Chapter 1

Varieties of Lattices

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In this chapter we discuss some of the more recent results and give a general overview of what is currently known about lattice varieties. Of course it is impossible to give a comprehensive account. Often we only cite recent or survey papers, which themselves have many more references. We would like to apologize in advance for any errors, omissions, or miscrediting of results.

For proofs of the results mentioned here, we refer the reader to the original papers. Details of many of the results from before 1992 can also be found in our monograph, P. Jipsen and H. Rose [39].

1-1. The lattice Λ

Recall from [LTF Section VI.2] that the lattice Λ of all lattice varieties is a dually algebraic, distributive lattice that has the variety $\mathbf{L}$ of all lattices at the top, the variety $\mathbf{T}$ of all trivial lattices at the bottom, and the variety $\mathbf{D} = \mathbf{Var}(C_2)$ of all distributive lattices as the unique atom. To conclude that $\mathbf{L}$ is join-irreducible and has no coatoms, B. Jónsson [40] argued as follows: Let $\mathbf{V}$, $\mathbf{W}$ be proper subvarieties of $\mathbf{L}$ and choose lattices $K \notin \mathbf{V}$, $L \notin \mathbf{W}$. Using P. M. Whitman's [79] result that every lattice can be embedded in a partition lattice, one obtains a subdirectly irreducible lattice S that extends $K \times L$. Since $S \notin \mathbf{Si}(\mathbf{V}) \cup \mathbf{Si}(\mathbf{W}) = \mathbf{Si}(\mathbf{V} \vee \mathbf{W})$, it follows that $\mathbf{V} \vee \mathbf{W}$ is a proper subvariety as well, hence $\mathbf{L}$ is join-irreducible. By R. A. Dean [13], $\mathbf{L}$ is generated by its finite members, so we may assume that K is finite. The distributivity of Λ and Jónsson's Lemma imply that the interval from $\mathbf{V}$ to

$\mathbf{V} \vee \mathbf{Var}(K)$ is finite, so every proper subvariety has at least one cover in Λ, and $\mathbf{L}$ has no co-atoms since $\mathbf{V} < \mathbf{V} \vee \mathbf{Var}(K) < \mathbf{L}$ (by join-irreducibility).

A substantial amount of research has been done on the structure near the bottom of Λ. One of the aims was to investigate this lattice by finding all varieties of a given finite height. By Jónsson's Lemma (LTF Theorem 475), a finite lattice generates a variety of finite height. The converse assertion, called the Finite Height Conjecture, was a longstanding open problem. Finally, J. B. Nation [60] found a counterexample (see LTF VI.2.2).

Specific lattices are labeled by capital letter and the varieties they generate are referred to by the corresponding boldface letter (for example, $\mathbf{N}_5 = \mathbf{Var}(N_5)$). We say that a variety $\mathbf{V}$ is *strongly covered* by a collection $\mathcal{C}$ of varieties, if every variety that properly contains $\mathbf{V}$ also contains at least one member of $\mathcal{C}$.

The first few levels above the trivial variety are described in [LTF Sections VI.2 and VI.3] (see LTF Figure 104). G. Grätzer [23] proved that any finitely generated modular variety is strictly above $\mathbf{M}_3$ is above $\mathbf{M}_4$ or $\mathbf{M}_{3^2}$ (or both). B. Jónsson [41] removed the restriction that the variety be finitely generated by proving the following result:

Theorem 1-1.1. *For a modular variety* $\mathbf{V}$ *the following conditions are equivalent:*

(i) $M_{3^2} \notin \mathbf{V}$,

(ii) *every subdirectly irreducible member of* $\mathbf{V}$ *has length* ≤ 2,

(iii) *the inequality* $x \wedge (y \vee (z \wedge w)) \wedge (z \vee w) \leq y \vee (x \wedge z) \vee (x \wedge w)$ *holds in* $\mathbf{V}$.

The only subdirectly irreducible lattices of length 2 are M_κ where $\kappa \geq 3$ is the cardinality of elements of height 1. For infinite κ these lattices all generate the same variety $\mathbf{M}_\omega$, and for $\kappa = n$ finite they generate (by Jónsson's Lemma) a covering chain of varieties $\mathbf{M}_n$ above $\mathbf{M}_3$ that joins to $\mathbf{M}_\omega$. The preceding theorem implies that any modular variety that does not contain M_{3^2} must be trivial, distributive or one of the varieties $\mathbf{M}_n$ $(n \geq 3)$ or $\mathbf{M}_\omega$. Jónsson [41] deduces the following result.

Theorem 1-1.2. *For* $n \geq 3$*, the covers of* $\mathbf{M}_n$ *are* $\mathbf{M}_{n+1}$*,* $\mathbf{M}_n \vee \mathbf{M}_{3^2}$ *and* $\mathbf{M}_n \vee \mathbf{N}_5$*. The variety* $\mathbf{M}_\omega$ *is strongly covered by* $\mathbf{M}_\omega \vee \mathbf{M}_{3^2}$ *and* $\mathbf{M}_\omega \vee \mathbf{N}_5$*.*

Let M_{3^n}, A_1, A_2, A_3 be the lattices in Figures 1 and suppose that M is a subdirectly irreducible modular lattice. The main technical result of D. X. Hong [33] is that if M_{3^n}, A_1, A_2, $A_3 \notin \mathbf{HS}\{M\}$, then M has length at most n. This is a typical *exclusion result* which is very useful when it comes to finding covers of varieties.

Let $\mathbf{M}_w^l$ be the variety generated by all modular lattices of length at most l and of width at most w ($1 \leq l,\, w \leq \infty$). For example, $\mathbf{M}_\infty^2 = \mathbf{M}_\omega$ and $\mathbf{M}_\infty^3$ is the variety generated by all subspace lattices of projective planes (see the proof of Theorem 444 in LTF). With this notation, Hong's result implies that for any variety $\mathbf{V}$ of modular lattices, M_{3^3}, A_1, A_2, $A_3 \notin \mathbf{V}$ if and only if $\mathbf{V} \subseteq \mathbf{M}_\infty^3$. It follows immediately that $\mathbf{M}_\infty^3$ has exactly five covers in Λ, given by $\mathbf{M}_\infty^3 \vee \mathbf{V}$ where $\mathbf{V} \in \{\mathbf{M}_{3^3}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{N}_5\}$.

It is easy to check that the varieties $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{M}_{3^3}$, $\mathbf{F}_2$ (generated by the corresponding lattices in Figures 1) each cover the variety $\mathbf{M}_{3^2}$. Using the above exclusion result and some added detail, D. X. Hong [32] proves that they are the only join-irreducible covers. More generally, he shows the following.

Theorem 1-1.3.

(i) *For $n \geq 2$, the covers of $\mathbf{M}_{3^n}$ are $\mathbf{M}_{3^{n+1}}$ and $\mathbf{M}_{3^n} \vee \mathbf{V}$, where $\mathbf{V} \in \{\mathbf{M}_4, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{F}_2, \mathbf{N}_5\}$.*

(ii) *Let $\mathbf{V}$ be a variety generated by a finite collection of finite modular lattices of length ≤ 3 and let $\mathbf{W}$ be a variety generated by a finite collection of lattices of the form $M_{n_1,\ldots,n_k}$ (see Figure 1). Then each of the following varieties is strongly covered by finitely many varieties that can be effectively found:*

$$\mathbf{V} \vee \mathbf{W}, \quad \mathbf{M}_\infty^2 \vee \mathbf{V} \vee \mathbf{W}, \quad \mathbf{M}_\infty^3 \vee \mathbf{V} \vee \mathbf{W}.$$

This result gives a fairly good description of the bottom of Λ on the modular side.

Problem 1. Find all the covers of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$.

Problem 2. Does the Finite Height Conjecture hold for modular varieties? Does it hold for the variety of modular 2-distributive lattices?

A stronger form of the Finite Height Conjecture for modular lattices asks whether every finite modular lattice has only finitely many modular covers, each generated by a finite lattice. C. Herrmann and A. Nurakunov [30] proved that for the class of modular lattices of finite height this stronger conjecture holds.

Problem 3. Does the variety of modular lattices or the variety of arguesian lattices have any dual covers?

A *planar lattice* is defined to be a finite lattice for which there exists a 2-dimensional Hasse diagram with no intersecting lines. The class $\mathbf{PM}$ of planar modular lattices and the variety $\mathbf{VPM} = \mathbf{Var}(\mathbf{PM})$ has been investigated by G. Grätzer and R. W. Quackenbush [27]. Given a planar modular lattice

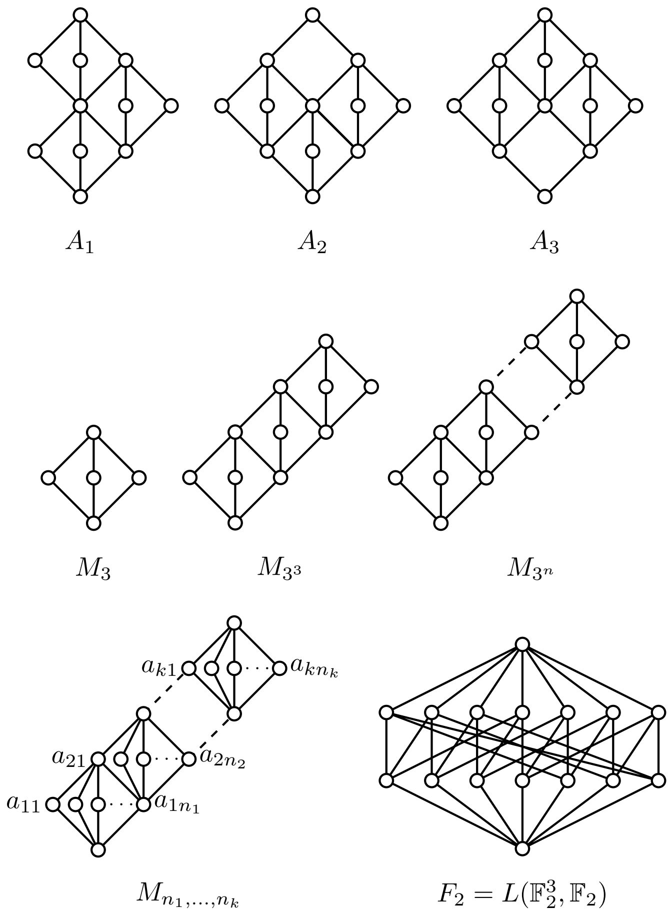


Figure 1: Modular lattices that generate varieties of finite height

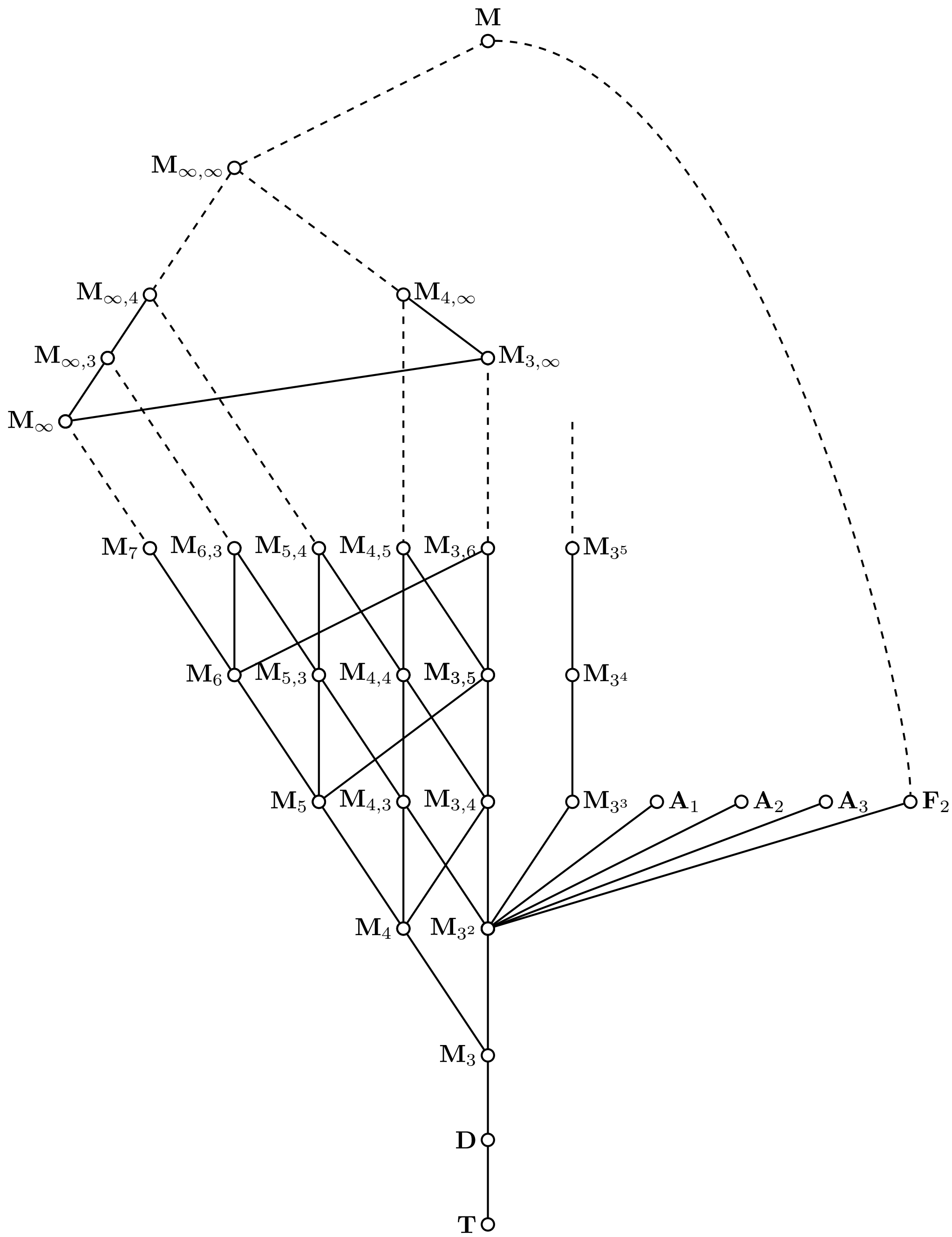


Figure 2: Some varieties of modular lattices ordered by inclusion

M, a *frame* of M is defined to be a maximal distributive sublattice. It is proved that for a planar modular lattice all maximal distributive sublattices are isomorphic, hence one can denote such a sublattice by Frame$M$. For example, the frame of $M_{n_1,\dots,n_k}$ is $C_{k+1} \times C_2$, whereas the frame of A_2 (or A_3) is $C_3 \times C_3$ (see Figure 1). The following distributive lattices that can occur as frames of subdirectly irreducible planar modular lattices.

Theorem 1-1.4 ([27])**.** *Let D be a planar distributive lattice with more than two elements. Then D is isomorphic to FrameM for some subdirectly irreducible planar modular lattice if and only if every element of $D - \{0,1\}$ is incomparable to some element in D, i.e. D is vertically indecomposable.*

To describe all subdirectly irreducible planar modular lattices, Grätzer and Quackenbush first observe that any M_n sublattices must occur as *covering sublattices*, which means that if $x \prec y$ holds in the sublattice, then it also holds in the original lattice. Given two comparable elements $a < b$ in a lattice, the *interval* b/a is defined to be the set $\{x \mid a \leq x \leq b\}$, and it is called a *prime interval* if it contains only a, b. Two intervals b/a, d/c are *perspective* if $a = b \wedge c$ and $b \vee c = d$. A *modular zig-zag connecting two prime intervals* b/a and d/c is a sequence $S_1, \dots, S_m$, $m \geq 1$, of covering M_3-s such that b/a is perspective to a prime interval in S_1, d/c is perspective to a prime interval in S_m, and any two adjacent members of this sequence form a sublattice isomorphic to M_{3^2} or $M_{3,2,3}$ (see Figure 1).

Theorem 1-1.5 (Structure theorem of subdirectly irreducible planar modular lattices [27])**.** *To construct every subdirectly irreducible planar modular lattice M, start with a vertically indecomposable planar distributive lattice D and insert doubly irreducible elements into covering squares of D so that in L any two prime intervals of D are connected by a modular zig-zag.*

The results in [27] are proved more generally for modular lattices of order-dimension 2, which is the analogue of planarity without the restriction to finite lattices.

For nonmodular varieties, B. Jónsson and I. Rival [45] proved that R. N. McKenzie's [52] list of 15 covers of $\mathbf{N}_5$ is complete (LTF Theorem 486). The lattices which generate these covers are shown in Figure 3.

The above result makes use of the semidistributive implications

$$\begin{array}{ll} (\mathrm{SD}_\vee) & x \vee y = x \vee z \implies x \vee (y \wedge z) = x \vee y \\ (\mathrm{SD}_\wedge) & x \wedge y = x \wedge z \implies x \wedge (y \vee z) = x \wedge y \end{array}$$

(see LTF Section VII). A variety of lattices is said to be *semidistributive*, if every member satisfies both laws. The *standard meet-sequence terms* $t_n(x,y,z)$ for variables x, y, z are defined by

$$\begin{aligned} t_0(x,y,z) &= y, \\ t_{n+1}(x,y,z) &= y \wedge (x \vee t_n(x,z,y)). \end{aligned}$$

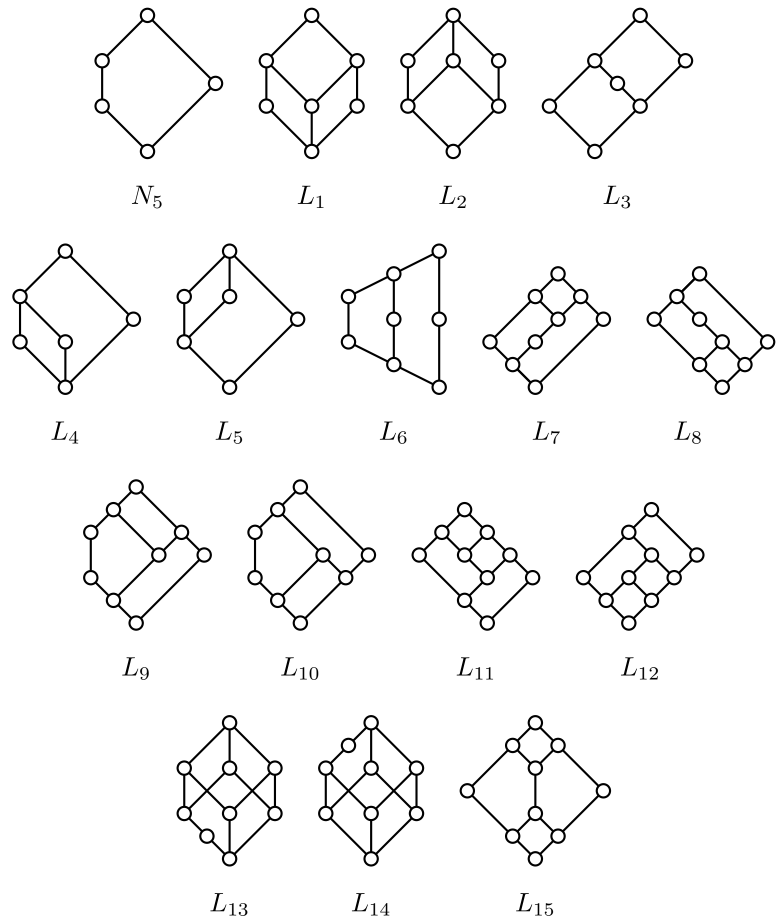


Figure 3: N_5 and the lattices that generate join-irreducible covers of $\mathbf{N}_5$

The key exclusion result by B. Jónsson and I. Rival [45] is the following.

Theorem 1-1.6. *For any variety* $\mathbf{V}$*, the following are equivalent.*

(i) $\mathbf{V}$ *is semidistributive.*

(ii) $M_3, L_1, L_2, L_3, L_4, L_5 \notin \mathbf{V}$.

(iii) *For some* n*, the equation*

$$(\mathrm{SD}^n_\vee) \qquad x \vee (y \wedge z) = x \vee t_n(x, y, z)$$

and its dual $(\mathrm{SD}^n_\wedge)$ *hold in* $\mathbf{V}$.

It follows from this result that semidistributivity is not an equational property.

The above equations define an increasing sequence of semidistributive varieties $\mathbf{SD}_n = \mathbf{Mod}((\mathrm{SD}^n_\vee), (\mathrm{SD}^n_\wedge))$. Since

$$(\mathrm{SD}^0_\vee) \qquad x \vee (y \wedge z) = x \vee y$$
$$(\mathrm{SD}^1_\vee) \qquad x \vee (y \wedge z) = x \vee (y \wedge (x \vee z))$$
$$(\mathrm{SD}^2_\vee) \qquad x \vee (y \wedge z) = x \vee (y \wedge (x \vee (z \wedge (x \vee y))))$$

$\mathbf{SD}_0 = \mathbf{T}$ and $\mathbf{SD}_1 = \mathbf{D}$. Lattices and subvarieties of $\mathbf{SD}_2$ are called *near distributive*. A useful characterization is given by the next exclusion result.

Theorem 1-1.7. *A lattice variety* $\mathbf{V}$ *is neardistributive if and only if it is semidistributive and* $L_{11}, L_{12} \notin \mathbf{V}$. *(J. G. Lee [49].)*

A lattice is said to be *almost distributive* if it is near distributive and satisfies the inequality

$$(\mathrm{AD}_\vee) \qquad u \wedge (w \vee (v \wedge ((x \vee y) \wedge (x \vee z)))) \leq v \vee (u \wedge w),$$

where $w = x \vee (y \wedge (x \vee z))$, and it's dual $(\mathrm{AD}_\wedge)$. The variety $\mathbf{AD}$ of all almost distributive lattices is studied by H. Rose [67] and J. G. Lee [49].

The main structural results about subdirectly irreducible almost distributive lattices require (a special case of) A. Day's doubling construction. The version described here is a generalization due to R. Freese, G. McNulty, and J. B. Nation [22] which will also be used later in the description of inherently nonfinitely based lattices. Given a lattice L, a convex subset C of L and a $\{0, 1\}$-lattice K, one defines a lattice $L \star_C K$, called the *inflation of* L *at* C *by* K, as follows. The underlying set is $(L - C) \cup (C \times K)$, and for elements x, y in this set, put $x \leq y$ if

(i) x, $y \in L - C$ and $x \leq y$ holds in L,

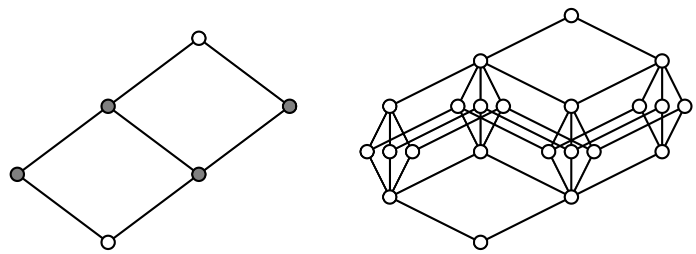


Figure 4: $C_2 \times C_3$ and $(C_2 \times C_3) \star_C M_3$

(ii) x, $y \in C \times K$ and $x \leq y$ holds in $C \times K$,

(iii) $x \in L - C$, $y = \langle c, k\rangle \in C \times K$, and $x \leq c$ holds in L, or

(iv) $x = \langle c, k\rangle \in C \times K$, $y \in L - C$, and $c \leq y$ holds in L.

Day's original doubling construction is obtained when $K = C_2$, in which case $L \star_C C_2$ is denoted by $L[C]$, and when $C = \{c\}$ this is further simplified to $L[c]$. For example, if we take $L = C_2 \times C_3$ and $C = L - \{0, 1\}$ then $L \star_C M_3$ is the lattice in Figure 4, and $(C_3 \times C_3)[d]$ gives the lattice L_{15} (Figure 3). The doubling construction for single elements was actually used in the context of transferable lattices before Day's construction.

For a variety $\mathbf{V}$, let $\Lambda_{\mathbf{V}}$ be the lattice of subvarieties of $\mathbf{V}$. If $\mathbf{V}$ is a lattice variety, then $\Lambda_{\mathbf{V}}$ is, of course, a principal ideal of Λ.

Theorem 1-1.8.

(i) *A subdirectly irreducible lattice L is almost distributive if and only if $L \cong D[d]$, for some distributive lattice D and $d \in D$.*

(ii) *A lattice variety $\mathbf{V}$ is almost distributive if and only if it is semidistributive and $L_6, \ldots, L_{12} \notin \mathbf{V}$.*

(iii) ***AD** is locally finite (that is, every finitely generated member is finite), hence the Finite Height Conjecture holds for almost distributive varieties and **AD** is generated by its finite members.*

(iv) *The cardinality of $\Lambda_{\mathbf{AD}}$ is $2^{\aleph_0}$.*

(v) *There exists an infinite descending chain in $\Lambda_{\mathbf{AD}}$.*

(vi) *There exists an almost distributive variety with infinitely many covers in $\Lambda_{\mathbf{AD}}$ and one with infinitely many dual covers.*

(H. Rose [67], J. G. Lee [49].)

Judging from the above results and additional details by Rose and Lee, one might say that the structure of $\Lambda_{\mathbf{AD}}$ is fairly well understood.

Problem 4. Is there a variety with uncountably many covers (or dual covers) in Λ or $\Lambda_{\mathbf{AD}}$?

Problem 5. Does **AD** have any dual covers?

We list below additional results about covers in Λ. In each case these results are established by long technical computations and the original papers contain further results that are of interest in their own right.

Theorem 1-1.9. *For $i = 6, 7, 8, 9, 10, 13, 14, 15$ and $n \geq 0$, the variety $\mathbf{L}_i^{n+1}$ is the only join-irreducible cover of $\mathbf{L}_i^n$ (where $\mathbf{L}_i^0 = \mathbf{L}_i$, see Figure 5). (H. Rose [67].)*

Theorem 1-1.10. *$\mathbf{L}_{12}$ has exactly two join-irreducible covers $\mathbf{L}_{12}^1$ and $\mathbf{G}^1$. For $n \geq 1$, $\mathbf{L}_{12}^{n+1}$ is the only join-irreducible cover of $\mathbf{L}_{12}^n$, and $\mathbf{G}^{n+1}$ is the only join-irreducible cover of $\mathbf{G}^n$. Above $\mathbf{L}_{11}$, the dual results hold (see Figure 5). (J. B. Nation [57].)*

Theorem 1-1.11. *The join-irreducible covers of $\mathbf{L}_1$ are $\mathbf{L}_{16}, \ldots, \mathbf{L}_{25}$. The covers of $\mathbf{L}_2$ are dual (see Figure 6). (J. B. Nation [58].)*

An approach to finding covers in Λ has been developed by J. B. Nation [59] (see also A. Day and J. B. Nation [12]).

C. Y. Wong [81] investigates weakened forms of distributivity similar to semidistributivity to find the covers of $\mathbf{L}_3$, $\mathbf{L}_4$ and $\mathbf{L}_5$. A lattice is said to be *weakly distributive* if it satisfies the following implications:

$$(\mathrm{WD}_\vee) \qquad x \wedge y = x \wedge z \quad \text{implies that} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$
$$(\mathrm{WD}_\wedge) \qquad x \vee y = x \vee z \quad \text{implies that} \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A variety of lattices is said to be *weakly distributive*, if this is true for every member. This property can also be characterized by an exclusion result.

Theorem 1-1.12. *For any variety $\mathbf{V}$, the following are equivalent.*

(i) *$\mathbf{V}$ is weakly distributive.*

(ii) *$M_3, L_1, L_2, L_4, L_5, L_{11}, L_{12}, L_{13}, L_{14}, T_1, T_2, T_3, T_4, P_4, P_5, P_{10} \notin \mathbf{V}$ (see Figures 3, 7 and 9).*

(iii) *For some n, the equation $x \wedge (t_n(x,y,z) \vee t_n(x,z,y)) \leq (x \wedge y) \vee (x \wedge z)$ and its dual hold in $\mathbf{V}$ ($t_n(x,y,z)$ are the standard meet sequence terms defined on page 6).*

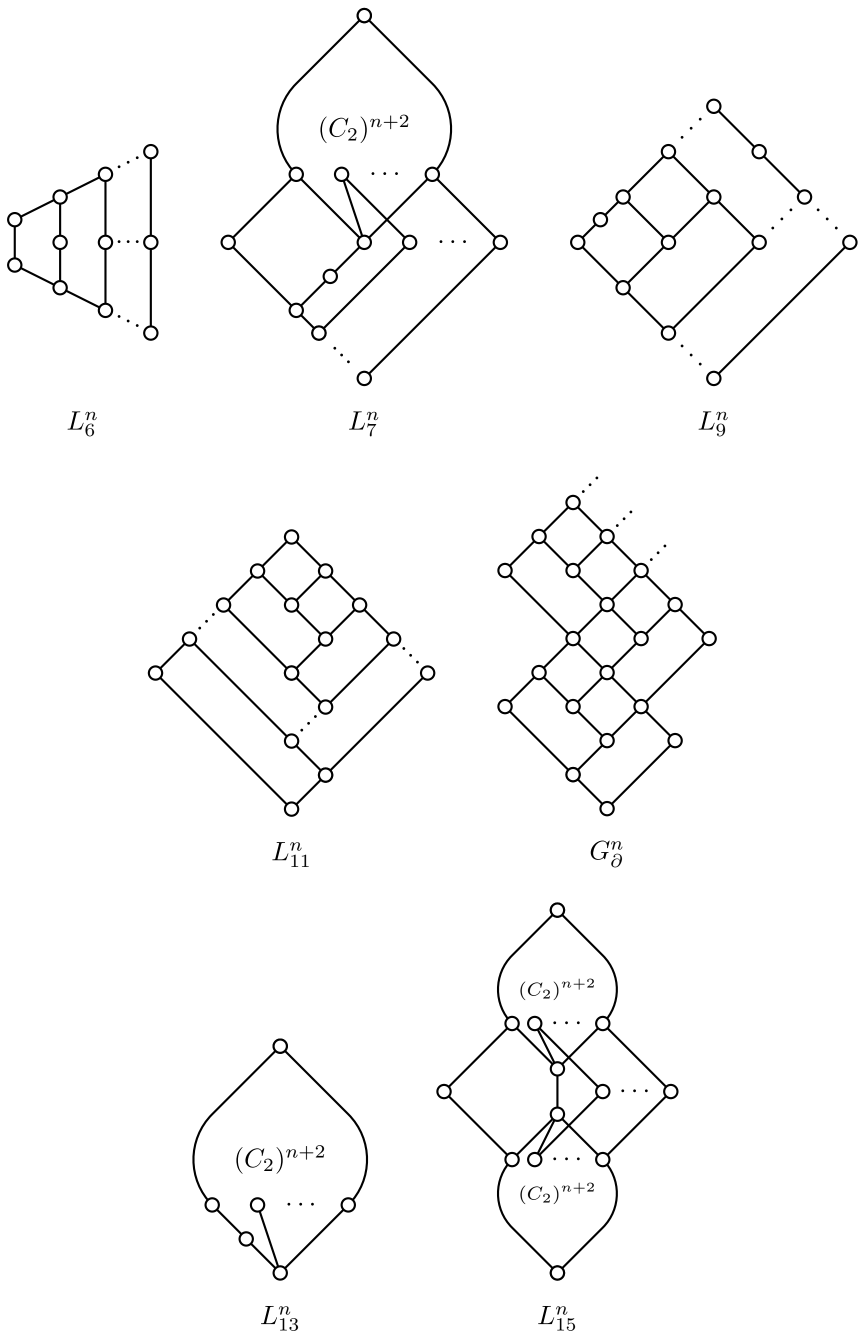


Figure 5: Sequences of lattices generating join-irreducible varieties. L_8^n, L_{10}^n, L_{12}^n, G^n, L_{14}^n are dual to L_7^n, L_9^n, L_{11}^n, G_∂^n, L_{13}^n respectively. (Here n is a superscript label, whereas $(C_2)^{n+2}$ is a power of C_2.)

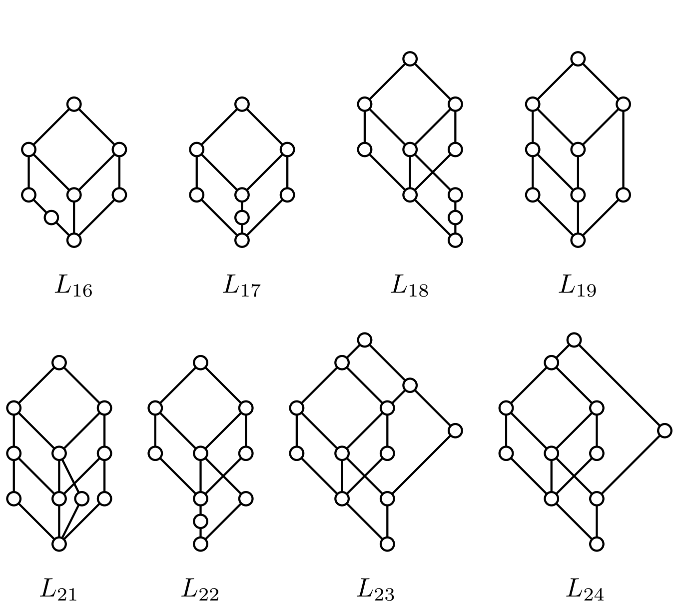


Figure 6: Lattices that generate covers of $\mathbf{L}_1$

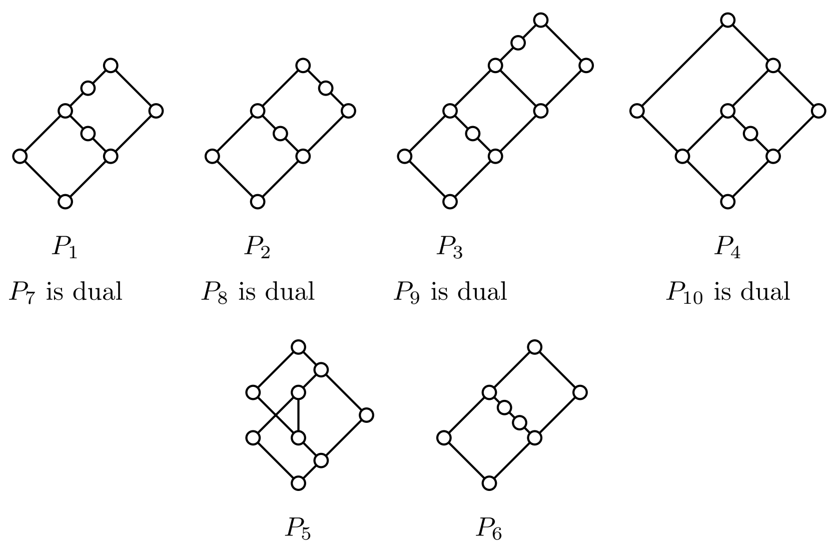
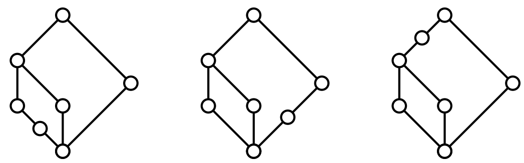
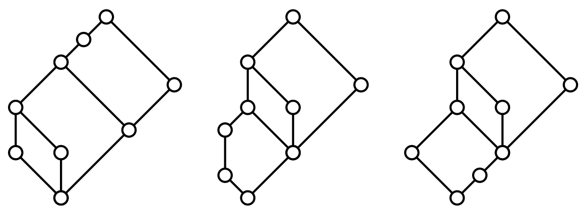


Figure 7: Lattices that generate covers of $\mathbf{L}_3$

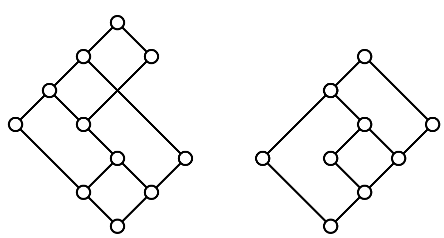


K_1 K_2 K_3



K_4 K_5 K_6

Figure 8: Lattices that generate covers of $\mathbf{L}_4$



T_1 — T_2 is dual

T_3 — T_4 is dual

Figure 9:

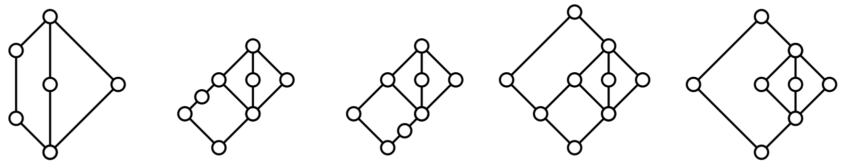


Figure 10: Lattices that generate finitely generated covers of $\mathbf{M}_3 \vee \mathbf{N}_5$

Note that $\mathbf{L}_4$ is not weakly distributive, but does satisfy ($\mathrm{WD}_\wedge$). Wong shows that ($\mathrm{WD}_\wedge$) cannot be characterized by an exclusion result, i.e., there is no finite list of finite subdirectly irreducible lattices such that a variety satisfies ($\mathrm{WD}_\wedge$) if and only if it contains none of these lattices. He then goes on to prove that ($\mathrm{WD}_\wedge$) is *weakly finitely definable with respect to* L_4 which means that there is a finite list of finite subdirectly irreducible lattices not in $\mathbf{L}_4$ such that if ($\mathrm{WD}_\wedge$) fails in a variety then it contains one of these lattices. Using this result together with the approach from J. B. Nation [59] and (lots of) additional details, he succeeds in proving the following result.

Theorem 1-1.13. *The join-irreducible covers of* $\mathbf{L}_3$ *are* $\mathbf{P}_1, \ldots, \mathbf{P}_{10}$ *(see Figure 7). The join-irreducible covers of* $\mathbf{L}_4$ *are* $\mathbf{K}_1, \ldots, \mathbf{K}_6$ *(see Figure 8). The covers of* $\mathbf{L}_5$ *are dual. (C. Y. Wong [81].)*

For the variety $\mathbf{M}_3 \vee \mathbf{N}_5$, only the finitely generated covers are known at this point.

Theorem 1-1.14. *The finitely generated join-irreducible covers of* $\mathbf{M}_3 \vee \mathbf{N}_5$ *are* $\mathbf{V}_1, \ldots, \mathbf{V}_8$ *(see Figure 10). (W. Ruckelshausen [68].)*

Problem 6. Does $\mathbf{M}_3 \vee \mathbf{N}_5$ have any nonfinitely generated covers?

All the preceding results support the Finite Height Conjecture in that every finitely generated lattice variety of height at most 4 has only finitely many covers, each generated by a finite lattice (see Figure 11). However, J. B. Nation [60] showed that the conjecture fails for lattices in general (see LTF VI.2.2). In the same paper it is also shown that there is a variety of finite height that has countably infinite many covers.

Calculations that enumerate small lattices have led to the results summarized here. The lattices of size n up to isomorphism were computed up to $n = 18$ by J. Heitzig and J. Reinhold [28] and extended to $n = 19$ by P. Jipsen

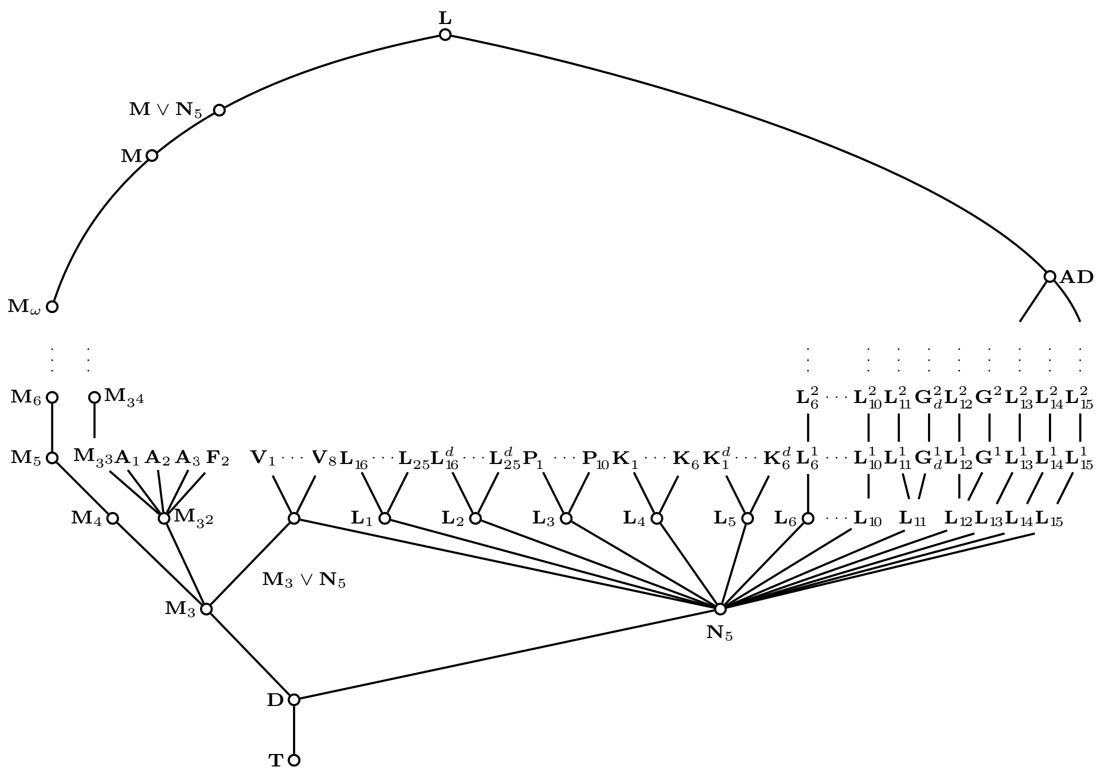


Figure 11: Known join-irreducible covers near the bottom of Λ

and N. Lawless [37]. The subdirectly irreducible lattices of size up to $n = 12$ were filtered out which produced the following result.

$n =$	5	6	7	8	9	10	11	12
Number of s.i. lattices	2	4	16	69	360	2103	13867	100853

Including the 1-element and 2-element lattice, there are 2556 subdirectly irreducible lattices up to size 10, and the collection of these lattices is denoted $\mathcal{L}_{10}$. For $L, M \in \mathcal{L}_{10}$, define $L \leq M$ if $L \in \mathbf{HS}\{M\}$. Then Jónsson's Lemma implies that $\mathbf{Var}(L) \subseteq \mathbf{Var}(M)$ if and only if $L \leq M$, hence $\mathcal{L}_{10}$ ordered by $\leq$ is isomorphic to the poset of completely join-irreducible varieties in Λ that are generated by a subdirectly irreducible lattices of size ≤ 10. This poset has been computed, but due to its size, it is not useful to draw a Hasse diagram. Since all varieties of height 3 in Λ are generated by a lattice of size ≤ 10, the poset shows the first few levels of Λ as in Figure 11, except that $\mathbf{V}_1, \ldots, \mathbf{V}_8$ are in the position of $\mathbf{M}_3 \vee \mathbf{N}_5$. So there are 2+8+15=25 elements of height 3 in this poset. The height of $\mathcal{L}_{10}$ is 7, with the following number of elements of each height:

Height in $\mathcal{L}_{10} =$	0	1	2	3	4	5	6	7
Number of s.i. lattices	1	1	2	25	143	575	1060	749

Problem 7. Is every variety of finite height finitely based?

Problem 8. Is every variety of finite height generated by a lattice of finite width?

Problem 9. Is there an algorithm to find the covers of a finitely generated variety?

1-2. Generating sets of varieties

It is well known that the variety of all lattices is generated by its finite members (R. A. Dean [13]). Using the doubling construction and R. N. McKenzie's [52] characterization of splitting lattices as finite subdirectly irreducible bounded lattices, A. Day [9] was able to prove the following sharper version of Dean's result.

Theorem 1-2.1. *The variety* $\mathbf{L}$ *of all lattices is generated by the class of all splitting lattices.*

The significance of this result is enhanced by the fact that it implies every finitely generated free lattice is weakly atomic (R. N. McKenzie [52] and A. Kostinsky [48] proved this condition equivalent to Day's theorem).

More recently, R. N. McKenzie [53] showed that $\mathbf{L}$ is also generated by the collection of all finite minimal simple lattices. (A simple lattice L is *minimal* if $L \ncong C_2$ and no simple lattice other than C_2 generates a proper subvariety of $\mathbf{Var}(L)$.)

For the variety of modular lattices, the situation is quite different.

Theorem 1-2.2.

(i) *The variety* $\mathbf{M}$ *of all modular lattices is not generated by its finite members. (R. Freese [16].)*

(ii) *Neither* $\mathbf{M}$ *nor the variety* $\mathbf{A}$ *of all arguesian lattices is generated by its members of finite length. (C. Herrmann [29].)*

Using P. Pudlák and J. Tůma's [66] result that every finite lattice can be embedded into a finite partition lattice, P. Bruyns and H. Rose [6] show that every lattice is embeddable into an ultraproduct of finite partition lattices, hence $\mathbf{L} = \mathbf{SP}_U(\{\mathrm{Part}\, n \mid n \in \omega\})$. Furthermore, since any lattice variety $\mathbf{V}$ satisfies the Embedding Property (see Section VI.4), there exists a lattice $L \in \mathbf{V}$ such that every member of $\mathbf{V}$ is embeddable into an ultrapower of $\mathbf{L}$, that is, $\mathbf{V} = \mathbf{SP}_U(L)$. Such lattices L are referred to as *ultra-universal* (see also C. Naturman and H. Rose [61]).

R. N. McKenzie [52] showed that splitting lattices in Λ are finite. However, splitting lattices can be defined in any lattice of varieties.

Problem 10. Is every splitting lattice in $\Lambda_{\mathbf{M}}$ finite?

Problem 11. Is $\mathbf{M}$ generated by all the splitting lattices in $\Lambda_{\mathbf{M}}$?

If L is a splitting lattice in Λ, then the largest variety that does not contain L is called the *conjugate variety* of L.

Problem 12. Is there a nontrivial conjugate variety in Λ that is generated by its finite members?

Problem 13. Is there a conjugate variety $\mathbf{V}$ with infinite subdirectly irreducible members that are projective in $\mathbf{V}$?

Note that if a variety $\mathbf{V}$ is generated by its finite members then every subdirectly irreducible projective member is finite. Thus a positive answer to the previous problem implies that $\mathbf{V}$ is not generated by its finite members.

1-3. Decidability of equational theories

A variety $\mathbf{V}$ is said to have a *decidable equational theory* if there is an algorithm that takes any lattice equation as input and outputs *true* if the equation holds in all members of $\mathbf{V}$, and otherwise it outputs false. This is equivalent to the solvability of the word problem for free lattices in the variety.

The variety **L** of all lattices has a decidable equational theory since Whitman [78] showed in 1941 that the word problem for free lattices is solvable (LTF Theorem 540). In fact, this result was proved already in 1920 by Skolem [74], as pointed out by S. Burris in [7]. Skolem's solution actually provides an algorithm that has polynomial complexity, whereas Whitman's solution, when implemented in a naive way, has exponential complexity.

The variety of all lattices satisfies the stronger *finite embeddability property* of T. Evans [14], whence it follows that the universal theory of lattices is decidable.

Recall that a variety is finitely generated if it has finitely many fundamental operations and is generated by a finite set of finite algebras. Such a variety has a decidable equational theory, since one can simply check whether a given equation holds in all of the finitely many finite generating algebras. Hence the variety **D** of distributive lattices has a decidable equational theory, and the same holds for the varieties $\mathbf{T}, \mathbf{M}_n, \mathbf{M}_{3^n}, \mathbf{M}_{n_1,\dots,n_k}, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{F}_2, \mathbf{L}_1, \dots, \mathbf{L}_{25}, \dots$. In fact, P. Bloniarz, H. B. Hunt and D. Rosenkrantz [5] have shown that the equational theory of any nontrivial finitely generated lattice variety is co-NP-complete.

If a variety is defined by finitely many equations and generated by its finite members then again one can decide any equation by an argument due to J. C. C. McKinsey [54]: either the equation can be derived from the finite equational basis or a finite counterexample can be found by examining the denumerable list of all finite members. More generally, it suffices that a variety is defined by a recursively enumerable set of identities and is generated by a recursively enumerable collection of finite algebras, to conclude that such a variety has a decidable equational theory. This result is useful for concluding, for example, that $\mathbf{M}_\infty$ has a decidable equational theory. It is also used by L. Santocanale and F. Wehrung [70] to show that the variety generated by all Tamari lattices and the variety generated by all permutohedra lattices have decidable equational theories (see Chapter **??** for definitions and some discussion).

As mentioned in LTF Theorem 542, R. Freese [17] proved that the word problem for the free modular lattice on 5 generators is unsolvable, hence the equational theory of modular lattices is undecidable. C. Herrmann [29] settled the word problem for the 4-generated free modular lattice by showing it is also unsolvable.

A lattice is *n-distributive* if it satisfies the identity

$$x \wedge \bigvee_{i=0}^{n} y_i = \bigvee_{i=0}^{n} \left(x \wedge \bigvee_{j=0, j\neq i}^{n} y_i \right).$$

Thus 1-distributivity is simply distributivity $x \wedge (y_0 \vee y_1) = (x \wedge y_0) \vee (x \wedge y_1)$, and 2-distributivity is the identity

$$x \wedge (y_0 \vee y_1 \vee y_2) = (x \wedge (y_1 \vee y_2)) \vee (x \wedge (y_0 \vee y_2)) \vee (x \wedge (y_0 \vee y_1)).$$

The variety $\mathbf{D}_n$ of all n-distributive lattices is distinct from the variety $\mathbf{D}_n^\partial$ of dually n-distributive lattices for $n > 1$. For example, a lattice of least cardinality in $\mathbf{D}_2$ that is not in $\mathbf{D}_2^\partial$ is given by L_{13} in Figure 3.

The varieties $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \ldots$ form an increasing chain of distinct varieties, as is shown by lattices of subspaces of n-dimensional vector spaces. Every lattice with n elements is $(n+1)$-distributive, and since the variety of all lattices is generated by its finite members, it follows that the union of this chain of varieties is the variety of all lattices.

Many fundamental results about n-distributive lattices were found by A. Huhn [35, 36], initially mostly under the additional assumption of modularity. An open problems in the latter paper asks if the variety $\mathbf{D}_n$ is generated by its finite members.

An element in a poset is *completely join-irreducible* if the set of elements strictly below it has a largest member, and a lattice is *spatial* if every element is a join of completely join-irreducible elements. Recall that an element in a lattice is *compact* if, whenever it is below the join of a set of elements, then it is less or equal to the join of a finite subset of these elements. An algebraic lattice is a complete lattice in which every element is the join of compact elements.

L. Santocanale and F. Wehrung [69] prove that, for fixed n, every n-distributive lattice can be embedded, within its variety, in an algebraic spatial lattice, which is thus also n-distributive. To show that n-distributivity is required in this result, they give an interesting counterexample of a join-semidistributive lattice that cannot be embedded, within its variety, into an algebraic spatial lattice. Using this geometric description of an n-distributive lattice, they are then able to prove the following result, thus answering Huhn's [36] question positively.

Theorem 1-3.1 (L. Santocanale and F. Wehrung [69]). *For each $n \in \omega$, the variety $\mathbf{D}_n$ of all n-distributive lattices is generated by its finite members, hence it has a decidable equational theory.*

The following property is used repeatedly in the proof: the nontrivial direction of n-distributivity implies that if a join-irreducible x is less or equal to $y_0 \vee \cdots \vee y_n$, then $x \leq y_0 \vee \cdots \vee y_{i-1} \vee y_{i+1} \vee \cdots \vee y_n$ for some i, and for spatial lattices the converse also holds.

Two decades earlier, C. Herrmann, D. Pickering, and M. Roddy [31] proved that every modular lattice can be embedded, within its variety, into an algebraic spatial lattice. However, as noted above, it does not follow that the variety of modular lattices is generated by its finite members.

For a poset P one can consider the lattice $\mathrm{Co}(P)$ of order-convex subsets of P. These lattices have been studied by M. V. Semenova and F. Wehrung [72, 71, 73], and it is proved that they generate a finitely based variety determined by the following identities (S), (U), (B) (named after the fictional characters Stirlitz, Udav and Bond)

$$\text{(S)}\qquad x \wedge (y' \vee z) = (x \wedge y') \vee \bigvee_{i=0}^{1} (x \wedge (y_i \vee z) \wedge ((y' \wedge (x \vee y_i)) \vee z))$$

where $y' = y \wedge (y_0 \vee y_1)$,

$$\begin{aligned} &x \wedge (y_0 \vee y_1) \wedge (y_1 \vee y_2) \wedge (y_0 \vee y_2) \\ \text{(U)}\qquad &= (x \wedge y_0 \wedge (y_1 \vee y_2)) \vee (x \wedge y_1 \wedge (y_0 \vee y_2)) \vee (x \wedge y_2 \wedge (y_0 \vee y_1)) \end{aligned}$$

$$\begin{aligned} &x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1) = \bigvee_{i=0}^{1} ((x \wedge y_i \wedge (z_0 \vee z_1)) \vee (x \wedge z_i \wedge (y_0 \vee y_1))) \\ \text{(B)}\qquad &\vee \bigvee_{i=0}^{1} (x \wedge (y_0 \vee y_1) \wedge (z_0 \vee z_1) \wedge (y_0 \vee z_i) \wedge (y_1 \vee z_{1-i})). \end{aligned}$$

The variety of lattices that satisfy these three identities is denoted by **SUB**.

Theorem 1-3.2 (M. V. Semenova and F. Wehrung [72])**.**

- *A lattice L is a member of* **SUB** *if and only if it is embeddable into* $\mathrm{Co}(P)$ *for some poset P.*
- *The embedding can be chosen to preserve the bounds of L (if any).*
- *If L is finite, P can be finite with size $\leq 2n^2 - 5n + 4$, where n is the number of join irreducibles of L.*
- *The variety* **SUB** *is generated by its finite members, hence it has a decidable equational theory.*

In a subsequent paper, Semenova and Wehrung [71] consider lattices $\mathrm{Co}(P)$ for posets P of height n. These lattices are also shown to generate a finitely based variety, denoted $\mathbf{SUB}_n$, and it is proved that $\mathbf{SUB}_2$ is locally finite while $\mathbf{SUB}_3$ is not. A third paper [73] in this series covers the case when the posets P are chains. In this situation the variety $\mathbf{SUB}_{LO}$ is generated as a quasivariety by the lattices $\mathrm{Co}(P)$, is locally finite, and has only finitely many subquasivarieties.

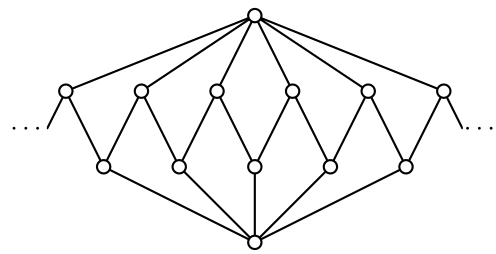


Figure 1: The lattice L_f

1-4. Equational Bases

Recall that an algebra is said to be *finitely based* if the variety which it generates is determined by finitely many equations. Nonfinitely based lattices were constructed by K. A. Baker [1], [2], R. Freese [15], C. Herrmann [29], R. N. McKenzie [51] and R. Wille [80]. One such lattice, due to McKenzie, is shown in Figure 1.

An algebra A is said to be *inherently nonfinitely based* if $\mathbf{Var}(A)$ is locally finite, and any locally finite variety to which A belongs is not finitely based. This concept was introduced independently by V. L. Murskiĭ [55] and P. Perkins [64]. Inspired by J. B. Nation's [60] counterexample to the Finite Height Conjecture, R. Freese, G. McNulty, and J. B. Nation [22] construct inherently nonfinitely based lattices. Here we only state a special case of their main result (see page 8 for the definition of $L \star_C K$).

Theorem 1-4.1. *Let L_f be the lattice in Figure 1 and define $C = L_f - \{0, 1\}$. Let K be a $\{0, 1\}$-lattice which belongs to a locally finite variety, and assume that K has an automorphism with an infinite orbit. Then $L_f \star_C K$ is inherently nonfinitely based.*

The two least complicated lattices K with the required automorphism are M_ω and B (a chain isomorphic to the integers with top and bottom elements added). The resulting lattices $L_f \star_C K$ are given in Figure 1. In the same paper, it is also shown that the lattice L_f is not inherently nonfinitely based.

Problem 14. Are there any modular lattices that are inherently nonfinitely based?

Analogously to the varieties $\mathbf{M}_w^l$ one defines $\mathbf{V}_w^l$ to be the variety generated by all lattices of height l and width w. We allow $l = \infty$ or $w = \infty$ in which case the respective parameter is not restricted. For $l, w < \infty$, the varieties $\mathbf{M}_w^l$ and $\mathbf{V}_w^l$ are finitely generated and hence finitely based.

Note that if a variety $\mathbf{V}$ is strongly covered by a finite set of varieties, then it is finitely based. Results about whether the varieties $\mathbf{M}_w^l$ and $\mathbf{V}_w^l$ are finitely based for $l = \infty$ or $w = \infty$, are as follows:

Theorem 1-4.2.

(i) $\mathbf{M}_\infty^2$ $(= \mathbf{V}_\infty^2)$ *and* $\mathbf{M}_\infty^3$ *are finitely based (see, Theorems 1-1.2 and 1-1.3),* $\mathbf{M}_\infty^n$ *is finitely based for all* n *(K. Baker, see C. Herrmann [1973]).*

(ii) $\mathbf{V}_\infty^3$ *is finitely based (C. Herrmann [1973]),* $\mathbf{V}_\infty^n$ *is not finitely based for* $n \geq 4$ *(K. Baker [2]).*

(iii) $\mathbf{M}_1^\infty = \mathbf{M}_2^\infty = \mathbf{D}$, $\mathbf{M}_3^\infty = \mathbf{M}_3$ *(see Theorem 1-1.2, since both its modular covers are generated by lattices of width 4),* $\mathbf{M}_4^\infty$ *is strongly covered by 10 varieties (each generated by* $\mathbf{M}_4^\infty$ *together with one of the lattices in LTF Figure 114,* F_2 *in Figure 1, or* N_5*) and hence finitely based (R. Freese [1977]),* $\mathbf{M}_n^\infty$ *is not finitely based for* $n \geq 5$ *(K. A. Baker [2]).*

(iv) $\mathbf{V}_2^\infty = \mathbf{N}_5$ *(O. T. Nelson [1968]), hence finitely based,* $\mathbf{V}_n^\infty$ *is not finitely based for* $n \geq 3$ *(*$n \geq 4$ *due to K. A. Baker [2],* $n = 3$ *due to Y.-C. Hsueh [34]).*

B. Jónsson [1974] showed that the join of two finitely based lattice varieties need not be finitely based, and K. A. Baker [2] did the same for two finitely based modular varieties. In view of these result, it is natural to look for sufficient conditions under which the join of two finitely based varieties remains finitely based.

Theorem 1-4.3. *Suppose that* $\mathbf{V}$ *and* $\mathbf{W}$ *are finitely based lattice varieties. If one of the following conditions holds, then* $\mathbf{V} \vee \mathbf{W}$ *is finitely based.*

(i) $\mathbf{V}$ *is modular and* $\mathbf{W}$ *is generated by a finite lattice that excludes* M_3*.*

(ii) $\mathbf{V}$ *and* $\mathbf{W}$ *are locally finite and the projective radius of* $\mathbf{V} \cap \mathbf{W}$ *is finite.*

(iii) $\mathbf{V}$ *and* $\mathbf{W}$ *are modular and* $\mathbf{W}$ *is generated by a lattice of finite length.*

(iv) $\mathbf{V}$ *is modular and* $\mathbf{W}$ *is generated by a finite lattice with finite projective radius.*

(v) $\mathbf{V} \cap \mathbf{W} = \mathbf{D}$*, the variety of all distributive lattices.*

(i) and (ii) are due to J. G. Lee [50], (iii) is due to Jónsson and the remaining statements are due to Y. Y. Kang [46].

Note that it follows from part (i) above that $\mathbf{M} \vee \mathbf{N}_5$ is finitely based. B. Jónsson [42] constructed an explicit basis of eight identities for this variety. The following problem was inspired by this result.

Problem 15. Is the unique cover of a conjugate variety in Λ always finitely based? (A. Day.)

Problem 16. Recall that **VPM** is the variety generated by all planar modular lattices. Is this variety finitely based?

1-5. Amalgamation and absolute retracts

G. Grätzer, B. Jónsson, and H. Lakser [25] showed that, besides the varieties **T** and **D**, no modular variety has the amalgamation property (see Section VI.4 for a discussion). A. Day and J. Ježek [11] finally extended this result to all lattice varieties.

Theorem 1-5.1. **T**, **D**, *and* **L** *are the only lattice varieties with the amalgamation property.*

For other varieties of algebras the amalgamation property also turned out to be rarely satisfied. A comprehensive survey about amalgamation for various types of algebras can be found in E. W. Kiss, L. Márki, P. Pröhle, and W. Tholen [47]. These results indicate that the concept of amalgamation does not mesh well with that of a variety. However the amalgamation class **Amal**(**V**) of a variety **V**, introduced by G. Grätzer and H. Lakser [26], has proved to be very fruitful. M. Yasuhara [82] showed that for any variety **V** of algebras, each member of **V** has an extension in **Amal**(**V**), hence **Amal**(**V**) is a proper class (Theorem VI.4.10). At present the main directions of study are to characterize the amalgamation class of a given variety and to decide whether it is (strictly) elementary, i.e., if it can be defined by a (finite) collection of first order sentences. Although we do not know anything about a single member of **Amal**(**M**), significant progress has been made with residually small lattice varieties. This started with a characterization of the amalgamation class of finitely generated lattice varieties by B. Jónsson [44], and was generalized by P. Jipsen and H. Rose [38] (see also P. Ouwehand and H. Rose [62]).

Many of the results below are valid for various congruence distributive varieties (not only lattice varieties), so we will state the more general results where applicable. A *retraction* of an embedding $f\colon A \to B$ is a homomorphism $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$. An algebra A in a class **K** is said to be an *absolute retract in* **K** if for every embedding $f\colon A \hookrightarrow B \in \mathbf{K}$, there is a retraction. The class of all absolute retracts of **K** is denoted by **Ar**(**K**). The concept of absolute retract is of interest here since C. Bergman [3] observed that for any variety **V** we have $\mathbf{Ar}(\mathbf{V}) \subseteq \mathbf{Amal}(\mathbf{V})$.

A variety is said to be *residually small*, if there is an upper bound on the cardinality of its subdirectly irreducible members. W. Taylor [75] proved that a variety **V** is residually small if and only if $\mathbf{V} = \mathbf{SAr}(\mathbf{V})$.

Theorem 1-5.2. *Let* $\mathbf{V}$ *be a residually small congruence distributive variety in which every member has a one-element subalgebra. Then* $A \in \mathbf{Amal}(\mathbf{V})$ *if and only if for any embedding* $f\colon A \hookrightarrow B \in \mathbf{V}$ *and any homomorphism* $h\colon A \to M \in \mathbf{Si}(\mathbf{Ar}(\mathbf{V}))$ *there exists a homomorphism* $g\colon B \to M$ *such that* $h = fg$.

The reverse implication is due to C. Bergman [3] and the forward direction is from P. Jipsen and H. Rose [38]. A useful corollary is that for finite algebras the condition in the preceding theorem can be checked.

Corollary 1-5.3. *Let* $\mathbf{V}$ *be a finitely generated congruence distributive variety in which every member has a one-element subalgebra. For finite algebras in* $\mathbf{V}$*, membership in* $\mathbf{Amal}(\mathbf{V})$ *is decidable. (B. Jónsson [44], P. Jipsen and H. Rose [38].)*

Since the amalgamation class of a variety is in general a proper subclass, it is interesting to ask whether it is an elementary class. Even for a finitely generated lattice variety this is a nontrivial problem.

Theorem 1-5.4. *The amalgamation class of any finitely generated nondistributive modular lattice variety is not elementary. (C. Bergman [4].)*

Problem 17. For which finitely generated varieties is the amalgamation class elementary?

Recent progress on this problem has been made by P. Ouwehand and H. Rose [63].

Theorem 1-5.5. *Let* $\mathbf{V}$ *be a finitely generated variety of lattices. Suppose that there is a lattice* $L \in \mathbf{Amal}(\mathbf{V})$ *with either a bottom or a top element, which does not have* C_2 *as homomorphic image, but some ultrapower* $L^I/\mathcal{U}$ *does have* C_2 *as homomorphic image. Then* $L^I/\mathcal{U} \notin \mathbf{Amal}(\mathbf{V})$*, and hence neither* $\mathbf{Amal}(\mathbf{V})$ *nor its complement are elementary.*

The lattice L is usually constructed by glueing countably many copies of a maximal subdirectly irreducible member on top of eachother (identifying the top of one member with the bottom of the next). Applications of this result include a simple proof of Theorem 1-5.4 as well as the result that any lattice variety generated by a finite simple lattice has a nonelementary amalgamation class. Further generalizations to nonfinitely generated varieties imply, for example, that $\mathbf{M}_\omega$ does not have an elementary amalgamation class.

Problem 18. If $\mathbf{Amal}(\mathbf{V})$ is an elementary class, does it follow that it is a Horn class?

P. Ouwehand and H. Rose [62] show that if an elementary class $\mathbf{K}$ is closed under updirected unions, then it is closed under finite direct products if and only if it is closed under reduced products (and hence definable by Horn sentences). This result applies to elementary amalgamation classes since M. Yasuhara [82] showed that they are closed under updirected unions. Hence the above problem is equivalent to asking if every elementary amalgamation class is closed under finite products.

Problem 19. Is there a nonfinitely generated variety other than $\mathbf{L}$ whose amalgamation class is elementary? In particular, is $\mathbf{Amal}(\mathbf{M})$ an elementary class?

Absolute retracts We now consider the problem of how the class of all absolute retracts of a variety can be constructed from its subdirectly irreducible members. Even for congruence distributive varieties, the product of two absolute retracts need not be an absolute retract (W. Taylor [76]), but fortunately lattices are well behaved.

Theorem 1-5.6. *Let* $\mathbf{V}$ *be a congruence distributive variety in which every member has a one-element subalgebra. Then the class of absolute retracts of* $\mathbf{V}$ *is closed under direct products and direct factors, that is,* $\prod_{i\in I} A_i \in \mathbf{Ar}(\mathbf{V})$ *iff* $\{ A_i \mid i \in I \} \subseteq \mathbf{Ar}(\mathbf{V})$. *(P. Jipsen and H. Rose [38], P. Ouwehand and H. Rose [62].)*

In fact, P. Ouwehand and H. Rose [62] show that for congruence distributive varieties, all finite absolute retracts can be obtained as products of subdirectly irreducible absolute retracts. The general case is more complicated and requires the concept of equational compactness (see also Section 1.9 of Appendix A). Here we only need the algebraic formulation: an algebra A is *equationally compact* if for every diagonal embedding of A into an ultrapower of A, there is a retraction. Clearly every finite algebra and every absolute retract with respect to some variety is equationally compact. Ouwehand and Rose also observe that equationally compact lattices are complete (a result implicit in B. Weglorz [77]). Hence absolute retracts in a lattice variety are complete lattices.

Consider the following characterization:

(∗) An algebra A is in $\mathbf{Ar}(\mathbf{V})$ if and only if A is a product of equationally compact reduced powers of $\mathbf{Si}(\mathbf{Ar}(\mathbf{V}))$.

Theorem 1-5.7. *Let* $\mathbf{V}$ *be a finitely generated variety of lattices.*

(i) *Every equationally compact reduced power of a finite absolute retract in* $\mathbf{V}$ *is an absolute retract in* $\mathbf{V}$ *(hence the reverse implication of* (∗) *holds).*

(ii) *If none of the subdirectly irreducible absolute retracts in* $\mathbf{V}$ *are homomorphic images of each other then* $\mathbf{V}$ *satisfies* (∗).

(iii) *Assume every proper subvariety satisfies* $(*)$. *If* $\mathbf{V}$ *is the join of its proper subvarieties or contains only one subdirectly irreducible absolute retract, then* $\mathbf{V}$ *satisfies* $(*)$.

(P. Ouwehand and H. Rose [62].)

Note that the previous theorem is a generalization of the well known result that the absolute retracts in $\mathbf{D}$ are precisely the complete Boolean lattices (since every complete Boolean lattice is a reduced power of C_2, which is the only subdirectly irreducible in $\mathbf{D}$). All finite lattices in $\mathbf{Si}(\mathbf{M})$ are simple, hence (ii) implies that every finitely generated modular variety satisfies $(*)$. It follows from Theorem 1-1.8(i) that any homomorphic image of a lattice in $\mathbf{Si}(\mathbf{AD})$ is distributive, whence $(*)$ also holds for all finitely generated almost distributive varieties.

1-6. Congruence varieties

A congruence variety is a variety of lattices which is generated by the congruence lattices of some variety of algebras. An account of this area of research can be found in B. Jónsson's appendix to G. Grätzer [24] (see also B. Jónsson [43]). In this section, we mention some more recent results and some additional results not included there.

1-6.1 The nonmodular case: Polin's variety

Contrary to the belief of many researchers, S. V. Polin [65] constructed a variety of algebras whose congruence variety is a proper nonmodular subvariety of $\mathbf{L}$. In the reconstruction of Polin's proof (from sketchy notes) A. Day showed that there are infinitely many distinct nonmodular congruence varieties, each of which contains no nondistributive modular lattices. Since the join of congruence varieties is again a congruence variety, there are infinitely many nonmodular congruence varieties. Moreover, we have the following results.

Theorem 1-6.1.

(i) *Any nonmodular congruence variety contains the variety of all almost distributive lattices. (A. Day 1977].)*

(ii) *Polin's congruence variety is the unique minimal nonmodular congruence variety. (A. Day and R. Freese [10].)*

For further information about Polin's variety see R. Freese [19].

Theorem 1-6.2. *Each minimal modular nondistributive congruence variety is determined by one of the varieties generated by all vector spaces of characteristic p (a prime or 0). (R. Freese, C. Herrmann and A. P. Huhn [20].)*

Since **D** is meet-irreducible in the lattice of modular varieties, it follows from this result that the meet of two congruence varieties does not have to be a congruence variety.

Corollary 1-6.3. *The set of all congruence varieties is not a sublattice of* Λ.

Problem 20. Is there a unique largest modular congruence variety?

We now turn to the question of congruence identities. Among the most significant results are the following.

Theorem 1-6.4.

(i) *There is a lattice equation strictly weaker than the modular law such that any congruence variety which satisfies this law is a modular variety. (J. B. Nation [56].)*

(ii) *Every modular congruence variety is arguesian. (R. Freese and B. Jónsson [21].)*

(iii) *No modular nondistributive congruence variety is finitely based. (R. Freese [18])*

(iv) *For each* $n \geq 0$*, the congruence lattice* $\operatorname{Con} F_n$ *of the free* n*-generated Polin algebra is a splitting lattice. Thus (by Theorem 1-6.1(ii)) a variety is congruence modular if and only if it satisfies the conjugate equation of one of these splitting lattices. (A. Day and R. Freese [10].)*

(v) *It is decidable whether a lattice equation implies congruence modularity (or distributivity). (G. Czédli and R. Freese [8].)*

Problem 21. Is there a nondistributive congruence variety which is finitely based?

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