedge y$. Conversely, if $\mathbf{A}$ satisfies that for all $x, y \leqslant 0$, $x \cdot y = x \wedge y$, then for any $x \leqslant 0$, $x \cdot x = x \wedge x = x$. □

The next result shows that ${\downarrow}0$-idempotence implies that all the elements in the positive cone are idempotent. The converse is not always true.

Theorem 5. *If* $\mathbf{A}$ *is an ipo-monoid so that* $0 \leqslant 1$ *and* $x \in {\downarrow}0$ *is idempotent, then both* ${\sim}x$ *and* $-x$ *are idempotent. Thus, all positive elements of a* ${\downarrow}0$*-idempotent ipo-monoid are idempotent.*

Proof. Suppose that $\mathbf{A}$ is an ipo-monoid so that $0 \leqslant 1$ and $x \leqslant 0$ is idempotent. By (ant), $1 \leqslant -x$ and so, $-x = -x \cdot 1 \leqslant -x \cdot -x$. Also, $x \leqslant x$ implies $-xx \leqslant 0 \leqslant 1$, and therefore, $-x \cdot x = -x \cdot x \cdot x \leqslant 1x = x$, and by (rot), $-x \cdot -x \leqslant -x$. Thus, $-x \cdot -x = -x$. By Lemma 2(2), ${\sim}x$ is also idempotent. Finally, if $\mathbf{A}$ is ${\downarrow}0$-idempotent, in particular $0 \cdot 0 = 0$, which implies that $0 \leqslant 1$ by (rot), and for every $1 \leqslant x$, we have that ${\sim}x \leqslant 0$ is idempotent and therefore so is $-{\sim}x = x$. □

3 Locally Integral IPO-Monoids and Involutive Semirings

An ipo-monoid is *integral* if it satisfies the inequality $x \leqslant 1$. Thus, integral ipo-monoids form a po-subvariety of the po-variety of ipo-monoids, in the sense of [9]. Notice that, since the inequality $x \leqslant 1$ can be expressed as $x \vee 1 = 1$ in the language of involutive semirings, the integral involutive semirings form a subvariety of the involutive semirings. We will introduce in what follows another po-subvariety of ipo-monoids and the corresponding subvariety of the variety of involutive semirings. We say that an ipo-monoid is *locally integral*[3] if

1. it satisfies the identity $-x \cdot x = x \cdot {\sim}x$,
2. multiplication is square-decreasing, that is, $x^2 \leqslant x$,
3. it is $\downarrow$0-idempotent.

The main goal of this section is a decomposition theorem stating that every locally integral ipo-monoid (involutive semirings, respectively) can be decomposed in a very particular way into integral involutive ipo-monoids (involutive semiring, respectively). Let's start by proving that integrality implies local integrality.

Proposition 6. *Every integral ipo-monoid is locally integral.*

Proof. Suppose that $\mathbf{A}$ is an integral ipo-monoid. The inequality $1 \cdot x \leqslant x$ implies that $1 \leqslant x\backslash x$, and therefore $x\backslash x = 1$, by the integrality of $\mathbf{A}$. Analogously, $x/x = 1$, and hence $x\backslash x = x/x$, which by Lemma 3 is equivalent to $-x \cdot x = x \cdot {\sim}x$.

The square decreasing property follows immediately from the monotonicity of multiplication, since $x \leqslant 1$ implies that $xx \leqslant 1x = x$.

Finally, ${\sim}x \leqslant 1$ implies that $0 \leqslant x$, for all x in $\mathbf{A}$, and in particular ${\downarrow}0 = \{0\}$. Furthermore, $1 \cdot 0 \leqslant 1$ implies that $0 \cdot 0 = 0 \cdot {\sim}1 \leqslant {\sim}1 = 0$, whence we deduce that $0 \cdot 0 = 0$, proving that $\mathbf{A}$ is $\downarrow$0-idempotent. □

Given a locally integral ipo-monoid $\mathbf{A}$, we define for every x in A the elements $0_x = x \cdot {\sim}x$ and $1_x = -0_x$. Local integrality implies that $0_x = -x \cdot x$ and $1_x = {\sim}0_x$, by Lemma 3, and hence ${\sim}1_x = -1_x = 0_x$. Notice also that $1_x = x\backslash x = x/x$, and hence $0_x \leqslant 0$ and $1 \leqslant 1_x$. Thus, both 0_x and 1_x are idempotent. We will use the interval notation $[0_x, 1_x] = \{y \in A : 0_x \leqslant y \leqslant 1_x\}$. The equivalence relation $x \equiv y$ if and only if $1_x = 1_y$ partitions every locally integral ipo-monoid in its equivalence classes $A_x = \{y \in A : 1_x = 1_y\}$ and, obviously, $x \in A_x$. The next lemma offers a very useful description of A_x.

Lemma 7. *For any locally integral ipo-monoid* $\mathbf{A}$ *and all* x *and* y *in* A*:*

1. $0_{\sim x} = 0_{-x} = 0_x$ *and* $1_{\sim x} = 1_{-x} = 1_x$,
2. $x \in [0_x, 1_x]$, *and therefore* $0_x \leqslant 1_x$,
3. $1_x \cdot y = y \iff 1_x \leqslant 1_y$,
4. $y \in [0_x, 1_x] \iff [0_y, 1_y] \subseteq [0_x, 1_x]$,
5. $y \in A_x \iff y \in [0_x, 1_x]$ *and* $1_x \cdot y = y$.

[3] This class forms a po-quasivariety, by definition. It is not known whether it is a po-variety or a proper po-quasivariety.

Proof. 1. A simple computation shows that $0_{\sim x} = -{\sim}x \cdot {\sim}x = x \cdot {\sim}x = 0_x$, and therefore $1_{\sim x} = -0_{\sim x} = -0_x = 1_x$. The proof that $0_{-x} = 0_x$ and $1_{-x} = 1_x$ is analogous.

2. The square-decreasing property, namely, $x \cdot x \leqslant x$, can also be expressed as $x \leqslant x\backslash x = 1_x$, by residuation. Thus, using part (1), we have that $0_x = 0_{\sim x} = -1_{\sim x} \leqslant -{\sim}x = x$. That is, $x \in [0_x, 1_x]$.
3. $1_x \leqslant 1_y = y/y$ is equivalent to $1_x \cdot y \leqslant y$, by residuation. And since $1 \leqslant 1_x$, we also have that $y \leqslant 1_x \cdot y$. Hence, $1_x \cdot y \leqslant y$ is equivalent to $1_x \cdot y = y$.
4. For the left-to-right implication, notice that $0_x \leqslant y \leqslant 1_x$ implies that $0_x \leqslant {\sim}y \leqslant 1_x$, by (ant), and then $0_x = 0_x \cdot 0_x \leqslant y \cdot {\sim}y = 0_y$. By (ant) again, we obtain that $1_y \leqslant 1_x$. The reverse implication is a consequence of part (2).
5. If $y \in A_x$, then $1_y = 1_x$, and thus $y \in [0_y, 1_y] = [0_x, 1_x]$, by part (2). Moreover, $1_x \cdot y = 1_y \cdot y = y$, by part (3). For the reverse implication, notice that if $y \in [0_x, 1_x]$ and $1_x \cdot y = y$, then $1_y \leqslant 1_x$, by part (4), and $1_x \leqslant 1_y$, by part (3). □

Next, we will use the description of A_x of the previous lemma in order to show that the sets A_x are closed under several operations of $\mathbf{A}$.

Lemma 8. *Let* $\mathbf{A}$ *be a locally integral ipo-monoid. For every* x *in* A*:*

1. A_x *is closed under the involutive negations,*
2. A_x *is closed under multiplication,*
3. A_x *is closed under all existing nonempty joins and nonempty meets.*

Proof. 1. By Lemma 7(1), if $y \in A_x$ then $1_{\sim y} = 1_y = 1_x$, and hence ${\sim}y \in A_x$.

2. If $y, z \in A_x$ then $y, z \in [0_x, 1_x]$, by Lemma 7(5). Hence, $0_x = 0_x \cdot 0_x \leqslant y \cdot z \leqslant 1_x \cdot 1_x = 1_x$. Also by Lemma 7(5), we have $1_x \cdot (y \cdot z) = (1_x \cdot y) \cdot z = y \cdot z$, since $y \in A_x$. Thus, again by Lemma 7(5), $y \cdot z \in A_x$.
3. Suppose that $\emptyset \neq Y \subseteq A_x$ and the join $\bigvee Y$ exists in A. Since for every y in Y, $y \in A_x \subseteq [0_x, 1_x]$, we obtain that also $\bigvee Y \in [0_x, 1_x]$. And since multiplication distributes with respect to all existing joins, we have that $1_x \cdot \bigvee Y = \bigvee_{y \in Y} 1_x \cdot y = \bigvee_{y \in Y} y = \bigvee Y$. Thus, by Lemma 7(5), $\bigvee Y \in A_x$. The closure under all existing nonempty meets can be obtained from the fact that A_x is also closed under negations and $\bigwedge Y = -\bigvee_{y \in Y} {\sim}y$. □

Our next goal is to find a canonical representative for each equivalence class A_x. But first, we will provide useful characterizations of A^+ and $\downarrow 0$.

Lemma 9. *Let* $\mathbf{A}$ *be a locally integral ipo-monoid. For all* p *and* a *in* A*, we have that* $p \in A^+$ *if and only if* $p = 1_p$ *and* $a \in \downarrow 0$ *if and only if* $a = 0_a$*. In particular, both involutive negations coincide for positive elements and for elements in* $\downarrow 0$*.*

Proof. We already know that $p \leqslant 1_p$ is valid for all p in A. If moreover $1 \leqslant p$, then $1_p = p/p \leqslant p/1 = p$. The other implication is trivial, since we know that $1 \leqslant 1_p$ is true for all p. The second part follows from the following equivalences:

$$a \in \downarrow 0 \iff {\sim}a \in A^+ \iff 1_a = 1_{\sim a} = {\sim}a \iff a = -1_a = 0_a.$$

The last part is true, since for every $p \in A^+$, ${\sim}p = {\sim}1_p = -1_p = -p$, and analogously for the elements of $\downarrow 0$. □

Lemma 10. *Let* $\mathbf{A}$ *be a locally integral ipo-monoid. For every* x *in* A, 1_x *is the only positive element of* A_x *and* 0_x *is the only element of* A_x *below* 0.

Proof. Obviously, $1_x \in [0_x, 1_x]$ and also $1_x \cdot 1_x = 1_x$, since $1_x \in A^+$. Thus, $1_x \in A_x$, by Lemma 7. Also, as we mentioned before, $1 \leqslant 1_x$. For any positive $p \in A_x$, we would have that $p = 1_p = 1_x$, by Lemma 9. The second part follows from (ant) and the fact that A_x is closed under the involutive negations. □

Remark 11. Notice that the previous lemma tells us that for every x in A, there is only one positive element p so that $A_x = A_p$. This means that the family $\{A_x : x \in A\}$ is actually indexed by A^+ and that for all $p, q \in A^+$, we have $A_p = A_q$ if and only if $p = q$. Furthermore, from Lemma 9 and the previous comment, ${\downarrow}0 = \{0_x : x \in A\} = \{0_p : p \in A^+\}$.

We can now show that the relation and operations of a locally integral ipo-monoid furnish each equivalence class A_x with the structure of an integral ipo-monoid, with a suitable identity.

Proposition 12. *If* $\mathbf{A}$ *is a locally integral ipo-monoid, then for every* p *in* A^+, *the structure* $\mathbf{A}_p = (A_p, \leqslant, \cdot, 1_p, \sim, -)$, *where the relation and the operations are the restrictions to* A_p *of the corresponding relation and operations of* $\mathbf{A}$, *is an integral ipo-monoid. If in addition* $\mathbf{A}$ *is a semiring, cyclic, or commutative, then* $\mathbf{A}_p$ *is also a semiring, cyclic, or commutative, respectively, for all* p *in* A^+.

Proof. By Lemma 8, every A_p is closed under multiplication and the involutive negations, and $1_p \in A_p$. Therefore, the structure $\mathbf{A}_p$ is well defined, $(A_p, \leqslant)$ is a poset, and since $1_p \cdot x = x$ for all $x \in A_p$ by Lemma 7, $(A_p, \cdot, 1_p)$ is a monoid. Moreover, since the only element of A_p below 0 is $0_p = -1_p$ by Lemma 10, we deduce from the property (ineg) of $\mathbf{A}$ that for all $x, y \in A_p$,

$$x \leqslant y \iff x \cdot {\sim}y \leqslant 0_p \iff -y \cdot x \leqslant 0_p,$$

which is precisely the property (ineg) for the structure $\mathbf{A}_p$. Finally, by Lemma 7 again, $A_p \subseteq [0_p, 1_p]$, and therefore $x \leqslant 1_p$ for all $x \in A_p$.

The proof for the locally integral involutive semirings follows from the fact that A_p is also closed under all binary joins and meets, by Lemma 8. □

We call every $\mathbf{A}_p$ an *integral component* of $\mathbf{A}$. As we saw in Proposition 12, some properties of $\mathbf{A}$ are inherited by every of its integral components. Sometimes the opposite is also true. We say that a property of ipo-monoids is *local* whenever an ipo-monoid has it if and only if all its integral components have it.

Given a locally integral ipo-monoid $\mathbf{A}$, the sets A^+ and ${\downarrow}0$ are obviously partially ordered by the order of $\mathbf{A}$. The next proposition describes these two posets.

Proposition 13. *Let* $\mathbf{A}$ *be a locally integral ipo-monoid. Then* $(A^+, \cdot, 1)$ *is a lower-bounded join-semilattice whose order coincides with the order of* $\mathbf{A}$. *Also,* $({\downarrow}0, \cdot, 0)$ *is an upper-bounded meet-semilattice, whose order coincides with the order of* $\mathbf{A}$, *and is dually isomorphic to* $(A^+, \cdot, 1)$. *If, in addition,* A^+ *is finite, then* $(A^+, \leqslant)$ *is a distributive lattice dual to* $({\downarrow}0, \leqslant)$.

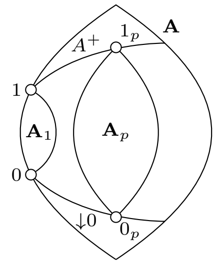

Fig. 1. Representation of the structure of a locally integral ipo-monoid

Proof. For the first part, notice that A^+ is closed under products. For all $p, q \in A^+$, we have that that $p = 1 \cdot p \leqslant pq$ and analogously $q \leqslant pq$. Furthermore, if $p \leqslant r$ and $q \leqslant r$, then $pq \leqslant r^2 \leqslant r$. This shows that $(A^+, \cdot, 1)$ is a join semilattice whose induced order is the restriction of $\leqslant$, and whose lower bound is 1.

As for the second part, the map $\eta \colon A^+ \to {\downarrow}0$ given by $\eta(p) = {\sim}p = 0_p$ is bijective (Remark 11) and for any two elements $p, q \in A^+$, we have that $p \leqslant q$ if and only if $\eta(q) = 0_q = {\sim}q \leqslant {\sim}p = 0_p = \eta(p)$, by (ant), and $\eta(1) = {\sim}1 = 0$. Therefore, the restriction of $\leqslant$ to ${\downarrow}0$ is a meet-semilattice ordering with upper bound 0. And by Lemma 4, given two elements $0_p, 0_q \in {\downarrow}0$, we have $0_p \cdot 0_q = 0_p \wedge 0_q$.

Finally, if A^+ is finite, then also ${\downarrow}0$ is finite and therefore a lattice with respect to the restricted order. Since meet and multiplication coincide in ${\downarrow}0$, and multiplication distributes with respect to joins, $({\downarrow}0, \leqslant)$ is distributive, and therefore also $(A^+, \leqslant)$ is distributive. □

Remark 14. Notice that the dual isomorphism $\eta \colon (A^+, \cdot, 1) \to ({\downarrow}0, \cdot, 0)$ sends joins to meets, and therefore, for any two positive elements p and q, we have that

$$0_p \cdot 0_q = \eta(p) \cdot \eta(q) = \eta(p \cdot q) = 0_{pq}.$$

Also, since we showed in the previous proposition that the product of two positive elements is their join, then we deduce that multiplication of positive elements is commutative. We can actually improve on this result.

Proposition 15. *All positive elements of locally integral ipo-monoids are central.*

Proof. Suppose that p is positive and let x be an arbitrary element. The equality $p \cdot 0_{px} = p(-(px) \cdot px) = p(px \cdot {\sim}(px)) = ppx \cdot {\sim}(px) = px \cdot {\sim}(px) = 0_{px}$ implies by (rot) that $-(px)x \leqslant -(px) \cdot px = 0_{px} \leqslant 0_{px} \cdot 1_{px} \leqslant {\sim}p = -p$, since $1 \leqslant p$ and $1 \leqslant 1_{px}$. Hence, $xp \leqslant px$, by (rot).

Now, applying (rot) to $xp \leqslant xp$, we obtain $-(xp)x \leqslant -p = {\sim}p$, and by (rot) again, $p \cdot -(xp) \leqslant -x$. Finally, since $xp \leqslant px$ is true for any x, in particular we have that $-(xp)p \leqslant p \cdot -(xp) \leqslant -x$, and by (rot) one last time, $px \leqslant xp$. □

As we saw in Lemma 8, every integral component of a locally integral ipo-monoid is closed under multiplication. But, what happens when we multiply elements from different components? The following lemma answers this question.

Lemma 16. *Given a locally integral ipo-monoid* $\mathbf{A}$*, positive elements* p *and* q*, and elements* $x \in A_p$ *and* $y \in A_q$*, the product* xy *is in* A_{pq}*.*

Proof. The inequalities $1_p = p \leqslant pq = 1_{pq}$ and $1_q = q \leqslant pq = 1_{pq}$ imply that $A_p \cup A_q \subseteq [0_p, 1_p] \cup [0_q, 1_q] \subseteq [0_{pq}, 1_{pq}]$, and therefore $x, y \in [0_{pq}, 1_{pq}]$, whence we deduce that $xy \in [0_{pq}, 1_{pq}]$. Moreover, $1_{pq} \cdot (xy) = pqxy = pxqy = 1_x \cdot x \cdot 1_y \cdot y = xy$, by Lemma 9, Proposition 15, and Lemma 7. Hence, by Lemma 7, $xy \in A_{pq}$. $\square$

All these results point toward the idea that locally integral ipo-monoids are built up from integral ones, or at least their monoid reducts are, by means of a Płonka sum. This construction was first introduced and studied in [10–12]; for more recent expositions see [13] and [3]. Given a compatible family of homomorphisms between algebras of the same type $\{\varphi_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j : i \leqslant j\}$, indexed by the order of a lower-bounded join-semilattice $(I, \vee, \bot)$, its *Płonka sum* is the algebra $\mathbf{S}$ of the same type defined on the disjoint union of their universes $S = \biguplus_{i \in I} A_i$, so that for every constant symbol c, $c^{\mathbf{S}} = c^{\mathbf{A}_\bot}$, and for every n-ary operation symbol σ and elements $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$, $\sigma^{\mathbf{S}}(a_1, \ldots, a_n) = \sigma^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n))$, where $j = i_1 \vee \cdots \vee i_n$. The compatibility condition of the family of homomorphisms says that for every $i \in I$, φ_{ii} is the identity on $\mathbf{A}_i$, and that if $i \leqslant j \leqslant k$ then $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$. One can readily prove that the Płonka sum of a compatible family of homomorphisms is well defined and it satisfies all regular equations that hold in all the algebras of the family. Recall that a *regular* equation is an equation in which the variables that appear on the left-hand side are the same as the variables that appear on the right-hand side.

Given a locally integral ipo-monoid, we would like to find a compatible family Φ of monoid homomorphisms indexed on the order of A^+, so that the monoid reduct of $\mathbf{A}$ can be reconstructed as the Płonka sum of Φ. Consider, for every pair of positive elements $p \leqslant q$, the map $\varphi_{pq} \colon A_p \to A_q$ given by $\varphi_{pq}(x) = qx$.

Lemma 17. *Let* $\mathbf{A}$ *be a locally integral ipo-monoid and* $p \leqslant q$ *two positive elements. Then* $\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q$ *is a well defined monoid homomorphism. Moreover, it respects arbitrary nonempty existing joins and therefore is monotone.*

Proof. For all positive elements p and q, and $x \in A_p$, we have that $qx \in A_{qp}$, by Lemma 16. Moreover, by Proposition 13, the inequality $p \leqslant q$ implies that $pq = q$. Hence, the map $\varphi_{pq} \colon A_p \to A_q$ is well defined. Furthermore, $\varphi_{pq}(1_p) = q1_p = qp = q = 1_q$ and for all $x, y \in A_p$,

$$\varphi_{pq}(x \cdot y) = qxy = qqxy = qxqy = \varphi_{pq}(x) \cdot \varphi_{pq}(y),$$

since q is positive and therefore idempotent and central, by Proposition 15. This shows that φ_{pq} is a monoid homomorphism. Finally, if $\emptyset \neq Y \subseteq A_p$ is such that $\bigvee Y$ exists, then $\varphi_{pq}\big(\bigvee Y\big) = q \cdot \bigvee Y = \bigvee_{y \in Y} qy = \bigvee_{y \in Y} \varphi_{pq}(y)$. $\square$

Proposition 18. *Let* $\mathbf{A}$ *be a locally integral ipo-monoid. Then, its associated family* $\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q\}$ *is compatible family of monoid homomorphisms indexed by the order of the join semilattice* $(A^+, \cdot, 1)$*.*

Proof. For every positive element p and $x \in A_p$, we have $\varphi_{pp}(x) = px = 1_p x = x$, by Lemma 7, since $x \in A_p$. That is, φ_{pp} is the identity homomorphism on $\mathbf{A}_p$. And if $p \leqslant q \leqslant r$ are positive elements, then $\varphi_{qr}(\varphi_{pq}(x)) = rqx = rx = \varphi_{pr}(x)$, since $rq = r$ by Proposition 13, because $q \leqslant r$. □

As we will show in the next result, the monoid reduct of a locally integral ipo-monoid is the Płonka sum of the family above. Although this is not the case for the rest of the structure, still we can recover it from its integral components. Recall that a property is *local* if it is satisfied by an ipo-monoid if and only if it is satisfied by all its local components.

Theorem 19. *Let* $\mathbf{A}$ *be a locally integral ipo-monoid and* Φ *its associated family of monoid homomorphisms defined above. Then, its Płonka sum* $\mathbf{S} = \left(\biguplus A_p, \cdot^{\mathbf{S}}, 1^{\mathbf{S}}\right)$ *is the monoid reduct of* $\mathbf{A}$*. Moreover, if we define* $\sim^{\mathbf{S}} x = \sim^{\mathbf{A}_p} x$ *and* $-^{\mathbf{S}} x = -^{\mathbf{A}_p} x$*, for every* $x \in A_p$ *with* p *positive, and*

$$x \leqslant^{\mathbf{S}} y \iff x \cdot^{\mathbf{S}} \sim^{\mathbf{S}} y = 0_{pq}, \quad \textit{for all } x \in A_p \textit{ and } y \in A_q,$$

then $\left(\biguplus A_p, \leqslant^{\mathbf{S}}, \cdot^{\mathbf{S}}, \sim^{\mathbf{S}}, -^{\mathbf{S}}\right)$ *is* $\mathbf{A}$*. Furthermore, cyclicity and commutativity are local properties.*

Proof. By Remark 11, the set $\{A_p : p \in A^+\}$ is a partition of A, and therefore $\biguplus A_p = A$. The element $1^{\mathbf{S}} = 1^{\mathbf{A}_1} = 1$, and given two elements $x \in A_p$ and $y \in A_q$, for arbitrary positive elements p and q, and $r = pq$, we have that

$$x \cdot^{\mathbf{S}} y = \varphi_{pr}(x) \cdot^{\mathbf{A}_r} \varphi_{qr}(x) = rx \cdot ry = rrxy = rxy = 1_r \cdot (xy) = xy,$$

since r is positive, and therefore central and idempotent, and $xy \in A_r$ by Lemma 16. The involutive negations of every integral component $\mathbf{A}_p$ are the restrictions of the corresponding operations of $\mathbf{A}$, by Proposition 12, and therefore $\sim^{\mathbf{S}} x = \sim^{\mathbf{A}_p} x = \sim x$ and $-^{\mathbf{S}} x = -^{\mathbf{A}_p} x = -x$.

Notice also that for every $x \in A_p$ and $y \in A_q$, for p and q positive, $x \leqslant y$ if and only if $x \cdot \sim y \leqslant 0$, by (ineg). Since $x \cdot \sim y \in A_{pq}$ and the only element below 0 in A_{pq} is 0_{pq} by Lemma 9, we have that

$$x \leqslant y \iff x \cdot \sim y \leqslant 0 \iff x \cdot \sim y = 0_{pq} \iff x \cdot^{\mathbf{S}} \sim^{\mathbf{S}} y = 0_{pq} \iff x \leqslant^{\mathbf{S}} y.$$

Finally, $\mathbf{A}$ is commutative if and only if all its integral components are commutative, since commutativity is expressible by the regular equation $x \cdot y = y \cdot x$. The same is true for cyclicity. □

Corollary 20. *A locally integral ipo-monoid* $\mathbf{A}$ *is idempotent if and only if all its integral components are Boolean algebras. In particular, any idempotent ipo-monoid is commutative if and only if it satisfies* $-x \cdot x = x \cdot \sim x$*.*

Proof. An integral ipo-monoid is idempotent if and only if it is a Boolean algebra, because if $\mathbf{A}$ is idempotent then for all $x, y \in A$, $x \cdot y = x \wedge y$. Indeed, $x \cdot y \leqslant 1 \cdot y = y$ and analogously $x \cdot y \leqslant x$. And if $z \leqslant x$ and $z \leqslant y$, then $z = z \cdot z \leqslant x \cdot y$. Hence, the result follows from the fact that a locally integral ipo-monoid is idempotent if and only if all its integral components are idempotent. □

The previous corollary covers the structural decomposition results in [8]. In this paper it is also shown that the variety of commutative idempotent involutive residuated lattices fails to be locally finite. Without the lattice operations, however, we have the following result.

Corollary 21. *Local finiteness is a local property of ipo-monoids.*

Proof. Suppose that the integral components of $\mathbf{A}$ are locally finite and let $X \subseteq A$ be a finite set and $J = \{1_x : x \in X\}$. Without loss of generality, we can assume that J is closed under binary joins (i.e., products), and that $J \subseteq X$. We will prove the proposition by induction on the cardinality of J. Let p be a minimal element in J and Y_p the closure of $X_p = X \cap A_p$ under products and involutive negations. Since $\mathbf{A}_p$ is locally finite, Y_p is also finite. Consider the finite set $X' = (X \smallsetminus X_p) \cup \{ry : y \in Y_p,\ p < r \in J\}$ and notice that $J' = \{1_x : x \in X'\} = J \smallsetminus \{p\}$, which is closed under binary joins, and $J' \subseteq X'$. By the inductive hypothesis, the subalgebra $\mathbf{B}$ generated by X' is finite. And since J' is closed under binary joins, $B \subseteq \bigcup_{q \in J'} A_q$. Now, for any $y \in Y_p$ and $x \in B$, $yx = (ry)x \in B$ and $xy = x(ry) \in B$, where $r = p \cdot 1_x \in J \smallsetminus \{p\}$. Since $1 \in B$ and both Y_p and B are closed under products and involutive negations, the universe of the subalgebra generated by X is $Y_p \cup B$, which is finite. The reciprocal is obvious. □

4 Glueing Constructions

The last theorem of the previous section shows how every ipo-monoid is an aggregate of its integral components. Our next question is, what are the conditions that a family of integral ipo-monoids and a family of homomorphisms should satisfy so that the construction of Theorem 19 is a (locally integral) ipo-monoid?

To make this question precise, let's assume that $\mathbf{D} = (D, \vee, 1)$ is a lower-bounded join semi-lattice, $\mathcal{A} = \{\mathbf{A}_p : p \in D\}$ is family of integral ipo-monoids, and $\varPhi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant^{\mathbf{D}} q\}$ is a compatible family of monoid homomorphisms. We call $(\mathbf{D}, \mathcal{A}, \varPhi)$ a *semilattice direct system of integral ipo-monoids.* Letting $\mathbf{A}_p = (A_p, \leqslant_p, \cdot_p, 1_p, \sim_p, -_p)$, for all p in D, we define the structure

$$\textstyle\int_{\varPhi} \mathbf{A}_p = \big(\biguplus_D A_p, \leqslant^{\mathbf{G}}, \cdot^{\mathbf{G}}, 1^{\mathbf{G}}, \sim^{\mathbf{G}}, -^{\mathbf{G}}\big),$$

where $\big(\biguplus_D A_p, \cdot^{\mathbf{G}}, 1^{\mathbf{G}}\big)$ is the Płonka sum of the family $\varPhi$, and therefore a monoid, and for all $p, q \in D$, $a \in A_p$, and $b \in A_q$, ${\sim^{\mathbf{G}}} a = {\sim_p} a$ and $-^{\mathbf{G}} a = -_p a$, and

$$a \leqslant^{\mathbf{G}} b \iff a \cdot^{\mathbf{G}} {\sim^{\mathbf{G}}} b = 0_{p \vee q}.$$

We call this structure $\int_{\varPhi} \mathbf{A}_p$ the *glueing of $\mathcal{A}$ along the family $\varPhi$.*

With this definition, one can restate Theorem 19 as saying that every locally integral ipo-monoid $\mathbf{A}$ is the glueing $\int_{\varPhi} \mathbf{A}_p$ of its integral components along the family of homomorphisms $\varPhi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q\}$ determined by $\varphi_{pq}(x) = qx$. Our question is, given a system $(\mathbf{D}, \mathcal{A}, \varPhi)$ of integral ipo-monoids, what are the conditions that $\varPhi$ must satisfy in order to ensure that $\int_{\varPhi} \mathbf{A}_p$ is an ipo-monoid?

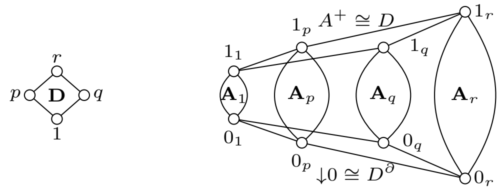

Fig. 2. Structure of a locally integral ipo-monoid

We start our analysis by identifying some relevant elements of $\int_\Phi \mathbf{A}_p$. But first, notice that since $\sim^{\mathbf{G}}$ and $-^{\mathbf{G}}$ are defined componentwise on the disjoint union $\biguplus_D A_p$, we can safely drop the superscripts and subscripts of these operations. Now, according to the definition of the Płonka sum, $1^{\mathbf{G}} = 1^{\mathbf{A}_1} = 1_1$. Let's set $0^{\mathbf{G}} = -1^{\mathbf{G}} = -1_1 = 0_1$.

The next lemma can be interpreted as saying that the glueing $\int_\Phi \mathbf{A}_p$ is indeed an "aggregate" of the integral ipo-monoids $\mathbf{A}_p$, although not necessarily an ipo-monoid itself, since the relation $\leqslant^{\mathbf{G}}$ could be not transitive. In the example of Figure 3, $0_r \leqslant^{\mathbf{G}} 0_p \leqslant^{\mathbf{G}} 1_p \leqslant^{\mathbf{G}} 1_q$, but $0_r \not\leqslant^{\mathbf{G}} 1_q$.

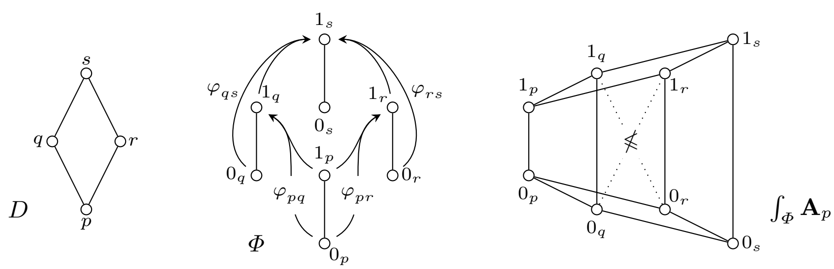

Fig. 3. A glueing of integral ipo-monoids that is not an ipo-monoid

Lemma 22. *If $\int_\Phi \mathbf{A}_p$ is the glueing of a system of integral ipo-monoids $(\mathbf{D}, \mathcal{A}, \Phi)$, then the restrictions of $\leqslant^{\mathbf{G}}$, $\cdot^{\mathbf{G}}$, $\sim^{\mathbf{G}}$, and $-^{\mathbf{G}}$ to A_p are $\leqslant_p$, $\cdot_p$, $\sim_p$, and $-_p$, respectively. Moreover, for all $p \leqslant^{\mathbf{D}} q$ and $a \in A_p$, we have that $\varphi_{pq}(a) = 1_q \cdot^{\mathbf{G}} a$.*

Proof. The fact that $\sim^{\mathbf{G}}$ and $-^{\mathbf{G}}$ restricted to A_p are $\sim_p$ and $-_p$ is immediate, by the definitions. Now, if $a, b \in A_p$, then $p \vee p = p$, and by the definition of $\leqslant^{\mathbf{G}}$ we have that $a \leqslant^{\mathbf{G}} b \iff a \cdot^{\mathbf{G}} {\sim} b = 0_p \iff \varphi_{pp}(a) \cdot_p \varphi_{pp}({\sim} b) = 0_p \iff a \cdot_p {\sim} b = 0_p \iff a \leqslant_p b$, since φ_{pp} is the identity on $\mathbf{A}_p$. For the same reason, $a \cdot^{\mathbf{G}} b = \varphi_{pp}(a) \cdot_p \varphi_{pp}(b) = a \cdot_p b$. Finally, if $p \leqslant^{\mathbf{D}} q$ and $a \in A_p$, then $\varphi_{pq}(a) = \varphi_{pq}(1_p \cdot_p a) = \varphi_{pq}(1_p) \cdot_q \varphi_{pq}(a) = 1_q \cdot_q \varphi_{pq}(a) = \varphi_{qq}(1_q) \cdot_q \varphi_{pq}(a) = 1_q \cdot^{\mathbf{G}} a$. ☐

Remark 23. An immediate consequence of this result is that $\leqslant^{\mathbf{G}}$ is a reflexive relation, since for every $p \in D$ and $a \in \mathbf{A}_p$, we have that $a \leqslant^{\mathbf{G}} a$ if and only if $a \leqslant_p a$, which we know is true. This result also implies that $\sim^{\mathbf{G}}$ and $-^{\mathbf{G}}$ satisfy (dn), since $\sim^{\mathbf{G}}-^{\mathbf{G}}a = \sim_p -_p a = a = -_p \sim_p a = -^{\mathbf{G}}\sim^{\mathbf{G}}a$.

It seems obvious that, for $\int_\Phi \mathbf{A}_p$ to be an ipo-monoid, the condition (ineg) has to be satisfied, what imposes on the family Φ the following *balance* condition:

$$\text{for all } p, q \in D, a \in A_p, b \in A_q, \quad a \cdot^{\mathbf{G}} \sim b = 0_{p\vee q} \iff -b \cdot^{\mathbf{G}} a = 0_{p\vee q}. \quad \text{(bal)}$$

One can readily check that the commutativity of $\int_\Phi \mathbf{A}_p$ implies that Φ is balanced. We will prove next that when Φ is balanced, the operations $\sim^{\mathbf{G}}$ and $-^{\mathbf{G}}$ are involutive with respect to the relation $\leqslant^{\mathbf{G}}$.

Lemma 24. *If $\int_\Phi \mathbf{A}_p$ is the glueing of a system of integral ipo-monoids $(\mathbf{D}, \mathcal{A}, \Phi)$ so that Φ satisfies* (bal)*, then for all $p, q \in D$, $a \in A$, and $b \in B$,*

$$a \leqslant^{\mathbf{G}} b \iff -b \leqslant^{\mathbf{G}} -a \iff \sim b \leqslant^{\mathbf{G}} \sim a.$$

Proof. The first equivalence can be proven as follows: $a \leqslant^{\mathbf{G}} b \iff a \cdot^{\mathbf{G}} \sim b = 0_{p\vee q} \iff -b \cdot^{\mathbf{G}} a = 0_{p\vee q} \iff -b \cdot^{\mathbf{G}} (\sim -a) = 0_{p\vee q} \iff -b \leqslant^{\mathbf{G}} -a$. For the other equivalence, just notice that $\sim b \leqslant^{\mathbf{G}} \sim a \iff a = -\sim a \leqslant^{\mathbf{G}} -\sim b = b$. □

Our next step should be to analyze the sets $G^+ = \{a \in \biguplus A_p : 1_1 \leqslant^{\mathbf{G}} a\}$ and $\downarrow^{\mathbf{G}} 0_1 = \{a \in \biguplus A_p : a \leqslant^{\mathbf{G}} 0_1\}$.[4] In particular we will show that the elements of G^+ are the elements of the form 1_p, and the elements of $\downarrow^{\mathbf{G}} 0_1$ are the ones of the form 0_p, for some $p \in D$.

Lemma 25. *If $\int_\Phi \mathbf{A}_p$ is the glueing of a system of integral ipo-monoids, then for all p in D and a in A_p, we have that*

$$1_1 \leqslant^{\mathbf{G}} a \iff a = 1_p \qquad \textit{and} \qquad a \leqslant^{\mathbf{G}} 0_1 \iff a = 0_p.$$

Proof. Since $1 \leqslant^{\mathbf{D}} p$, for all p, in particular $p = 1 \vee p$. Hence, $1_1 \leqslant^{\mathbf{G}} a \iff \varphi_{1p}(1_1) \cdot_p \varphi_{pp}(\sim_p a) = 0_p \iff 1_p \cdot_p \sim_p a = 0_p \iff \sim_p a = 0_p \iff a = -_p 0_p = 1_p$. The proof of the second equivalence is analogous. □

Reflecting on Proposition 13, we would like to show that the relation $\leqslant^{\mathbf{G}}$ endows G^+ with a structure of join-semilattice isomorphic to D, and $\downarrow^{\mathbf{G}} 0_1$ with a structure of meet-semilattice dually isomorphic to D. In general, this will not be true. For this to hold, it will be necessary to assume an extra property of Φ. We will prove first that this property is valid for the family of monoid homomorphisms associated to a locally integral ipo-monoid.

Lemma 26. *Let $\mathbf{A}$ be a locally integral ipo-monoid and $p \leqslant q$ positive elements. Then, $\varphi_{pq}(0_p) = 0_q$ if and only if $p = q$.*

Proof. The implication from left to right is obvious, since $p = q$ implies that φ_{pq} is the identity map. As for the other implication, just notice that $p < q$ implies that $0_q < 0_p \leqslant q \cdot 0_p = \varphi_{pq}(0_p)$, and therefore $\varphi_{pq}(0_p) \neq 0_q$. □

[4] Notice that, even though we don't know whether $\leqslant^{\mathbf{G}}$ is a partial order (and actually, it will not be one in general), these definitions still make sense.

This suggests the following condition for Φ, which we call *zero avoidance*:

$$\text{for all } p \leqslant^{\mathbf{D}} q, \quad \varphi_{pq}(0_p) = 0_q \iff p = q. \tag{za}$$

Lemma 27. *If $\int_\Phi \mathbf{A}_p$ is the glueing of a system of integral ipo-monoids $(\mathbf{D}, \mathcal{A}, \Phi)$ and Φ satisfies* (za)*, then for all p and q in D, $1_p \leqslant^{\mathbf{G}} 1_q \iff p \leqslant^{\mathbf{D}} q$ and $0_p \leqslant^{\mathbf{G}} 0_q \iff q \leqslant^{\mathbf{D}} p$.*

Proof. For all $p, q \in D$, with $r = p \vee q$, we have the following equivalences: $1_p \leqslant^{\mathbf{G}} 1_q \iff \varphi_{pr}(1_p) \cdot_r \varphi_{qr}(0_q) = 0_r \iff 1_r \cdot_r \varphi_{qr}(0_q) = 0_r \iff \varphi_{qr}(0_q) = 0_r \iff q = r = p \vee q \iff p \leqslant^{\mathbf{D}} q$. The proof of the second equivalence is analogous. □

The previous lemma seems to capture the spirit of Proposition 13. Notice though that this proposition is more specific, as it says that the join of two positive elements, as well as the meet of two elements below 0, is their product. We can readily see that, in the glueing $\int_\Phi \mathbf{A}_p$ along a family Φ satisfying (za), for any two elements p and q in D, with $r = p \vee q$, we have

$$1_p \cdot^{\mathbf{G}} 1_q = \varphi_{pr}(1_p) \cdot_r \varphi_{qr}(1_q) = 1_r \cdot_r 1_r = 1_r = 1_{p\vee q} = 1_p \vee 1_q.$$

But, for the case of the elements of ${\downarrow}^{\mathbf{G}} 0_1$, this will not always be true: for instance, in the example of Figure 3, $0_q \cdot 0_r = 1_s \neq 0_s = 0_{q\vee r}$. We will need to impose an extra condition on Φ:

$$\text{for all } p, q \in D, \quad 0_p \cdot^{\mathbf{G}} 0_q = 0_{p\vee q}. \tag{$*$}$$

Notice that condition $(*)$ is not spurious, as it is equivalent to the fact that for all $p, q \in D$, $0_p \leqslant^{\mathbf{G}} 1_q$, which is a desirable property, since we know that $0_p \leqslant^{\mathbf{G}} 0_1$, $0_1 \leqslant^{\mathbf{G}} 1_1$, and $1_1 \leqslant^{\mathbf{G}} 1_q$, and we want $\leqslant^{\mathbf{G}}$ to be a partial order, and in particular transitive. Thus, the condition $(*)$ will be a consequence of a much more general condition on Φ:

$$\text{for all } a, b, c \in \biguplus A_p, \quad \text{if } a \leqslant^{\mathbf{G}} b \text{ and } b \leqslant^{\mathbf{G}} c, \text{ then } a \leqslant^{\mathbf{G}} c. \tag{tr}$$

Our next result characterizes the condition (tr) in simpler terms.

Lemma 28. *If $\int_\Phi \mathbf{A}_p$ is the glueing of a system of integral ipo-monoids $(\mathbf{D}, \mathcal{A}, \Phi)$ and Φ satisfies* (bal)*, then Φ satisfies* (tr) *if and only if it satisfies:*

1. *for all $p \leqslant^{\mathbf{D}} q$, and $a, b \in A_p$, $a \leqslant_p b \implies \varphi_{pq}(a) \leqslant_q \varphi_{pq}(b)$;* (mon)
2. *for all $p \leqslant^{\mathbf{D}} q$, $p \leqslant^{\mathbf{D}} r$, and $a \in A_p$, ${\sim}\varphi_{pq}(a) \leqslant^{\mathbf{G}} \varphi_{pr}({\sim}a)$;* (lax)
3. *for all $p \vee r \leqslant^{\mathbf{D}} v$, $a \in A_p$, and $b \in A_r$,*

$$\varphi_{rv}({\sim}b) \leqslant_v {\sim}\varphi_{pv}(a) \implies a \leqslant^{\mathbf{G}} b. \tag{${\sim}$lax}$$

Proof. First, notice that for all $p \leqslant^{\mathbf{D}} q$ and $a \in A_p$, we have that $a \cdot^{\mathbf{G}} {\sim}\varphi_{pq}(a) = \varphi_{pq}(a) \cdot_q \varphi_{qq}({\sim}\varphi_{pq}(a)) = \varphi_{pq}(a) \cdot_q {\sim}\varphi_{pq}(a) = 0_q$, what implies that $a \leqslant^{\mathbf{G}} \varphi_{pq}(a)$. We will use this property several times in what follows. Suppose now that Φ satisfies both (bal) and (tr).

(mon) Suppose that $a, b \in A_p$ are such that $a \leqslant_p b$, and let $p \leqslant^{\mathbf{D}} q$. Then, by the property above, $a \leqslant^{\mathbf{G}} b \leqslant^{\mathbf{G}} \varphi_{pq}(b)$, and by (tr), we obtain that $a \leqslant^{\mathbf{G}} \varphi_{pq}(b)$. Hence, $\varphi_{pq}(a) \cdot_q {\sim}\varphi_{pq}(b) = a \cdot^{\mathbf{G}} {\sim}\varphi_{pq}(b) = 0_q$, and therefore $\varphi_{pq}(a) \leqslant_q \varphi_{pq}(b)$.

(lax) By the property above, we have that $a \leqslant^{\mathbf{G}} \varphi_{pq}(a)$ and ${\sim}a \leqslant^{\mathbf{G}} \varphi_{pr}({\sim}a)$, and by Lemma 24, ${\sim}\varphi_{pq}(a) \leqslant^{\mathbf{G}} {\sim}a$. We deduce by (tr) that ${\sim}\varphi_{pq}(a) \leqslant^{\mathbf{G}} \varphi_{pr}({\sim}a)$.

(${\sim}$lax) By the property above, we have that $a \leqslant^{\mathbf{G}} \varphi_{pv}(a)$ and ${\sim}b \leqslant^{\mathbf{G}} \varphi_{rv}({\sim}b)$, and by Lemma 24, ${\sim}\varphi_{pv}(a) \leqslant^{\mathbf{G}} {\sim}a$. If in addition we have $\varphi_{rv}({\sim}b) \leqslant_v {\sim}\varphi_{pv}(a)$, then $\varphi_{rv}({\sim}b) \leqslant^{\mathbf{G}} {\sim}\varphi_{pv}(a)$ and we deduce by (tr) that ${\sim}b \leqslant^{\mathbf{G}} {\sim}a$, and so $a \leqslant^{\mathbf{G}} b$.

In order to prove the reverse implication, suppose that Φ satisfies (bal) and the three above conditions, and $p, q, r \in D$, with $s = p \vee q$, $t = q \vee r$, $u = p \vee r$, $a \in A_p$, $b \in A_q$, and $c \in A_r$ are such that $a \leqslant^{\mathbf{G}} b$ and $b \leqslant^{\mathbf{G}} c$. Then, by definition of $\leqslant^{\mathbf{G}}$, we have that $\varphi_{ps}(a) \cdot_s \varphi_{qs}({\sim}b) = 0_s$ and $\varphi_{qt}(b) \cdot_t \varphi_{rt}({\sim}c) = 0_t$, whence we deduce that $\varphi_{ps}(a) \leqslant_s -\varphi_{qs}({\sim}b)$ and $\varphi_{rt}({\sim}c) \leqslant_t {\sim}\varphi_{qt}(b)$. Taking $v = s \vee t$, we deduce by (mon) that $\varphi_{pv}(a) \leqslant_v \varphi_{sv}(-\varphi_{qs}({\sim}b))$ and $\varphi_{rv}({\sim}c) \leqslant_v \varphi_{tv}({\sim}\varphi_{qt}(b))$. Moreover, by (lax), we have that ${\sim}\varphi_{qt}(b) \leqslant^{\mathbf{G}} \varphi_{qs}({\sim}b)$ and by Lemma 24, we deduce that $-\varphi_{qs}({\sim}b) \leqslant^{\mathbf{G}} -{\sim}\varphi_{qt}(b) = \varphi_{qt}(b)$, and therefore

$$\varphi_{pv}(a) \cdot_v \varphi_{rv}({\sim}c) \leqslant_v \varphi_{sv}(-\varphi_{qs}({\sim}b)) \cdot_v \varphi_{tv}({\sim}\varphi_{qt}(b)) = 0_v,$$

which implies that $\varphi_{pv}(a) \cdot_v \varphi_{rv}({\sim}c) = 0_v$ and hence $\varphi_{rv}({\sim}c) \leqslant_v {\sim}\varphi_{pv}(a)$, and applying (${\sim}$lax), $a \leqslant^{\mathbf{G}} c$. □

Remark 29. Notice that if a compatible family Φ satisfies (bal), then it satisfies (lax) if and only if for all $p \leqslant^{\mathbf{D}} q$, $p \leqslant^{\mathbf{D}} r$, and $a \in A_p$, $-\varphi_{pq}(a) \leqslant^{\mathbf{G}} \varphi_{pr}(-a)$.

Lemma 30. *If $\int_\Phi \mathbf{A}_p$ is the glueing of a system of integral ipo-monoids $(\mathbf{D}, \mathcal{A}, \Phi)$ and Φ satisfies* (bal)*,* (za)*, and* (lax)*, then $\leqslant^{\mathbf{G}}$ is antisymmetric.*

Proof. Suppose that $p, q \in D$ with $r = p \vee q$, and $a \in A_p$ and $b \in A_q$ are such that $a \leqslant^{\mathbf{G}} b$ and $b \leqslant^{\mathbf{G}} a$. That is, $\varphi_{pr}(a) \cdot_r \varphi_{qr}({\sim}b) = 0_r$ and $\varphi_{qr}(b) \cdot_r \varphi_{pr}({\sim}a) = 0_r$, or equivalently $\varphi_{pr}(a) \leqslant_r -\varphi_{qr}({\sim}b)$ and $\varphi_{qr}(b) \leqslant_r -\varphi_{pr}({\sim}a)$. By (lax), we get

$$\varphi_{pr}(a) \leqslant_r -\varphi_{qr}({\sim}b) \leqslant_r \varphi_{qr}(-{\sim}b) = \varphi_{qr}(b) \leqslant_r -\varphi_{pr}({\sim}a).$$

Hence, we would have that $\varphi_{pr}(0_p) = \varphi_{pr}(a \cdot_p {\sim}_p a) = \varphi_{pr}(a) \cdot_r \varphi_{pr}({\sim}_p a) = 0_r$. By (za), this only is possible if $p = r$. By a symmetric argument, we also obtain that $q = r$, and therefore $p = q$. Thus, by Lemma 22, we have that $a \leqslant_p b$ and $b \leqslant_p a$, and therefore $a = b$. □

We are now in the position to prove our main result.

Theorem 31. *A structure $\mathbf{A}$ is a locally integral ipo-monoid if and only if there is a system $(\mathbf{D}, \mathcal{A}, \Phi)$ of integral ipo-monoids satisfying* (bal)*,* (za)*, and* (tr) *so that $\mathbf{A} = \int_\Phi \mathbf{A}_p$.*

Proof. As we showed in Theorem 19, if $\mathbf{A}$ is a locally integral ipo-monoid, then $(A^+, \cdot, 1)$ is a lower-bounded join-semilattice, its integral components form a family $\{\mathbf{A}_p : p \in A^+\}$ of integral ipo-monoids, and we have a compatible family

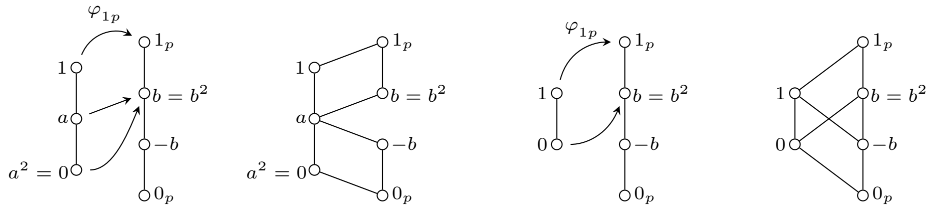

Fig. 4. Two glueings, one being a semiring, the other just an ipo-monoid

of monoid homomorphisms $\Phi = \{\varphi_{pq}\colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant q\}$ given by $\varphi_{pq}(x) = qx$, so that $\mathbf{A} = \int_\Phi \mathbf{A}_p$. Moreover, Φ satisfies condition (bal) since $\mathbf{A}$ satisfies (ineg), condition (za) by Lemma 26, and condition (tr) since $\leqslant$ is a partial order.

Conversely, if $(D, \vee, 1)$ is a lower-bounded join-semilattice, $\{\mathbf{A}_p : p \in D\}$ is a family of integral ipo-monoids, and $\Phi = \{\varphi_{pq}\colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant^{\mathbf{D}} q\}$ is a compatible family of monoid homomorphisms satisfying (bal), (za), and (tr), then $\leqslant^{\mathbf{G}}$ is a reflexive binary relation on $\biguplus A_p$ by Remark 23, which is also transitive since it satisfies (tr), and antisymmetric by Lemma 30. That is, $\left(\biguplus A_p, \leqslant^{\mathbf{G}}\right)$ is a poset. By construction, $\left(\biguplus A_p, \cdot^{\mathbf{G}}, 1^{\mathbf{G}}\right)$ is a monoid. Furthermore, since Φ satisfies (bal) and the only element in ${\downarrow^{\mathbf{G}}}0_1 \cap A_p$ is 0_p, for every $p \in D$, by Lemma 25, we deduce that $\int_\Phi \mathbf{A}_p$ satisfies (ineg) and therefore it is an ipo-monoid. It can be readily checked that for all $p \in D$ and $x \in A_p$, $-^{\mathbf{G}}x \cdot^{\mathbf{G}} x = x \cdot^{\mathbf{G}} {\sim^{\mathbf{G}}}x$, since these involutive negations and products are computed inside $\mathbf{A}_p$, which is integral. For the same reasons, one can check that $x \cdot^{\mathbf{G}} x \leqslant^{\mathbf{G}} x$, since the product is computed inside $\mathbf{A}_p$ and the restriction of $\leqslant^{\mathbf{G}}$ to $\mathbf{A}_p$ is $\leqslant_p$, by Lemma 22. And since ${\downarrow^{\mathbf{G}}}0_1 = \{0_p : p \in D\}$ by Lemma 25 and $0_p \cdot^{\mathbf{G}} 0_p = 0_{p \vee p} = 0_p$ by $(*)$, which is a consequence of (tr), we also have that $\int_\Phi \mathbf{A}_p$ is ${\downarrow}0$-idempotent. In summary, $\int_\Phi \mathbf{A}_p$ is locally integral. Since for all $p \in D$ and $x \in A_p$, $1_x = {\sim^{\mathbf{G}}}(-^{\mathbf{G}}x \cdot^{\mathbf{G}} x) = {\sim_p}(-_p x \cdot_p x) = 1_p$, we deduce that $\{\mathbf{A}_p : p \in D\}$ is the family of integral components of $\int_\Phi \mathbf{A}_p$. Also, by Lemma 22, we know that $\varphi_{pq}(x) = 1_q \cdot^{\mathbf{G}} x$, for all $p \leqslant^{\mathbf{D}} q$ and $x \in A_p$, that is, Φ is the family of homomorphisms of the decomposition of Theorem 19. □

Corollary 32. *Given any nonempty family of nontrivial integral ipo-monoids (involutive semirings, respectively) there is a locally integral ipo-monoid (involutive semiring, respectively) whose integral components are the given ones.*

Proof. If $\{\mathbf{A}_p : p \in D\}$ is a nonempty family of nontrivial ipo-monoids, let's choose a lower-bounded linear order on D and let $\mathbf{D} = (D, \vee, 1)$ be the associated lower-bounded join-semilattice. Then, the set $\Phi = \{\varphi_{pq}\colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant^{\mathbf{D}} q\}$ of maps so that $\varphi_{pq}(x) = 1_q$ if $p < q$ and $\varphi_{pq}(x) = x$ if $p = q$ is a compatible family of monoid homomorphisms satisfying (bal), (za), (mon), (lax), and ($\sim$lax). By Theorem 31, $\int_\Phi \mathbf{A}_p$ is a locally integral ipo-monoid whose integral components are $\{\mathbf{A}_p : p \in D\}$. If in addition all the integral components are involutive semirings, then $\int_\Phi \mathbf{A}_p$ is also an involutive semiring, since the join of $a \in A_p$ and $b \in \mathbf{A}_q$ in $\int_\Phi \mathbf{A}_p$ is either their join in $\mathbf{A}_p$, if $p = q$, or $1_{p \vee q}$, if $p \neq q$. □

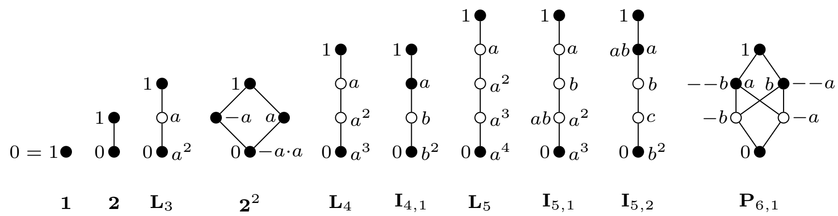

Fig. 5. All integral involutive semirings up to size 5 and an integral ipo-monoid of size 6, as components for constructing locally integral idempotent semirings and ipo-monoids.

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