Chapter 1
On the representation of Boolean magmas and Boolean semilattices

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Abstract

A magma is an algebra with a binary operation $\cdot$, and a Boolean magma is a Boolean algebra with an additional binary operation $\cdot$ that distributes over all finite Boolean joins. We prove that all square-increasing ($x \leq x^2$) Boolean magmas are embedded in complex algebras of idempotent ($x = x^2$) magmas. This solves a problem in a recent paper [3] by C. Bergman. Similar results are shown to hold for commutative Boolean magmas with an identity element and a unary inverse operation, or with any combination of these properties.

A Boolean semilattice is a Boolean magma where $\cdot$ is associative, commutative, and square-increasing. Let SL be the class of semilattices and let $\mathbf{S}(\mathsf{SL}^+)$ be all subalgebras of complex algebras of semilattices. All members of $\mathbf{S}(\mathsf{SL}^+)$ are Boolean semilattices and we investigate the question of which Boolean semilattices are representable, i.e., members of $\mathbf{S}(\mathsf{SL}^+)$. There are 79 eight-element integral Boolean semilattices that satisfy a list of currently known axioms of $\mathbf{S}(\mathsf{SL}^+)$. We show that 72 of them are indeed members of $\mathbf{S}(\mathsf{SL}^+)$, leaving the remaining 7 as open problems.

1.1 Introduction

The study of complex algebras of relational structures is a central part of algebraic logic and has a long history. In the classical setting, complex algebras connect Kripke semantics for polymodal logics to Boolean algebras with normal operators. Several varieties of Boolean algebras with operators are generated by the complex algebras of a standard class of relational structures. For example it is well known that

- the variety of relation algebras is generated by complex algebras of ternary atom structures,

- the variety of representable relation algebras is generated by complex algebras of Brandt groupoids,
- the variety of group relation algebras is generated by complex algebras of groups,
- the variety of modal algebras is generated by complex algebras of binary relations, and many subvarieties are determined by first-order definable subclasses,
- the variety of tense algebras is generated by complex algebras of a binary relation and its converse,
- the variety of 2-dimensional cylindric algebras is generated by complex algebras of two orthogonal equivalence relations and the identity relation, and for the variety of n-dimensional cylindric algebras similar first-order classes of n-ary relations exist.

Hajnal Andreka and Istvan Nemeti have published many deep and important results about these and related varieties of Boolean algebras with operators. In this chapter we discuss several varieties that are determined by complex algebras of well-known classes of algebras and partial algebras.

Given a class $\mathcal{K}$ of structures of the same signature, we denote the class of complex algebras by $\mathcal{K}^+$, and the variety it generates by $\mathbf{V}(\mathcal{K}^+) = \mathbf{HSP}(\mathcal{K}^+)$.

In general it is a difficult problem to decide if this variety is finitely axiomatizable, or to find a specific equational basis for it. Even when the class $\mathcal{K}$ is a variety of algebras with a straightforward finite equational basis and a decidable equational theory, the variety $\mathbf{V}(\mathcal{K}^+)$ may not be finitely based and may have an undecidable equational theory. The variety of group relation algebras is such an example [2, 8, 9].

We will consider classes $\mathcal{K}$ of algebras with a binary operation $\cdot$ and possibly constants e, o and a unary operation $^{-1}$ determined by a subset of the following six common equational properties:

(a) *associative* $(xy)z = x(yz)$,
(c) *commutative* $xy = yx$,
(i) *idempotent* $x^2 = x$,
(n) *inverse* $xx^{-1} = e = e^{-1}$ and $x^{-1-1} = x$,
(u) *unital* $xe = x = ex$.

Combinations of these identities define several varieties of algebras denoted by $\mathcal{K}_{a...u}$ where the subscripts show which identities are assumed. They include the variety of groups ($\mathcal{K}_{anu}$), semigroups ($\mathcal{K}_a$), semilattices ($\mathcal{K}_{aci}$), unital semilattices ($\mathcal{K}_{aciu}$), as well as many varieties of algebras with a nonassociative binary operation. If the constant e or the unary operation $^{-1}$ do not occur in the defining identities, we assume they are not in the signature of the algebras. Hence the variety $\mathcal{K}_\emptyset$ contains all groupoids, also known as magmas (since the term "groupoid" more commonly refers to a small category in which every morphism is an isomorphism). All the varieties determined by these identities are distinct, except $\mathcal{K}_{ainu}$ and $\mathcal{K}_{acinu}$ since they are both equal the variety of one-element algebras: $x = xe = xxx^{-1} = xx^{-1} = e$. Hence there are a total of 31 varieties determined by subsets of $\{a, c, i, n, u\}$.

The aim is to determine which of these classes $\mathsf{V}(\mathcal{K}^+)$ is finitely based and/or has a decidable equational theory. In many cases this is still an open problem, but in a few cases the answers are known, and some of these results are due to the work of Hajnal and Istvan and their collaborators. We also investigate when it is decidable whether a finite algebra is a member of $\mathsf{V}(\mathcal{K}^+)$.

A *partial magma* $\mathbf{M} = \langle M, \cdot \rangle$ is a set with a partial binary operation $\cdot$. A partial magma is

(t) *total* (or a *magma*) if the binary operation $\cdot$ is totally defined and
(f) *finite* if the set M is finite.

We allow (partial) magmas to have an extended signature with constants e, o and with a unary total operation ${}^{-1}$. The class of all magmas is denoted Mag and the class of all partial magmas is denoted PMag. A partial magma $\mathbf{M}$ satisfies an identity $s = t$, written $\mathbf{M} \models s = t$, if for all assignments to the variables in s, t, either both sides are defined and equal, or both sides are undefined. For total magmas this agrees with the usual interpretation of satisfaction.

Given a set Σ of identities (using possibly the extended signature of magmas),

$$\begin{aligned}\mathrm{Mod}(\Sigma) &= \{\mathbf{M} \in \mathsf{Mag} : \mathbf{M} \models \Sigma\} \text{ and}\\ \mathrm{PMod}(\Sigma) &= \{\mathbf{M} \in \mathsf{PMag} : \mathbf{M} \models \Sigma\}\end{aligned}$$

A *Boolean magma* is an algebra of the form $\mathbf{B} = \langle B, \wedge, \vee, -, 0, 1, \cdot \rangle$ such that $\langle B, \wedge, \vee, -, 0, 1 \rangle$ is a Boolean algebra and $\cdot$ is a *binary operator*, i.e., distributes over joins in each argument:

$$\begin{aligned}x \cdot (y \vee z) &= (x \cdot y) \vee (x \cdot z),\\ (x \vee y) \cdot z &= (x \cdot z) \vee (y \cdot z)\end{aligned}$$

and is *normal*: $x \cdot 0 = 0 \cdot x = 0$. Hence Boolean magmas form a variety of Boolean algebras with a binary operator. A Boolean magma is *complete* and *atomic* if the Boolean algebra is complete and atomic.

A Boolean magma with a constant $\mathbf{e}$ that is an identity element is called a *unital* Boolean magma, and if it has a unary *normal operator* ${}^{-1}$ that satisfies

(n') $x \neq 0 \implies \mathbf{e} = \mathbf{e}^{-1} \leq xx^{-1}$, and $x^{-1-1} = x$

then it is called an *inverse* Boolean magma. A Boolean magma is *integral* if it satisfies

(t') $x \cdot y = 0 \implies x = 0$ or $y = 0$.

As usual, Boolean magma homomorphisms preserve all the Boolean operations and the operator $\cdot$, as well as $\mathbf{e}$ and $^{-1}$ if they are present.

A partial magma $\mathbf{M}$ is called *associative*, *commutative*, *idempotent*, *inverse* or *unital* if it satisfies the identities (a), (c), (i), (n), (u) respectively. In the case of (a), (c) and (u) it follows that $\mathbf{M}^+$ also satisfies these identities, but the formulas (i) and (n) need not be preserved by the complex algebra construction. However, for any idempotent magma, the complex algebra is

(i') *square-increasing* $x \le x \cdot x$,

and for any inverse magma the complex algebra satisfies

(n') $x \neq 0 \implies \mathbf{e} = \mathbf{e}^{-1} \le xx^{-1}$, and $x^{-1-1} = x$,

where $\le$ is interpreted as $\subseteq$ in the complex algebra. Let $\mathcal{C}$ denote the set of conditions $\{(a),(c),(f),(i),(n),(t),(u)\}$, and for any $(\phi) \in \mathcal{C} - \{(i),(n),(t)\}$, let $(\phi') = (\phi)$. For any $S \subseteq \mathcal{C}$ we define S' to be the corresponding subset of $\mathcal{C}' = \{(\phi') \mid (\phi) \in \mathcal{C}\}$.

For a partial magma $\mathbf{M} = \langle M, \cdot \rangle$, define the *complex algebra*

$$\mathbf{M}^+ = \langle \mathcal{P}(M), \cap, \cup, -, \emptyset, M, \cdot \rangle$$

where $X \cdot Y = \{x \cdot y : x \in X, y \in Y\}$ for $X, Y \subseteq M$ is the complex operation in $\mathbf{M}^+$. If $\mathbf{M}$ has an additional constant e or a unary operation $^{-1}$, then the complex algebra has a constant $\mathbf{e} = \{e\}$ or a unary operator $X^{-1} = \{x^{-1} : x \in X\}$.

More generally, the complex algebra construction applies to ternary relational structures $\mathbf{U} = \langle U, R \rangle$ where $R \subseteq U^3$, by defining $X \cdot Y = \{z \in U \mid (x,y,z) \in R$ for some $x \in X$ and $y \in Y\}$ and $\mathbf{U}^+ = \langle \mathcal{P}(U), \cap, \cup, -, \emptyset, U, \cdot \rangle$.

It is straightforward to check that $\mathbf{M}^+$ and $\mathbf{U}^+$ are Boolean magmas, possibly with constants and inverse, and if $\mathbf{M}$ is total then $\mathbf{M}^+$ is integral.

Let $\mathbf{U} = \langle U, R \rangle$ and $\mathbf{V} = \langle V, S \rangle$ be ternary relational structures. From modal logic it is well known that a Boolean magma homomorphism from $\mathbf{V}^+$ to $\mathbf{U}^+$ is uniquely determined by a map $h : \mathbf{U} \to \mathbf{V}$ that is a *bounded morphism*, i.e., for all $x, y, z \in U$ and $x', y' \in V$

$$\begin{aligned} (x,y,z) \in R &\implies (h(x), h(y), h(z)) \in S \text{ and} \\ (x', y', h(z)) \in S &\implies \exists x, y \in U (h(x) = x', h(y) = y' \text{ and } (x,y,z) \in R). \end{aligned}$$

Moreover, the correspondence between complete and atomic Boolean magmas with homomorphisms and ternary relational structures with bounded morphisms is a categorical equivalence that is essentially due to [6].

A Boolean magma $\mathbf{B}$ is said to be *represented* by a (partial) magma $\mathbf{M}$ if there exists an embedding of $\mathbf{B}$ into the complex algebra of $\mathbf{M}$. A Boolean magma is *represented by a class* $\mathcal{K}$ of (partial) magmas if it is represented by some member of $\mathcal{K}$.

Theorem 1 ([3, 4])

- *Boolean magmas are represented by partial magmas.*
- *Integral Boolean magmas are represented by magmas.*
- *Finite Boolean magmas are represented by finite magmas.*
- *Commutative Boolean magmas are represented by commutative partial magmas.*

More generally, Boolean magmas that satisfy any subset S' of $\{(c'), (f'), (t')\}$ are represented by partial magmas that satisfy the corresponding subset S.

The following connection between complex algebras of partial magmas and total magmas is useful for extending representability results to axiomatizing varieties generated by classes of magmas. For any partial magma $\mathbf{M}$, define the *total one-point extension* magma $\mathbf{M}_o = \mathbf{M} \cup \{o\}$ by

$$x \cdot^{\mathbf{M}_o} y = \begin{cases} xy & \text{if } xy \text{ is defined in } \mathbf{M} \\ o & \text{otherwise.} \end{cases}$$

Lemma 2

1. *The map $h : \mathbf{M}_o^+ \to \mathbf{M}^+$ defined by $h(X) = X \cap M$ is a Boolean magma homomorphism, hence $(\mathrm{PMod}(\Sigma))^+ \subseteq \mathsf{H}((\mathrm{Mod}(\Sigma))^+)$.*
2. *Let Σ be a set of magma identities. Then $\mathbf{M} \models \Sigma \implies \mathbf{M}_o \models \Sigma$.*
3. *Suppose $\mathcal{V}$ is a subvariety of Boolean magmas such that every member of $\mathcal{V}$ is representable by $(\mathrm{PMod}(\Sigma))^+$ and $(\mathrm{Mod}(\Sigma))^+ \subseteq \mathcal{V}$. Then $\mathcal{V}$ is the variety generated by complex algebras of models of Σ.*

We now extend Theorem 1 to the remaining 14 varieties of non-associative magmas determined by subsets of $\{(c), (i), (n), (u)\}$.

1.2 Representable Boolean magmas

Problem 3.7 in [3] asks whether the variety generated by complex algebras of idempotent magmas is finitely based, and whether square-increasing Boolean magmas can be represented by idempotent partial magmas.

Our first result shows that both of these questions have positive answers. Similar results for unital and inverse magmas are considered subsequently. The concept of *canonical extension* for Boolean algebras with operators is due to [6] and a brief definition and relevant properties can be found in [3]. A formula is called *canonical* if it is preserved by taking canonical extensions.

Theorem 3 *Every square-increasing Boolean magma $\mathbf{B}$ is representable by an idempotent partial magma $\mathbf{M}$. Moreover,*

1. *if $\mathbf{B}$ is finite we can take $\mathbf{M}$ to be finite,*

2. if **B** *is integral then we can take* **M** *to be a total magma,*
3. if **B** *is commutative we can take* **M** *to be commutative, and*
4. if **B** *is commutative and integral we can take* **M** *to be a total commutative magma.*

Proof Let **B** be a square-increasing Boolean magma. By Theorem 2.3, 2.4 of [3] we can assume that **B** is complete and atomic since the square-increasing law is canonical.

Let A be the set of atoms of B, and let $+$ be a group operation on A, with identity $0 \in A$ and inverse of $a \in A$ written as $-a$. Since any set can be the carrier of a group, this is always possible. Define a partial magma operation $\cdot$ on $M = A \times A$ by

$$(a,x)\cdot(b,y) = \begin{cases} (a-x+y,x) & \text{if } a-x+y \leq a \cdot^{\mathbf{B}} b \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This definition implies that $(a,x)\cdot(a,x) = (a-x+x,x) = (a,x)$ since $a-x+x = a \leq a \cdot^{\mathbf{B}} a$ follows from **B** being square-increasing. Hence $\mathbf{M} = \langle M, \cdot \rangle$ is an idempotent partial magma.

A map $e : \mathbf{B} \to \mathbf{M}^+$ is defined on atoms by $e(a) = \{(a,x) : x \in A\}$ and lifts to all elements of **B** by mapping joins in **B** to unions in $\mathbf{M}^+$. Since $A \times A$ is partitioned by the sets $e(a)$ for $a \in A$, this map is a Boolean algebra embedding, and it suffices to show that $e(a \cdot^{\mathbf{B}} b) = e(a)\cdot e(b)$ for all $a, b \in A$. Equivalently we need to show that the first projection $\pi_1 : M \to A$ is a bounded morphism, i.e., for all $u, v \in A$

$$(u,v) \in e(a)\cdot e(c) \text{ if and only if } u \leq a \cdot^{\mathbf{B}} b.$$

For the forward implication assume $(u,v) \in e(a)\cdot e(b)$. Then $(a,x)\cdot(b,y) = (u,v)$ for some $x, y \in A$, so from the definition of $\cdot$ in **M** we deduce $u = a - x + y \leq a \cdot^{\mathbf{B}} b$. For the reverse implication, assume $u \leq a \cdot^{\mathbf{B}} b$ and let $v \in A$ be given. Take $x = v$ and $y = v - a + u$. Then $u = a - v + v - a + u = a - x + y \leq a \cdot^{\mathbf{B}} b$, hence $(a,x)\cdot(b,y) = (u,v)$ from which $(u,v) \in e(a)\cdot e(b)$ follows.

Now (1) holds by construction. For (2) observe that if **B** is integral then we can define a function $g : M \to A$ such that $g(a,b) \leq a \cdot^{\mathbf{B}} b$ for all $a, b \in A$. Redefine $\cdot$ on M by

$$(a,x)\cdot(b,y) = \begin{cases} (a-x+y,x) & \text{if } a-x+y \leq a \cdot^{\mathbf{B}} b \\ (g(a,b),0) & \text{otherwise} \end{cases}$$

and repeat the above argument to show that the map e is still an embedding.

(3) requires redefining $M = A \times A \times \{0,1\}$. Let $\preceq$ be any total order on A, and extend it lexicographically to an order on $A \times \{0,1\}$ by

$$(a,i) \preceq (a',i') \iff a \prec a' \text{ or } (a = a' \text{ and } i \preceq i').$$

Then the set $U = \{((a,i),(a',i')) : (a,i) \preceq (a',i')\}$ has cardinality $|U| \geq |M|$, since if $|A|$ is finite then

$$|U| = \frac{1}{2}(|A \times \{0,1\})^2 + |A \times \{0,1\}|) = 2|A|^2 + |A| = |M| + |A|$$

and otherwise $|A| = |M|$. Hence for every $a \in A$ we can define a surjective function $f_a : U \to M$ such that $f_a((b,i),(b',i')) = (a,b,i)$ if $b = b'$ and $i = i'$ (f_a can be arbitrary on other elements of U, as long as the map is surjective).

We now define $\cdot : M^2 \to M$ by $(a,b,i) \cdot (a',b',i') =$

$$\begin{cases} f_a((b,i),(b',i')) & \text{if } (b,i) \preceq (b',i') \text{ and } \pi_1(f_a((b,i),(b',i'))) \leq a \cdot a' \\ f_a((b',i'),(b,i)) & \text{if } (b',i') \preceq (b,i) \text{ and } \pi_1(f_a((b',i'),(b,i))) \leq a' \cdot a \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is easy to check that this binary operation is commutative and idempotent. The embedding of $\mathbf{B}$ into $\mathbf{M}^+$ is defined on atoms by $e(a) = \{(a,x,i) : x \in A, i \in \{0,1\}\}$. The proof that this is a Boolean semilattice embedding is similar to the argument for (1).

Finally, (4) is proved by modifying (3), in the same way that (1) was modified to obtain a proof of (2). □

$\cdot$	e	a	b	$^{-1}$
e	e	a	b	e
a	a	$a \vee b$	$e \vee a \vee b$	b
b	b	$e \vee a \vee b$	b	a

B

$\hookrightarrow$

$\cdot$	e	$aa0$	$aa1$	$ab0$	$ab1$	$bb0$	$bb1$	$ba0$	$ba1$	$^{-1}$
e	e	$aa0$	$aa1$	$ab0$	$ab1$	$bb0$	$bb1$	$ba0$	$ba1$	e
$aa0$	$aa0$	$aa0$	$bb0$	$bb1$	$ba0$	e	$bb0$	$bb1$	$ba0$	$bb1$
$aa1$	$aa1$	$bb0$	$aa1$	$ba1$	$bb0$	$aa0$	e	$ba1$	$bb0$	$bb0$
$ab0$	$ab0$	$bb1$	$ba1$	$ab0$	$bb1$	$aa1$	$ab0$	e	$bb1$	$ba1$
$ab1$	$ab1$	$ba0$	$bb0$	$bb1$	$ab1$	$ab1$	$aa0$	$aa1$	e	$ba0$
$bb0$	$bb0$	e	$aa0$	$aa1$	$ab1$	$bb0$	$bb0$	$bb1$	$ba0$	$aa1$
$bb1$	$bb1$	$bb0$	e	$b0$	$aa0$	$bb0$	$bb1$	$ba1$	$bb0$	$aa0$
$ba0$	$ba0$	$bb1$	$ba1$	e	$aa1$	$bb1$	$ba1$	$ba0$	$bb1$	$ab1$
$ba1$	$ba1$	$ba0$	$bb0$	$bb1$	e	$ba0$	$bb0$	$bb1$	$ba1$	$ab0$

M

Table 1.1 An integral commutative square-increasing inverse unital Boolean magma **B** and the corresponding commutative idempotent inverse unital magma **M** (a triple (x,y,i) is denoted by xyi).

Corollary 4 *The variety of square-increasing Boolean magmas is generated by all complex algebras of idempotent magmas.*

The variety of commutative square-increasing Boolean magmas is generated by all complex algebras of commutative idempotent magmas.

The technique of Theorem 3 is easily extended to cover the formulas (n) and (u) which leads to the following general result.

Theorem 5 *Let* BMag *denote the variety of Boolean magmas. For any* $S \subseteq \mathcal{C} - \{(a)\}$ *we have* $\mathsf{BMag} \cap \mathrm{Mod}(S') = \mathbf{S}(\mathrm{Mod}(S)^+)$.

For any $S \subseteq \{(c), (i), (u)\}$ *we have* $\mathsf{BMag} \cap \mathrm{Mod}(S') = \mathbf{HSP}(\mathrm{Mod}(S)^+)$.

We now discuss what happens if associativity (a) is included as one of the axioms. As mentioned before, magmas that satisfy $(a), (n), (u)$ are groups, hence the complex algebras generate the variety GRA of group relation algebras. However the variety of Boolean magmas generated by models of $(a), (n'), (u)$ is the variety IRA of integral relation algebras. Since the complex algebra construction preserves (a) and (u), and replaces (n) by (n'), clearly $\mathsf{GRA} \subseteq \mathsf{IRA}$, but Monk [9] showed that GRA is not even finitely axiomatizable over IRA. Monk's result also shows that this situation persists if commutativity (c) is added. In a comprehensive monograph [2], Hajnal Andreka, Steve Givant and Istvan Nemeti show that many subvarieties of GRA have undecidable equational theories. This includes GRA, commutative GRA and the variety generated by complex algebras of groups of exponent 2.

In [5] a result of Hajnal Andreka [1] is used to show that the variety generated by complex algebras of semigroups (or commutative semigroups) is not finitely axiomatizable. Building on work of P. Reich [10], it is also shown in [5] that all Boolean semigroups with ≤ 4 elements are representable in the complex algebra of a semigroup, but for larger Boolean semigroups there is so far no algorithm for deciding if a particular Boolean semigroup is representable by some (perhaps infinite) semilattice. For other subsets of $\mathcal{C}$ that include (a), the situation is not so clear.

1.3 Representable Boolean semilattices

A *Boolean semilattice* is a Boolean magma satisfying the additional axioms

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \qquad x \cdot y = y \cdot x \quad \text{and} \quad x \leq x \cdot x.$$

The complex algebra of a semilattice is always a Boolean semilattice since associativity and commutativity are preserved by passing from a magma to its complex algebra, while idempotence of the magma operation implies that the complex operation is square-increasing. A Boolean semilattice $\mathbf{B}$ is *representable* if there exists a semilattice $\mathbf{S}$ such that $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{S}^+$.

Let $\mathsf{S}(\mathsf{SL}^+)$ be the class of subalgebras of complex algebras of semilattices. Algebras in $\mathsf{S}(\mathsf{SL}^+)$ are of the form $\langle A, \vee, \wedge, -, 0, 1, \cdot \rangle$ such that $(A, \vee, \wedge, -, 0, 1)$ is a Boolean algebra and $\cdot$ is a commutative, associative operator (distributes over $\vee$ and $x0 = 0$). It is an open problem to find an axiomatization for the variety $\mathsf{V}(\mathsf{SL}^+)$ generated by all complex algebras of semilattices.

Let LSL be the class of linearly ordered semilattices.

Theorem 6 (Bergman [3]) *The variety* $\mathbf{V}(\mathsf{LSL}^+)$ *generated by complex algebras of linearly ordered semilattices is the variety of Boolean algebras with a commutative associative idempotent binary operator.*

Hence $\mathbf{V}(\mathsf{LSL}^+)$ is finitely axiomatized relative to all Boolean semilattices by the single identity $xx = x$.

The next lemma shows that representable Boolean semilattices have some unexpected equational and quasiequational properties. The notation $x - y$ abbreviates $x \wedge -y$.

Lemma 7 *Representable Boolean semilattices satisfy the following formulas:*

(1) $x \wedge y1 \leq xy$
(2) $x(xy - x) \leq x^2 \vee (xy - x)^2$
(3) $x \leq yw \implies xz \leq x(yz \wedge v) \vee w(yz - v)$
(3') $(x \wedge yw)z \leq (x \wedge yw)(yz \wedge v) \vee w(yz - v)$
(4) $xy \leq x \vee y \implies x^2 \wedge y^2 \leq xy$
(5) $yz \leq u \vee v \implies xy \wedge zw \leq xu \vee wv$
(5') $xy \wedge zw \leq x(yz \wedge u) \vee w(yz - u)$
(6) $x \leq xy,\ yz \leq v \vee y,\ xv \wedge xz = 0,\ v \leq xy \wedge yz \implies v \leq y^2$
(7) $x \leq xy,\ xv \wedge xz = 0 \implies x1 \wedge wz \leq w(yz - v)$

Moreover (3) $\Leftrightarrow$ (3') $\Rightarrow$ (2), *and* (5) $\Leftrightarrow$ (5') $\Rightarrow$ (4), *whereas* (1) $\not\Rightarrow$ (2), (1)–(2) $\not\Rightarrow$ (3), *and* (1)–(3) $\not\Rightarrow$ (4).

Proof Let $\mathbf{B}$ be representable and $x, y, z, u, v, w \in B$. Then $\mathbf{B}$ is a subalgebra of $\mathbf{S}^+$ for some semilattice $\mathbf{S}$, hence x, y, z, u, v, w are subsets of S.

(1) Assuming $p \in x \wedge y1$, we have $p \in x$ and $p = y'q$ for some $y' \in y$ and $q \in S$. Hence $p \leq y'$ in the semilattice S, from which it follows that $p = py' \in xy$.

For (2), let $p \in x(xy - x)$. Then $p = x'q$ for some $x' \in x$ and $q \in xy - x$. Hence $q = x''y'$ for some $x'' \in x$, $y' \in y$. If $x'y' \in x$ then $p = x'q = (x'y')x'' \in x^2$. If $x'y' \notin x$ then $x'y' \in xy - x$, so $p = (x'y')q \in (xy - x)^2$.

For (3), assume $x \leq yw$ and let $p \in xz$. Then $p = x'z'$ for some $x' \in x$ and $z' \in z$. From $x \leq yw$ we deduce $x' = y'w'$ for some $y' \in y$, $w' \in w$. If $y'z' \notin v$ then $y'z' \in yz - v$, so $p = (y'w')z' = w'(y'z') \in w(yz - v)$. If $y'z' \in v$ then $p = (y'w')z' = (y'w')(y'z') \in x(yz \wedge v)$.

(3') is an identity equivalent to (3), obtained by replacing x in (3) by $x \wedge yw$. To see that (3) implies (2), replace in (3) x by $xy - x$, w by x, z by x and v by $-x$.

To prove (4), assume $xy \leq x \vee y$ and let $p \in x^2 \wedge y^2$. Then $p = x'x'' = y'y''$ for some $x', x'' \in x$, $y', y'' \in y$. Since $x'y' \in xy$ we have $x'y' \in x$ or $x'y' \in y$. In the first case, $p = (x'y')y'' \in xy$ and in the second case $p = x''(x'y') \in xy$.

(5) is a generalization of (4), since if we replace x, y, z, u, v, w in (5) by x, x, y, y, x, y then the result is (4). To see that (5) implies the identity (5'), take $u := yz \wedge v$ and $v := yz - v$ in (5), then $yz \leq u \vee v$ and we get the

desired conclusion. Conversely, if (5') holds and $yz \le u \vee v$, then $yz \wedge u \le u$ and $yz - u \le (u \vee v) - u \le v$ so (5') implies (5).

(6) is left as an exercise, with the hint that its proof is similar to (7).

To prove (7), assume $x \le xy$, $xv \wedge xz = 0$ and $p \in x1 \wedge wz$. Then $p = w'z' \le x'$ for some $w' \in w, z' \in z$ and $x' \in x$. Since $x \le xy$, we have $x' \le y'$ for some $y' \in y$. Hence $x'z' \le y'z'$, so $x'z' = x'y'z'$. Now $xv \wedge xz = 0$ implies $x'z' \notin xv$, whence $y'z' \in yz - v$. Finally $p \le x' \le y'$ implies $p = py' = w'z'y' = w'(y'z') \in w(yz - v)$.

To see that part of the list is irredundant, we provide Boolean semigroups $\mathbf{C}_i$ that satisfy (1)–(i) but fail $(i+1)$ (for $i \le 4$):

$\cdot$	a	b
a	a	b
b	b	1

$\mathbf{C}_1$
$x = a$
$y = b$

$\cdot$	a	b	c
a	$a \vee b$	1	1
b	1	$a \vee b$	1
c	1	1	1

$\mathbf{C}_2$
$x = a$
$y = a$

$\cdot$	a	b	c
a	1	$a \vee b$	1
b	$a \vee b$	1	1
c	1	1	1

$\mathbf{C}_3$
$x = u = a$
$y = z = b$
$v = b \vee c$

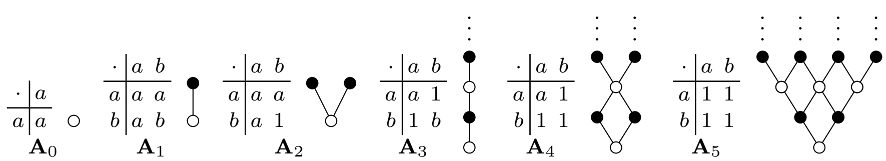
$\cdot$	a	b	c
a	$a \vee c$	$a \vee b$	1
b	$a \vee b$	$b \vee c$	1
c	1	1	1

$\mathbf{C}_4$
$x = a$
$y = b$

□

There exist 16-element counterexamples showing that (6),(7) are not implied by (1)–(5). It is not known at this point whether (5) is strictly stronger than (4), or whether (6), (7) are independent (in the presence of (1)–(5)).

Up to isomorphism, there is a unique Boolean semilattice with 2 elements, denoted by $\mathbf{A}_0$. It is the complex algebra of the 1-element semilattice, and is term-equivalent to the two element Boolean algebra since $x \cdot y = x \wedge y$. Figure 1.2 shows all integral Boolean semilattices with 4 elements ($1 = a \vee b$) and their semilattice representations ($a = \circ, b = \bullet$):



$\mathbf{A}_0$:

$\cdot$	a
a	a

$\mathbf{A}_1$:

$\cdot$	a	b
a	a	a
b	a	b

$\mathbf{A}_2$:

$\cdot$	a	b
a	a	a
b	a	1

$\mathbf{A}_3$:

$\cdot$	a	b
a	a	1
b	1	b

$\mathbf{A}_4$:

$\cdot$	a	b
a	a	1
b	1	1

$\mathbf{A}_5$:

$\cdot$	a	b
a	1	1
b	1	1

Fig. 1.2 Nontrivial Boolean semilattices with ≤ 4 elements (hence ≤ 2 atoms)

Note that $\mathbf{A}_1, \mathbf{A}_2$ are represented by finite semilattices, while the other three cannot be represented by a finite semilattice. The following lemmas provide a simple criterion for a Boolean semilattice that implies any representing semilattice is necessarily infinite.

Lemma 8 *Suppose* $\mathbf{B}$ *is a Boolean semilattice and there exist two non-zero elements* $a, b \in B$ *such that* $a \wedge b = 0$ *and* $a \vee b \le ab$. *If* $\mathbf{S}$ *is any semilattice and* $e : \mathbf{B} \to S^+$ *is an embedding then* S *is infinite.*

Proof From the assumption that $a \wedge b = 0$ and $a, b \leq ab$ it follows that $e(a) \cap e(b) = \emptyset$ and $e(a), e(b) \subseteq e(a)e(b)$. Hence for any $a_1 \in e(a)$ there exists $b_1 \in e(b)$ such that $a_1 < b_1$, and for any $b_1 \in e(b)$ there exists $a_2 \in e(a)$ such that $b_1 < a_2$. Continuing by induction, we obtain two infinite sequences of distinct elements. Since all these elements are in the semilattice $\mathbf{S}$, the set S is infinite. □

For any Boolean semilattice, define the relation $\sqsubseteq$ by $x \sqsubseteq y$ if and only if $x \leq xy$.

Lemma 9 *Let $\mathbf{B}$ be a Boolean semilattice.*

1. *$\sqsubseteq$ is reflexive on B.*
2. *$\sqsubseteq$ is transitive on B if and only if $\mathbf{B}$ satisfies $x \wedge 1y = x \wedge xy$.*
3. *If $\mathbf{B}$ is representable by a finite semilattice then $\sqsubseteq$ is antisymmetric on the atoms of $\mathbf{B}$.*

Proof (1) holds since $x \leq xx$. For (2), assume $\sqsubseteq$ is transitive and $x \leq y1$. Then $x \leq xx \leq xy1$, and $y1 \leq yy1 = y1y$. Hence $x \sqsubseteq y1 \sqsubseteq y$, so by transitivity $x \sqsubseteq y$, i.e. $x \leq xy$.

Conversely, assume $x \leq y1 \implies x \leq xy$ holds for all x, y, and let $x \sqsubseteq y \sqsubseteq z$. Then $x \leq xy$ and $y \leq yz$, whence $x \leq xyz \leq 11z = 1z$. By assumption, $x \leq xz$.

Since distinct atoms are always disjoint, (3) follows from Lemma 8. □

1.4 Constructions of representable Boolean magmas

We now consider some constructions on Boolean magmas that preserve representability. As in [3], the *atom-structure* of a complete and atomic Boolean semilattice $\mathbf{B}$ is denoted by $\mathbf{B}_+ = \langle B_+, R \rangle$ where B_+ is the set of atoms of $\mathbf{B}$ and $R = \{(a, b, c) \in (B_+)^3 : c \leq a \cdot b\}$.

The *tensor product* $\mathbf{A} \otimes \mathbf{B}$ of two complete and atomic Boolean magmas $\mathbf{A}, \mathbf{B}$ is defined as $(\mathbf{A}_+ \times \mathbf{B}_+)^+$. Note that if $\mathbf{A}$ and $\mathbf{B}$ are Boolean semilattices, so is $\mathbf{A} \otimes \mathbf{B}$.

Theorem 10 *If $\mathbf{A}$ is representable by a magma $\mathbf{M_A}$, and $\mathbf{B}$ is representable by a magma $\mathbf{M_B}$ then $\mathbf{A} \otimes \mathbf{B}$ is representable by $\mathbf{M_A} \times M_{\mathbf{B}}$.*

Proof Assume $e : \mathbf{A} \hookrightarrow \mathbf{M}_{\mathbf{A}}^+$ and $f : \mathbf{B} \hookrightarrow \mathbf{M}_{\mathbf{B}}^+$ are the embeddings that show $\mathbf{A}, \mathbf{B}$ are representable. Then there are bounded epimorphisms $e_+ : \mathbf{M_A} \twoheadrightarrow \mathbf{A}_+$ and $f_+ : \mathbf{M_B} \twoheadrightarrow \mathbf{B}_+$, hence we can define a surjective map $g : \mathbf{M_A} \times \mathbf{M_B} \twoheadrightarrow \mathbf{A}_+ \times \mathbf{B}_+$ by $g(x, y) = (e_+(x), f_+(y))$. This is easily seen to be a bounded morphism, whence g^+ is the required embedding into the complex algebra of the semilattice $\mathbf{M_A} \times \mathbf{M_B}$. □

Table 1.3 shows the tensor product $A_3 \otimes A_3$ and its representation in the semilattice $\mathbb{N} \times \mathbb{N}$. Note that A_5 is a subalgebra of this tensor product, hence A_5 is also representable in $\mathbb{N} \times \mathbb{N}$.

$\cdot$	a	b
a	a	1
b	1	b

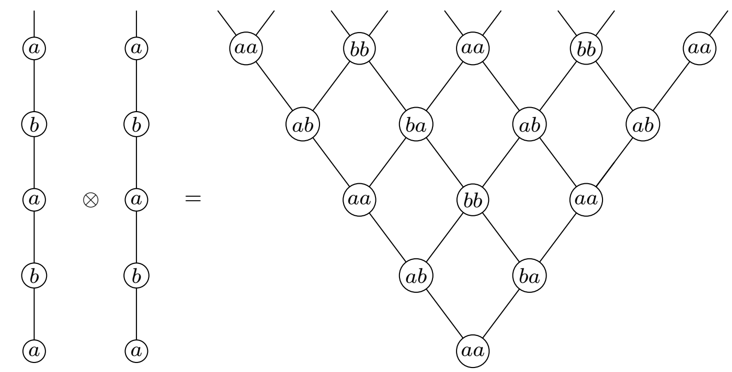
A_3 $\otimes$

$\cdot$	a	b
a	a	1
b	1	b

A_3 $=$

$\cdot$	aa	ab	ba	bb
aa	aa	$aa \vee ab$	$aa \vee ba$	1
ab	$aa \vee ab$	ab	1	$ab \vee bb$
ba	$aa \vee ba$	1	ba	$ba \vee bb$
bb	1	$ab \vee bb$	$ba \vee bb$	bb

$A_3 \otimes A_3$



A_3 A_3 $A_3 \otimes A_3$

Table 1.3 Tensor product of two Boolean magmas and their representations. The notation aa, ab, etc. abbreviates $(a, a), (a, b) \ldots$.

The notation $\mathbf{A}^{(n)}$ is used for the repeated tensor product with n factors of $\mathbf{A}$, i.e., $\mathbf{A}^{(1)} = \mathbf{A}$ and $\mathbf{A}^{(n+1)} = \mathbf{A}^{(n)} \otimes \mathbf{A}$. Table 1.4 gives the operation table on the atoms of $A_3^{(3)}$, which is representable in $\mathbb{N}^3$.

This representation was used to represent the algebra $\mathbf{B}_{41}$ (see Figure 1.5). Previously this Boolean magma was not known to have a semilattice representation.

The *ordinal sum* of two magmas $\mathbf{M}, \mathbf{N}$ is $\mathbf{M} \oplus \mathbf{N} = \langle M \uplus N, \cdot \rangle$ where

$$x \cdot y = \begin{cases} xy & \text{if } x, y \in S \text{ or } x, y \in T \\ x & \text{if } x \in S, y \in T \\ y & \text{if } y \in S, x \in T. \end{cases}$$

If both $\mathbf{M}, \mathbf{N}$ are associative, commutative or idempotent then so is their ordinal sum. In particular, if both $\mathbf{M}, \mathbf{N}$ are semilattices then the ordinal sum is a semilattice that stacks a copy of $\mathbf{M}$ below a copy of $\mathbf{N}$.

$\cdot$	a	b	c	d	f	g	h	k
a	a	$a{\vee}b$	$a{\vee}c$	$a{\vee}b{\vee}c{\vee}d$	$a{\vee}f$	$a{\vee}b{\vee}f{\vee}g$	$a{\vee}c{\vee}f{\vee}h$	$\mathbf{1}$
b		b	$a{\vee}b{\vee}c{\vee}d$	$b{\vee}d$	$a{\vee}b{\vee}f{\vee}g$	$b{\vee}g$	$\mathbf{1}$	$b{\vee}d{\vee}g{\vee}k$
c			c	$c{\vee}d$	$a{\vee}c{\vee}f{\vee}h$	$\mathbf{1}$	$c{\vee}h$	$c{\vee}d{\vee}h{\vee}k$
d				d	$\mathbf{1}$	$b{\vee}d{\vee}g{\vee}k$	$c{\vee}d{\vee}h{\vee}k$	$d{\vee}k$
f					f	$f{\vee}g$	$f{\vee}h$	$a{\vee}b{\vee}c{\vee}d$
g						g	$a{\vee}b{\vee}c{\vee}d$	$g{\vee}k$
h							h	$h{\vee}k$
k								k

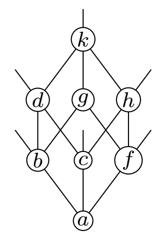


Table 1.4 Operation table and representation of $A_3 \otimes A_3 \otimes A_3 = A_3^{(3)}$

$\cdot$	a	b	c
a	a	1	1
b		b	1
c			1

$\mathbf{B}_{41}$

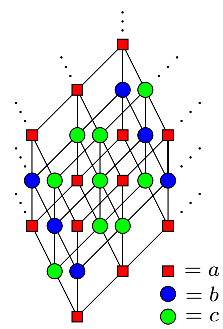


Fig. 1.5 The semilattice representation of $\mathbf{B}_{41}$

The construction can be adapted for Boolean magmas, Boolean semigroups or Boolean semilattices. The *ordinal sum* $\mathbf{A} \oplus \mathbf{B}$ of Boolean magmas $\mathbf{A}$ and $\mathbf{B}$ is an algebra $\mathbf{C}$ such that $C = A \times B$, where the Boolean operations are defined pointwise and

$$(a,b) \cdot^{\mathbf{C}} (c,d) = \begin{cases} (ac \vee a \vee c, bd) & \text{if } b,d \neq 0 \\ (ac \vee a, bd) & \text{if } b = 0, d \neq 0 \\ (ac \vee c, bd) & \text{if } b \neq 0, d = 0 \\ (ac, bd) & \text{if } b = 0 = d. \end{cases}$$

Theorem 11 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are Boolean semigroups or semilattices then* $\mathbf{A} \oplus \mathbf{B}$ *is a Boolean semigroup or a Boolean semilattice respectively.*

Furthermore, if $\mathbf{A}$ *is representable by a semigroup or semilattice* $\mathbf{S_A}$*, and* $\mathbf{B}$ *is representable by a semigroup or semilattice* $\mathbf{S_B}$ *then the ordinal sum* $S_{\mathbf{A}} \oplus S_{\mathbf{B}}$ *gives a representation of* $\mathbf{A} \oplus \mathbf{B}$*.*

Proof Assume $\mathbf{A}$ and $\mathbf{B}$ are Boolean semilattices. Then clearly the magma operation $\cdot^{\mathbf{C}}$ is idempotent and commutative. To see that it is associative consider the following calculation for $b, d, g \neq 0$

$$\begin{aligned}((a,b)\cdot^{\mathbf{C}}(c,d))\cdot^{\mathbf{C}}(f,g) &= ((ac\vee a\vee c)f\vee ac\vee a\vee c\vee f, bdg)\\ &= (acf\vee af\vee cf\vee ac\vee a\vee c\vee f, bdg) \text{ and}\\ (a,b)\cdot^{\mathbf{C}}((c,d)\cdot^{\mathbf{C}}(f,g)) &= (a(cf\vee c\vee f)\vee a\vee cf\vee c\vee f, bdg)\\ &= (acf\vee ac\vee af\vee cf\vee a\vee c\vee f, bdg)\end{aligned}$$

The remaining 7 cases when one or more of b, d, g are 0 can be checked similarly. Moreover, $(a,b)\cdot^{\mathbf{C}}(0,0) = (a0\vee 0, b0) = (0,0)$ and assuming $b, d, g \neq 0$ we have

$$\begin{aligned}(a,b)\cdot^{\mathbf{C}}((c,d)\vee(f,g)) &= (a(c\vee f)\vee a\vee c\vee f, b(d\vee g))\\ &= (ac\vee a\vee c\vee af\vee a\vee f, bd\vee bg)\\ &= ((a,b)\cdot^{\mathbf{C}}(c,d))\vee((a,b)\cdot^{\mathbf{C}}(f,g)).\end{aligned}$$

Now suppose $h : \mathbf{A} \hookrightarrow \mathbf{S}^+_{\mathbf{A}}$ and $k : \mathbf{B} \hookrightarrow \mathbf{S}^+_{\mathbf{B}}$ are the embeddings that show $\mathbf{A}, \mathbf{B}$ are representable. An embedding $l : A \oplus B \hookrightarrow S^+_{\mathbf{A}} \oplus S^+_{\mathbf{B}}$ is defined by $l(a,b) = (h(a), k(b)$. This is clearly an injective Boolean morphism, and we compute

$$\begin{aligned}l((a,b)\cdot(c,d)) &= l(ac\vee a\vee c, bd)\\ &= (h(ac\vee a\vee c), k(bd))\\ &= (h(a)h(c)\vee h(a)\vee h(c), k(b)k(d))\\ &= (h(a), k(b))\cdot(h(c), k(d))\\ &= l(a,b)\cdot l(c,d)\end{aligned}$$

So to show that $\mathbf{A} \oplus \mathbf{B}$ is representable, it suffices to show that $S^+_{\mathbf{A}} \oplus S^+_{\mathbf{B}}$ is isomorphic to $(S_{\mathbf{A}} \oplus S_{\mathbf{B}})^+$. The isomorphism maps a pair (U, V) to $U \uplus V$, and it is straight forward to check that this preserves $\cdot$ and the Boolean operations. □

An example of this construction is given in Figure 1.6. The lower part of the diagram is a representation of $\mathbf{A}_4$ in an infinite semilattice. The upper part is $\mathbf{A}_0$, a 2-element Boolean algebra, represented by a 1-element semilattice. The 8-element Boolean magma $\mathbf{B}_1$ is isomorphic to $\mathbf{A}_0 \oplus \mathbf{A}_4$, hence $\mathbf{B}_1$ is representable.

Let $\mathbf{B}$ be a Boolean semigroup and define the *countable ordinal sum* $\mathbf{B}^\oplus$ to have the same elements and Boolean operations as $\mathbf{B}$ and

$$x \cdot^{\mathbf{B}^\oplus} y = \begin{cases}(x\cdot^B y)\vee x\vee y & \text{if } x, y \neq 0\\ 0 & \text{if } x = 0 \text{ or } y = 0.\end{cases}$$

$\cdot$	a	b	c
a	a	b	c
b		b	$b{\vee}c$
c			$b{\vee}c$

$\mathbf{B}_1$

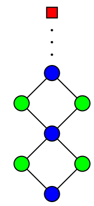


Fig. 1.6 An ordinal sum representation of $\mathbf{B}_1 \cong \mathbf{A}_0 \oplus \mathbf{A}_4$

Theorem 12 (Countable ordinal sum of B) *For any Boolean semigroup* $\mathbf{B}$, $\mathbf{B}^{\oplus}$ *is a Boolean semigroup and if* $\mathbf{B}$ *is commutative then* $\mathbf{B}^{\oplus}$ *is a Boolean semilattice.*

Moreover, if $S_{\mathbf{B}}$ *is a semigroup that represents* $\mathbf{B}$ *then the countable ordinal sum* $\oplus_{k\in\mathbb{N}} S_{\mathbf{B}}$ *is a semigroup representation of* $\mathbf{B}^{\oplus}$.

Proof If suffices to show associativity, so let $x, y, z \in B$ and denote $x \cdot^{\mathbf{B}^{\oplus}} y$ by xy. If any one of x, y, z is 0, then $(xy)z = 0 = x(yz)$, and if they are all nonzero, then

$$\begin{aligned}(xy)z &= (x{\cdot}y \vee x \vee y){\cdot}z \vee (x{\cdot}y \vee x \vee y) \vee z \\ &= x{\cdot}y{\cdot}z \vee x{\cdot}z \vee y{\cdot}z \vee x{\cdot}y \vee x \vee y \vee z \\ &= x{\cdot}(y{\cdot}z \vee y \vee z) \vee x \vee (y{\cdot}z \vee y \vee z) = x(yz).\end{aligned}$$

Assume $e : \mathbf{B} \to S_{\mathbf{B}}^{+}$ is an embedding that represents $\mathbf{B}$. Let T be the countable ordinal sum of $S_{\mathbf{B}}$, and for $s \in S_{\mathbf{B}}$, let s_i be the copy of s in the i^{th} disjoint summand of T. The representation of $\mathbf{B}^{\oplus}$ is defined by $e' : \mathbf{B}^{\oplus} \to T^{+}$ where $e'(x) = \{s_i \mid s \in e(x) \text{ and } i \in \mathbb{N}\}$. It is straight forward to check that this is a Boolean semigroup homomorphism. □

As an application, we note that in Figure 1.2 $\mathbf{A}_3 = \mathbf{A}_1^{\oplus}$ and $\mathbf{A}_4 = \mathbf{A}_2^{\oplus}$. Similarly several of the 8-element Boolean semilattices in the Appendix have representations based on the countable ordinal sum construction.

For the last general construction, let $\mathbf{B}$ be a Boolean semigroup and define $\mathbf{B2}$ to be the Boolean algebra $\mathbf{B} \times \mathbf{2}$ with binary operation

$$(x, i) \cdot (y, j) = (xy \vee xj \vee iy, i \wedge j)$$

where $\mathbf{2}$ is identified with the subset $\{0, 1\}$ of $\mathbf{B}$.

Theorem 13 (The doubling extension of B) *If* $\mathbf{B}$ *is a Boolean semigroup or Boolean semilattice then the same holds for the algebra* $\mathbf{B2}$.

If $\mathbf{B}$ *is represented by a semigroup* $S_{\mathbf{B}}$*, then* $\mathbf{B2}$ *is represented by the product* $S_{\mathbf{B}} \times 2$*, where* 2 *is the semilattice with 2 elements.*

Proof Associativity of $\cdot$ follows from associativity of $\mathbf{B}$ and the calculation

$$\begin{aligned}((x,i)(y,j))(z,k) &= (xy \vee xj \vee iy, i \wedge j)(z,k)\\ &= ((xy \vee xj \vee iy)z \vee (xy \vee xj \vee iy)k \vee ijz, i \wedge j \wedge k)\\ &= (xyz \vee xjz \vee iyz \vee xyk \vee xjk \vee iyk \vee ijz, i \wedge j \wedge k)\\ &= x(yz \vee jz \vee yk) \vee xjk \vee i(yz \vee yk \vee jz), i \wedge j \wedge k)\\ &= (x,i)(yz \vee yk \vee jz, j \wedge k) = ((x,i)(y,j))(z,k).\end{aligned}$$

Similarly one can check that $\cdot$ is an operator that also preserves commutativity and the square-increasing law. Given an embedding $e : \mathbf{B} \to S_{\mathbf{B}}^{+}$, define $e' : \mathbf{B2} \to (S_{\mathbf{B}} \times 2)^{+}$ by $e'(x,i) = (e(x) \times \{0\}) \cup \delta(i)$, where $\delta(0) = \emptyset$ and $\delta(1) = S_{\mathbf{B}} \times \{1\}$. Then e' is injective, and to see that it preserves $\cdot$ we compute

$$\begin{aligned}e'((x,i)(y,j)) &= e'(xy \vee xj \vee iy, i \wedge j)\\ &= e(xy \vee xj \vee iy) \times \{0\} \cup \delta(i \wedge j)\\ &= e(xy) \times \{0\} \cup e(xj) \times \{0\} \cup e(iy) \times \{0\} \cup \delta(i)\delta(j)\\ &= (e(x)e(y)) \times \{0\} \cup (e(x) \times \{0\})\delta(j) \cup \delta(i)(e(y) \times \{0\}) \cup \delta(i)\delta(j)\\ &= (e(x) \times \{0\} \cup \delta(i)) \cdot (e(y) \times \{0\} \cup \delta(j)) = e'(x,i) \cdot e'(y,j).\end{aligned}$$

Similar calculations show that e' is a Boolean homomorphism. □

Note that if $\mathbf{B}$ is a Boolean semilattice then $S_{\mathbf{B}}$ is a semilattice, hence $\mathbf{B2}$ is also a Boolean semilattice. The Boolean semigroup $\mathbf{B2}$ differs from $\mathbf{B} \otimes \mathbf{2}$ since the atom-structure of $\mathbf{B2}$ has an extra atom while $\mathbf{B} \otimes \mathbf{2}$ is isomorphic to $\mathbf{B}$.

As an application we observe that the Boolean semilattice $\mathbf{B}_{13}$ is representable since it is isomorphic to $\mathbf{A}_4\mathbf{2}$ (see Figure 1.7).

$\cdot$	a	b	c
a	a	$b\vee c$	$b\vee c$
b		b	$b\vee c$
c			$b\vee c$

$\mathbf{B}_{13}$

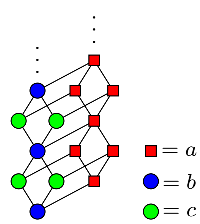


Fig. 1.7 A representation given by the doubling construction of $\mathbf{B}_{13} \cong \mathbf{A}_4\mathbf{2}$

A semilattice is a *tree-semilattice* if its partial order is a tree, i.e., has a bottom element and every principal downset is a chain. So a tree-semilattice satisfies $x, y \le z \implies x \le y$ or $y \le x$.

A Boolean semilattice is *tree-representable* if it is embedded in the complex algebra of a tree-semilattice.

Theorem 14 *If* $\mathbf{B}$ *is a tree-representable Boolean semilattice then the identity* $(x \wedge y1)z - x \leq yz$ *holds in* $\mathbf{B}$.

Proof Assume $\mathbf{B}$ is a subalgebra of $\mathbf{T}^+$ for some tree-semilattice $\mathbf{T}$. Let $p \in (x \wedge y1)z - x$. Then $p \notin x$ and $p = x'z'$ for some $x' \in x$, $z' \in z$ and $x' \leq y'$ for some $y' \in y$. If $x' \leq y'z'$ then $x' = x'y'z' = py' = p$, contradicting $p \notin x$. Since T is a tree-semilattice and y' is an upper bound for both x' and $y'z'$, it follows that $y'z' \leq x'$. Hence $p = x'z' = x'y'z' = y'z' \in yz$. □

We note that $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are tree-representable (Figure 1.2). In the appendix all 8-element tree-representable Boolean semilattices are given in Figures 1.12, 1.11.

A computer calculation shows that there are (up to isomorphism) 79 integral Boolean semilattices with 8 elements that satisfy the formulas in Lemma 7. Of these, 72 are known to be representable (listed with representations in the Appendix). The representation of $\mathbf{B}_{40}$ (Figure 1.8) was found by Miklos Maroti [7]. It is an open problem whether the remaining 7, shown in Table 1.9, are representable.

$\cdot$	a	b	c
a	a	1	1
b		b	1
c			c

$\mathbf{B}_{40}$

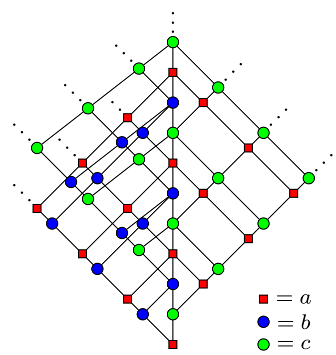


Fig. 1.8 A Boolean semilattice with a representation found by M. Maroti [7]

$\cdot$	a	b	c
a	a	$a\vee c$	$a\vee c$
b		$b\vee c$	$a\vee c$
c			$a\vee c$

B_{17}

$\cdot$	a	b	c
a	a	$a\vee b$	1
b		$b\vee c$	1
c			1

B_{32}

$\cdot$	a	b	c
a	a	$a\vee b$	1
b		$a\vee b$	1
c			$b\vee c$

$B_{34} = B_{17}^{\oplus}$

$\cdot$	a	b	c
a	$a\vee c$	$b\vee c$	$b\vee c$
b		$b\vee c$	$b\vee c$
c			$b\vee c$

B_{45}

$\cdot$	a	b	c
a	$a\vee c$	1	$a\vee c$
b		$b\vee c$	1
c			$a\vee c$

$B_{46} = B_{45}^{\oplus}$

$\cdot$	a	b	c
a	$a\vee c$	1	1
b		$a\vee b$	1
c			$b\vee c$

B_{48}

$\cdot$	a	b	c
a	$a\vee c$	1	1
b		$a\vee b$	1
c			1

B_{49}

Table 1.9 Seven 8-element Boolean semilattices without known representations.

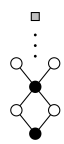
This research was done while the second and third author were undergraduate students at Chapman University. We gratefully acknowledge the very helpful efforts of the referee.

1.5 Appendix: Known representations for 8-element Boolean semilattices

For each of the tables below, we provide (where known) a semilattice with the elements partitioned into three disjoint subsets a, b, c such that the complex products of these subsets give the values in the table. Note: $a = ▫$, $b = ●$, $c = ○$.

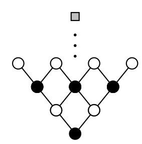
$\cdot$	a	b	c
a	a	b	c
b		b	$b\vee c$
c			$b\vee c$

$B_1 = A_4 \oplus A_0$



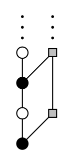
$\cdot$	a	b	c
a	a	b	c
b		$b\vee c$	$b\vee c$
c			$b\vee c$

$B_2 = A_5 \oplus A_0$



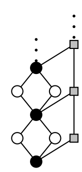
$\cdot$	a	b	c
a	a	b	$b\vee c$
b		b	$b\vee c$
c			c

B_3



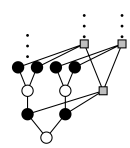
$\cdot$	a	b	c
a	a	b	$b\vee c$
b		b	$b\vee c$
c			$b\vee c$

B_4



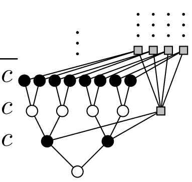
$\cdot$	a	b	c
a	a	b	$b\vee c$
b		$b\vee c$	$b\vee c$
c			c

B_5



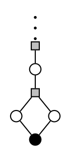
$\cdot$	a	b	c
a	a	b	$b\vee c$
b		$b\vee c$	$b\vee c$
c			$b\vee c$

B_6



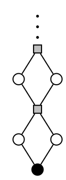
$\cdot$	a	b	c
a	a	b	$a\vee c$
b		b	b
c			$b\vee c$

B_7



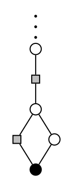
$\cdot$	a	b	c
a	a	b	$a\vee c$
b		b	b
c			1

B_8



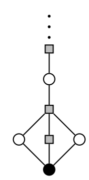
$\cdot$	a	b	c
a	a	b	1
b		b	b
c			c

B_9



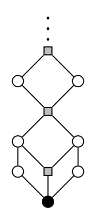
$\cdot$	a	b	c
a	a	b	1
b		b	b
c			$b\vee c$

B_{10}



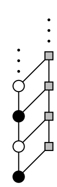
$\cdot$	a	b	c
a	a	b	1
b		b	b
c			1

B_{11}



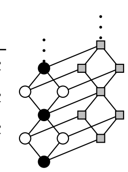
$\cdot$	a	b	c
a	a	$b\vee c$	$b\vee c$
b		b	$b\vee c$
c			c

B_{12}



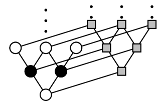
$\cdot$	a	b	c
a	a	$b\vee c$	$b\vee c$
b		b	$b\vee c$
c			$b\vee c$

B_{13}



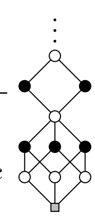
$\cdot$	a	b	c
a	a	$b\vee c$	$b\vee c$
b		$b\vee c$	$b\vee c$
c			$b\vee c$

B_{14}



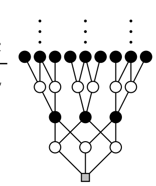
$\cdot$	a	b	c
a	a	a	a
b		$b\vee c$	1
c			$a\vee c$

B_{15}



$\cdot$	a	b	c
a	a	a	a
b		$b\vee c$	1
c			1

B_{16}

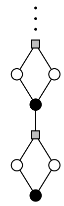


$\cdot$	a	b	c
a	a	$a\vee c$	$a\vee c$
b		$b\vee c$	$a\vee c$
c			$a\vee c$

B_{17} Rep.?

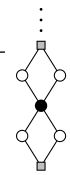
$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		b	$b\vee c$
c			$b\vee c$

$B_{18} = B_1^{f\oplus}$



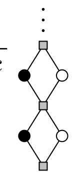
$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		b	$b\vee c$
c			1

B_{19}



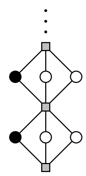
$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		b	1
c			c

$B_{20} = B_1^{ft\oplus}$



$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		b	1
c			$a\vee c$

$B_{21} = B_2^{ft\oplus}$



$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		b	1
c			1

$B_{22} = B_6^{ft\oplus}$

$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		$b\vee c$	$b\vee c$
c			$b\vee c$

$B_{23} = B_4^{t\oplus}$

$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		$b\vee c$	1
c			$a\vee c$

$B_{24} = B_4^{f\oplus}$

$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		$b\vee c$	1
c			1

$B_{25} = B_{16}^{\oplus}$

$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		$a\vee b$	1
c			1

$B_{26} = B_9^{ft\oplus}$

$\cdot$	a	b	c
a	a	$a\vee b$	$a\vee c$
b		1	1
c			1

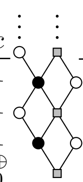
$B_{27} = B_7^{t\oplus}$

$\cdot$	a	b	c
a	a	$a\vee b$	1
b		b	1
c			c

$B_{28} = B_{12}^{\oplus}$

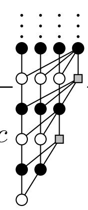
$\cdot$	a	b	c
a	a	$a\vee b$	1
b		b	1
c			1

$B_{29} = B_{10}^{t\oplus}$



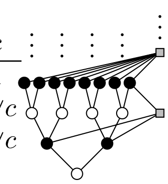
$\cdot$	a	b	c
a	a	$a\vee b$	1
b		$b\vee c$	$b\vee c$
c			c

B_{30}



$\cdot$	a	b	c
a	a	$a\vee b$	1
b		$b\vee c$	$b\vee c$
c			$b\vee c$

B_{31}



$\cdot$	a	b	c
a	a	$a\vee b$	1
b		$b\vee c$	1
c			1

B_{32} Rep.?

$\cdot$	a	b	c
a	a	$a\vee b$	1
b		$a\vee b$	1
c			c

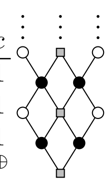
$B_{33} = B_{13}^{\oplus}$

$\cdot$	a	b	c
a	a	$a\vee b$	1
b		$a\vee b$	1
c			$b\vee c$

$B_{34} = B_{17}^{\oplus}$

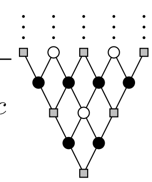
$\cdot$	a	b	c
a	a	$a\vee b$	1
b		$a\vee b$	1
c			1

$B_{35} = B_{11}^{t\oplus}$



$\cdot$	a	b	c
a	a	$a\vee b$	1
b		1	$b\vee c$
c			c

B_{36}



$\cdot$	a	b	c
a	a	$a\vee b$	1
b		1	1
c			c

$B_{37} \le A_3^{(3)}$

$\cdot$	a	b	c
a	a	$a\vee b$	1
b		1	1
c			$b\vee c$

$B_{38} \le A_3^{(2)} \otimes A_4$

$\cdot$	a	b	c
a	a	$a\vee b$	1
b		1	1
c			1

$B_{39} \le A_3^{(4)}$

$\cdot$	a	b	c
a	a	1	1
b		b	1
c			c

B_{40} Maroti[7]

$\cdot$	a	b	c
a	a	1	1
b		b	1
c			1

$B_{41} \le A_3^{(3)}$

·	a	b	c
a	a	1	1
b		$b\vee c$	$b\vee c$
c			$b\vee c$

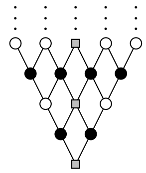
$B_{42} = B_{14}^{\oplus}$

·	a	b	c
a	a	1	1
b		$b\vee c$	1
c			1

$B_{43} \le A_3^{(4)}$

·	a	b	c
a	a	1	1
b		1	1
c			1

B_{44}



·	a	b	c
a	$a\vee c$	$b\vee c$	$b\vee c$
b		$b\vee c$	$b\vee c$
c			$b\vee c$

B_{45} Rep.?

·	a	b	c
a	$a\vee c$	1	$a\vee c$
b		$b\vee c$	1
c			$a\vee c$

$B_{46} = B_{45}^{\oplus}$

·	a	b	c
a	$a\vee c$	1	1
b		$b\vee c$	1
c			1

$B_{47} \le A_3^{(3)} \otimes B_{23}$

·	a	b	c
a	$a\vee c$	1	1
b		$a\vee b$	1
c			$b\vee c$

B_{48} Rep.?

·	a	b	c
a	$a\vee c$	1	1
b		$a\vee b$	1
c			1

B_{49} Rep.?

·	a	b	c
a	$a\vee c$	1	1
b		1	1
c			1

$B_{50} \le A_4^{(3)}$

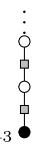
Fig. 1.10 Representations for B_1–B_{50}, except $B_{17}, B_{32}, B_{34}, B_{45}, B_{46}, B_{48}, B_{49}$

·	a	b	c
a	a	b	c
b		b	$b\vee c$
c			c

$B_1^t = A_3 \oplus A_0$

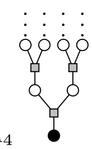
·	a	b	c
a	a	b	$a\vee c$
b		b	b
c			c

$B_2^t = A_0 \oplus A_3$



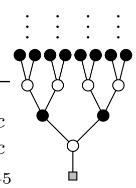
·	a	b	c
a	a	b	$a\vee c$
b		b	b
c			$a\vee c$

$B_3^t = A_0 \oplus A_4$



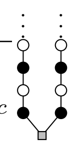
·	a	b	c
a	a	a	a
b		$b\vee c$	$b\vee c$
c			$b\vee c$

$B_4^t = A_0 \oplus A_5$



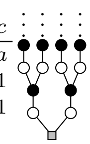
·	a	b	c
a	a	a	a
b		$a\vee b$	1
c			$a\vee c$

B_5^t



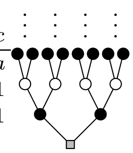
·	a	b	c
a	a	a	a
b		$a\vee b$	1
c			1

B_6^t



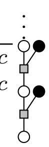
·	a	b	c
a	a	a	a
b		1	1
c			1

B_7^t



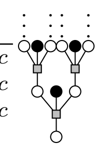
·	a	b	c
a	a	a	$a\vee c$
b		$a\vee b$	$a\vee c$
c			c

B_8^t



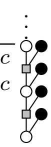
·	a	b	c
a	a	a	$a\vee c$
b		$a\vee b$	$a\vee c$
c			$a\vee c$

B_9^t



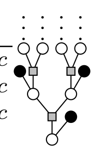
·	a	b	c
a	a	$a\vee c$	$a\vee c$
b		1	$a\vee c$
c			c

B_{10}^t



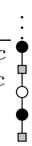
·	a	b	c
a	a	$a\vee c$	$a\vee c$
b		1	$a\vee c$
c			$a\vee c$

B_{11}^t



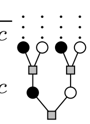
·	a	b	c
a	a	$a\vee b$	$a\vee c$
b		b	$b\vee c$
c			c

$B_{12}^t = B_3^{ft\oplus}$



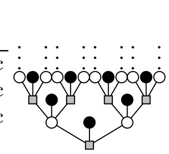
·	a	b	c
a	a	$a\vee b$	$a\vee c$
b		$a\vee b$	1
c			$a\vee c$

B_{13}^t



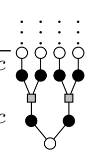
·	a	b	c
a	$a\vee c$	$a\vee c$	$a\vee c$
b		1	$a\vee c$
c			$a\vee c$

B_{14}^t



·	a	b	c
a	$a\vee c$	1	$a\vee c$
b		1	1
c			$a\vee c$

B_{15}^t



·	a	b	c
a	1	1	1
b		1	1
c			1

B_{16}^t

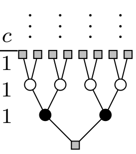
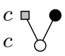


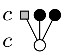
Fig. 1.11 Algebras representable by infinite tree semilattices

$\cdot$	a	b	c
a	a	c	c
b		b	c
c			c



B_1^{ft}

$\cdot$	a	b	c
a	a	c	c
b		$b\vee c$	c
c			c

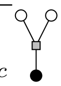


B_2^{ft}

$\cdot$	a	b	c
a	a	b	c
b		b	c
c			c

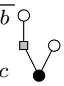
B_3^{ft}

$\cdot$	a	b	c
a	a	b	a
b		b	b
c			$a\vee c$



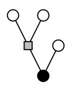
B_4^{ft}

$\cdot$	a	b	c
a	a	b	$a\vee b$
b		b	b
c			$b\vee c$



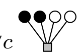
B_5^{ft}

$\cdot$	a	b	c
a	a	b	$a\vee b$
b		b	b
c			1



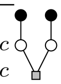
B_6^{ft}

$\cdot$	a	b	c
a	a	a	a
b		$a\vee b$	a
c			$a\vee c$



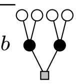
B_7^{ft}

$\cdot$	a	b	c
a	a	a	a
b		$a\vee b$	$a\vee c$
c			$a\vee c$



B_8^{ft}

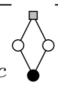
$\cdot$	a	b	c
a	a	a	a
b		$a\vee b$	$a\vee b$
c			1



B_9^{ft}

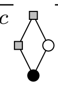
Fig. 1.12 Algebras representable by finite tree semilattices

$\cdot$	a	b	c
a	a	b	c
b		b	b
c			$b\vee c$



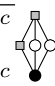
B_1^{f}

$\cdot$	a	b	c
a	a	b	$b\vee c$
b		b	b
c			c



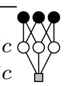
B_2^{f}

$\cdot$	a	b	c
a	a	b	$b\vee c$
b		b	b
c			$b\vee c$



B_3^{f}

$\cdot$	a	b	c
a	a	a	a
b		$b\vee c$	$a\vee c$
c			$a\vee c$



B_4^{f}

Fig. 1.13 Algebras representable by finite (non-tree) semilattices

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