Topological Duality and Lattice Expansions Part I: A Topological Construction of Canonical Extensions

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1. Introduction

The two main objectives of this paper are (a) to prove topological duality theorems for semilattices and bounded lattices, and (b) to show that the topological duality from (a) provides a construction of canonical extensions of bounded lattices. The paper is the first of two parts. The main objective of the sequel [11] is to establish a characterization of lattice expansions, i.e., lattices with additional operations, in the topological setting built in this paper.

Regarding objective (a), consider the following simple question:

Is there a subcategory of **Top** that is dually equivalent to **Lat**?

Here, **Top** is the category of topological spaces and continuous maps and **Lat** is the category of bounded lattices and lattice homomorphisms.

To date, the question has been answered positively either by specializing **Lat** or by generalizing **Top**. The earliest examples are of the former sort.

Tarski [16] (treated in English, e.g., in [1]) showed that every complete atomic Boolean lattice is represented by a powerset. Taking some historical license, we can say this result shows that the category of complete atomic Boolean lattices with complete lattice homomorphisms is dually equivalent to the category of discrete topological spaces. Birkhoff [2] showed that every finite distributive lattice is represented by the lower sets of a finite partial order. Again, we can now say that this shows that the category of finite distributive lattices is dually equivalent to the category of finite T_0 spaces and continuous maps. In the seminal papers, [14, 15], Stone generalized Tarski and then Birkhoff, showing that (a) the category of Boolean lattices and lattice homomorphisms is dually equivalent to the category of zero-dimensional, regular spaces and continuous maps and then (b) the category of distributive lattices and lattice homomorphisms is dually equivalent to the category of *spectral spaces* and *spectral maps*. We will describe spectral spaces and spectral maps below. For now, notice that all of these results can be viewed as specializing **Lat** and obtaining a subcategory of **Top**. In the case of distributive lattices,

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the topological category is not full because spectral maps are special continuous maps.

As a conceptual bridge, Priestley [13] showed that distributive lattices can also be dually represented in a category of certain topological spaces augmented with a partial order. This is an example of the latter sort of result, namely, a duality between lattices and a subcategory of a generalization of **Top**.

Urquhart [17], Hartung [8] and Hartonas [7] developed similar dualities for arbitrary bounded lattices. It is fair to say that they follow in the spirit of Priestley duality for distributive lattices in that their dual objects are certain topological spaces equipped with additional (partial order) structure. The dual morphisms are continuous maps that suitably preserve the additional structure. This is in contrast to the spirit of Stone duality, in which the dual category is simply a subcategory of **Top**.

Urquhart's construction equips the dual spaces with two quasi-orders in such a way that Priestley duality is precisely the special case where the two orders are converses of each other. Hartung takes a slightly different approach via the theory of concept lattices. His construction yields two topological spaces and a binary relation between them. Again, Priestley duality is a special case. Whereas Urquhart and Hartung must appeal to the axiom of choice to show that their spaces are inhabited with enough points, Hartonas avoids this in his duality and develops some interesting applications. His spaces are certain Stone spaces equipped with an auxiliary binary relation. So the sense in which this follows Priestley is clear.

Another approach to dualities for arbitrary lattices is given an exposition in Chapters 1 and 4 of Gierz *et al*, [4]. There, the duality between inf complete semilattices and sup complete semilattices arising from adjoint pairs of maps is specialized to various categories of algebraic and arithmetic lattices. Since algebraic and arithmetic lattices are precisely the ideal completions of join semilattices and lattices respectively, the general duality specializes to categories of lattices.

We take a different path via purely topological considerations that simplifies the duality of Hartonas by eliminating the need for an auxiliary binary relation. At the end of this path, we find algebraic and arithmetic lattices characterized as topological spaces. This establishes an affirmative answer to our original question with no riders: the dual category to **Lat** is a subcategory of **Top** *simpliciter*.

Like Stone, we find subcategories of **Top** (actually, of spectral spaces) that are dually equivalent to the categories of arbitrary semilattices with unit and arbitrary bounded lattices. The results makes explicit the relation between the duality of Hartonas and the duality via arithmetic lattices.

Because the sequel paper applies topological duality to problems of lattices with additional operations such as modal operators, residuals, etc., the sense in which a map between lattices is "structure-preserving" must be considered carefully. We consider here meet semilattice homomorphisms (Halmos' word for these is *hemimorphisms* [6]), and lattice homomorphisms. In addition, there is an obvious functor $(-)^\partial$ sending a lattice to its order opposite. This allows us to consider order-reversing, or *antitone*, maps that send meets to joins and so on.

2. Background and Definitions

In this paper, lattices are always bounded; semilattices always have a unit. Also, we designate semilattices as meet or join semilattices according to which order we intend. Lattice and semilattice homomorphisms preserve bounds. In sympathy with this view, for a collection of subsets of a universal set X to be "closed under finite intersections" includes empty intersection, so that X belongs to the collection.

Since our main concern is an interplay between ordered structures and topological structures, we can lay some ground rules at the start.

- Order-theoretic terminology and notation, when applied to a T_0 space, refer to the specialization order (which we denote by $\sqsubseteq$ when X is understood). For our purpose, the simplest characterization of specialization is $x \sqsubseteq y$ if and only if $\mathsf{N}^\circ(x) \subseteq \mathsf{N}^\circ(y)$ where $\mathsf{N}^\circ(x)$ is the filter of open neighborhoods of x. For example, for $x \in X$, $\downarrow x$ and $\uparrow x$ denote the sets of elements, respectively, below or equal to x and above or equal to x in the specialization order. Evidently, $\downarrow x$ is the closure of the singleton $\{x\}$. Also, a set is *directed* if it has non-empty intersection with $\uparrow x \cap \uparrow y$ for any members x, y of the set. In general, we will reserve "square" symbols for topological situations. For example $x \sqcap y$ will mean the meet with respect to specialization (if it exists).
- Topological terminology and notation, when applied to a partial order, refer to the *Scott topology*: where open sets are upper sets U that are inaccessible by directed joins, i.e. if $\bigvee^{\uparrow} D$ exists for a directed set D and $\bigvee^{\uparrow} D \in U$, then $D \cap U \neq \varnothing$. Note that the specialization order of the Scott topology coincides with the original order.

For a partially ordered set P, P^∂ denotes the order opposite. This notation is used mostly with respect to lattices. So L^∂ is again a lattice.

In a topological space X, say that a point $a \in X$ is *finite* if $\uparrow a$ is open. The term agrees with usage in lattice and domain theory, where an element a of a dcpo (or complete lattice) is called finite if and only if $\uparrow a$ is open in the Scott topology. Finite points of a complete lattice were first called *compact* by Nachbin [12] and that usage continues in order theory today, but so does the term "finite." Let $\mathsf{Fin}(X)$ denote the collection of finite points of X. We take $\mathsf{Fin}(X)$ to be ordered by restriction of the specialization order on X. Again, if C is a complete lattice, then $\mathsf{Fin}(C)$ is to be understood relative to the Scott topology on C.

A topological space is said to be *sober* if the map $x \mapsto \mathsf{N}^\circ(x)$ is a bijection between X and the collection of completely prime filters in the lattice of opens. Equivalently, a space is sober if every closed irreducible set is of the form $\downarrow x$ for a unique point x (recall a set A is *irreducible* if $A \subseteq B \cup C$ for closed sets B, C implies $A \subseteq B$ or $A \subseteq C$). Sobriety is a topological condition that ensures the space is T_0 and has some nice order-theoretic behavior. We will use the following well-known fact about sober spaces.

Lemma 2.1. *In a sober space, any directed set D has a supremum $\bigsqcup^{\uparrow} D$, which is in the closure of D. Moreover, any continuous function between sober spaces preserves directed suprema: $f(\bigsqcup^{\uparrow} D) = \bigsqcup^{\uparrow} f(D)$.*

In our deliberations, we will construct canonical extensions of lattices and show that they too are topologically representable.

A complete lattice C is a *completion* of a lattice L if L is a sublattice of C (more generally, L is embedded in C). L is *lattice dense* in C if

$$\mathsf{Meets}_C(\mathsf{Joins}_C(L)) = C = \mathsf{Joins}_C(\mathsf{Meets}_C(L)),$$

where

$$\mathsf{Meets}_C(A) = \{\bigwedge A' \mid A' \subseteq A\}$$
$$\mathsf{Joins}_C(A) = \{\bigvee A' \mid A' \subseteq A\}$$

Furthermore L is *lattice compact* in C if for all $U, V \subseteq L$, if $\bigwedge_C U \leq \bigvee_C V$ then there exist finite $U_0 \subseteq U$, $V_0 \subseteq V$ for which $\bigwedge U_0 \leq \bigvee V_0$.

Notice that lattice density and lattice compactness are *not* the same as topological density and compactness with respect to the Scott topology, hence the extra qualifier. In most work on canonical extension, these two properties are referred to simply as density and compactness.

A completion C is a *canonical extension of* L if L is lattice dense and lattice compact in C. In Section 4, we give a proof of the following theorem, originally due to Gehrke and Harding [3].

Theorem 2.2 ([3]). *Every lattice L has a canonical extension, denoted by L^δ, unique up to isomorphism, i.e. if C is also a canonical extension of L, then there is a lattice isomorphism between L^δ and C that keeps L fixed.*

Our touchstone for topological duality is Stone's representation theorem for bounded distributive lattices:

Theorem 2.3. *The category* **DL** *of distributive lattices and lattice homomorphisms is dually equivalent to the category* **Spec** *of spectral spaces and spectral functions.*

A *spectral space* is a sober space X in which the compact open sets form a basis that is closed under finite intersections (in particular, X is itself compact). A *spectral function* is a continuous f for which f^{-1} also preserves the way below relation on opens, where U *is way below* V means that any open cover of V contains a finite subcover of U. The way below relation is denoted by $U \ll V$. On spectral spaces, this is equivalent to requiring that f^{-1} preserves compact opens. Spectral functions (often in more general settings) are also known as *perfect* functions. We prefer to avoid this terminology because perfect has an entirely different meaning in lattice theory. Letting $\mathsf{KO}(X)$ denote the collection of compact open subsets of X, Stone's Theorem establishes that KO extends to a contravariant equivalence functor from **Spec** to **DL**. The inverse equivalence functor is denoted by $\mathsf{spec}(L)$. It takes a distributive lattice L to the space of its prime filters with topology generated by the sets $\{P \in \mathsf{spec}(L) \mid a \in P\}$ for $a \in L$.

For a space X, a *filter of* X is a filter in the usual order-theoretic sense: a set F so that (i) $x \in F$ and $x \sqsubseteq y$ implies $y \in F$ (ii) F is non-empty and (iii) $x, y \in F$ implies there exists $z \in F$ so that $z \sqsubseteq x, y$. Note that this is *not* the same as the more familiar notion of a *filter on* X, i.e., a filter of subsets of X.

A set satisfying (i) is an *upper set* with respect to specialization. In the topological setting, such sets are said to be *saturated*. Evidently, any open set is saturated and any intersection of saturated sets is again saturated. Moreover, suppose A is saturated and $x \notin A$. Then for each $y \in A$, we find an open set U_y containing y but not x. The union of all such U_y covers A and excludes x. Thus, A is exactly the intersection of its open neighborhoods.

We will be interested in special sorts of saturated sets: compact saturated sets, open sets and filters. In that light, we define

- $\mathsf{K}(X)$: the collection of compact saturated subsets of X;
- $\mathsf{O}(X)$: the collection of open subsets of X; and
- $\mathsf{F}(X)$: the collection of filters of X.

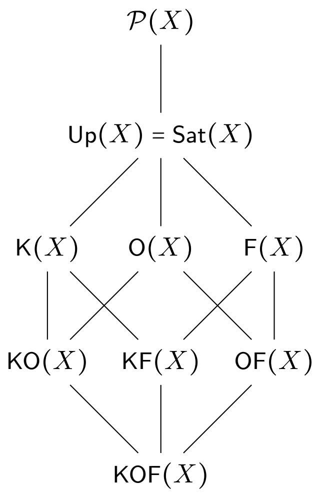
Intersections of these are denoted by concatenation, e.g., $\mathsf{OF}(X) = \mathsf{O}(X) \cap \mathsf{F}(X)$. In particular, OF, KO and KOF will be important. As already noted, spectral spaces are characterized by having $\mathsf{KO}(X)$ as a basis that is closed under finite intersection. On spectral spaces, spectral maps are those maps $f: X \to Y$ for which f^{-1} sends $\mathsf{KO}(Y)$ into $\mathsf{KO}(X)$. We take each of these collections to be ordered by inclusion.

The following technical observation is useful.

Lemma 2.4. *In a topological space X, let $F_1, \ldots, F_m$ be pairwise incomparable filters. Then $F_1 \cup \cdots \cup F_m$ is compact if and only if each F_i is a principal filter.*

Proof. Clearly, a principal filter $\uparrow x$ is compact, so a finite union of principal filters is compact.

Suppose $F_1, \ldots, F_m$ are pairwise incomparable filters and F_m is not principal. Let $\mathcal{D}$ be the collection of opens U such that $F_m \setminus U \neq \varnothing$. For $x \in F_m$, there is an element $y \in F_m$ so that $x \not\sqsubseteq y$. So there is an U for which $x \in U$ and $y \notin U$. For $x \in F_i$ $(i < m)$, the filters are pairwise incomparable, so there is an element $y \in F_m$ so that $x \not\sqsubseteq y$. Again there is an open U separating x from y. Thus $\mathcal{D}$ is an open cover of $F_1 \cup \cdots \cup F_m$. Suppose $U, V \in \mathcal{D}$. Then there are



X:	T_0 space ordered by specialization
$\mathcal{P}(X)$:	**all** subsets of X
$\mathsf{Up}(X)$:	**up-closed** subsets of X
$\mathsf{Sat}(X)$:	**saturated** subsets of X
$\mathsf{K}(X)$:	**compact saturated** subsets of X
$\mathsf{O}(X)$:	**open** subsets of X
$\mathsf{F}(X)$:	**filters** of X

FIGURE 1. Inclusion relations among collections of subsets

elements $x \in F_m \setminus U$ and $y \in F_m \setminus V$. Because F_m is a filter, there is also an element $z \in F_m$ below both x and y. Hence $z \in F_m \setminus (U \cup V)$. So $\mathcal{D}$ is directed. By construction, no $U \in \mathcal{D}$ covers $F_1 \cup \cdots \cup F_m$. □

In particular, the compact filters are principal, and $\mathsf{KOF}(X)$ is in an order-reversing bijection with $\mathsf{Fin}(X)$. For $F \in \mathsf{KOF}(X)$, we let $\min F$ denote the generator of F.

Theorem 2.5. *For a topological space X, the following are equivalent:*

(1) *X is spectral and $\mathsf{OF}(X)$ forms a basis that is closed under finite intersection;*
(2) *X is spectral, $\mathsf{OF}(X)$ forms a basis, X is a meet semilattice with respect to specialization and X has a least element;*
(3) *X is sober and $\mathsf{KOF}(X)$ forms a basis that is closed under finite intersection.*

Proof. Suppose (3) holds, then the compact opens and the open filters separately form bases. Furthermore, if K and H are compact opens, then $K = F_1 \cup \cdots \cup F_m$ for some compact open filters F_i and likewise $H = G_1 \cup \cdots \cup G_n$. Since each set $F_i \cap G_j$ is compact open, so is $K \cap H$. Similarly $X = \bigcap \varnothing$ is a compact open filter. So X is spectral and has a least element. Since $\mathsf{KOF}(X)$ is closed under finite intersection, $\mathsf{Fin}(X)$ is itself a directed subset of X. By sobriety the supremum exists, which must be the greatest element of X. For $x_0, x_1 \in X$, consider $B_{x_0,x_1} = \{a \in \mathsf{Fin}(X) \mid x_0, x_1 \in {\uparrow}a\}$. Because of (3), this is a directed set which has a supremum, $y := \bigsqcup^{\uparrow} B_{x_0,x_1}$. If y is in an open set U, then $y \in {\uparrow}a \subseteq U$ for some $a \in B_{x_0,x_1}$. So $y \sqsubseteq x_0, x_1$. Now consider y', a lower bound of x_0 and x_1. Then $y' \in U$ implies that $y' \in \uparrow a \subseteq U$ for some finite a. But $a \in B_{x_0,x_1}$, so $y' \sqsubseteq y$.

Suppose (2) holds. The least element of X ensures that X itself is a filter. Suppose F and G are open filters. Then $F \cap G$ is open and is a filter because X is a meet semilattice.

Suppose (1) holds. Spectral spaces are sober. Any compact open K equals $F_1 \cup \cdots \cup F_m$ for some open filters F_i. These can be chosen to be pairwise incomparable. So $\mathsf{KOF}(X)$ forms a basis. Evidently, a finite intersection of compact open filters is compact open because X is spectral. Separately, a finite intersection of open filters is an open filter. Hence (3) holds. □

We refer to a topological space satisfying these conditions as an *HMS* space in honor of Hofmann, Mislove and Stralka [9]. This naming is justified by Theorem 3.7 below because these are precisely the spaces that arise as algebraic lattices with the Scott topology.

3. F-saturation

Saturation can be relativized to any special class of opens in place of arbitrary opens. In particular, any intersection of open filters is saturated and is either empty or is a filter. Because of the greatest element, in an *HMS* space an intersection of open filters is never empty. Say that a set is *F-saturated* if it is an intersection of open filters. We have just noted that F-saturated subsets of an *HMS* space are always filters. (In general spaces, the only possible F-saturated non-filter is $\varnothing$). We let $\mathsf{FSat}(X)$ denote the complete lattice of F-saturated subsets of X ordered by inclusion, and define

$$\mathsf{fsat}(A) := \bigcap\{F \in \mathsf{OF}(X) \mid A \subseteq F\}.$$

Thus arbitrary meets in $\mathsf{FSat}(X)$ are intersections, and joins are defined by $\bigvee \mathcal{A} := \mathsf{fsat}(\bigcup \mathcal{A})$. In short, in any space, fsat is a *closure operator*; in any space with a greatest element, fsat produces a filter.

Lemma 3.1. *If X is an HMS space, then X is a complete lattice with respect to specialization. Moreover, for a compact set A, $\mathsf{fsat}(A)$ is compact, hence is a principal filter, and* $\min \mathsf{fsat}(A) = \bigsqcap A$.

Proof. The earlier proof that X is a meet semilattice generalizes to arbitrary meets. That is, for $A \subseteq X$, let $B^*_A := \{F \in \mathsf{KOF}(X) \mid A \subseteq F\}$, writing it as B^*_x for singletons. Each $F \in B^*_A$ is principal, so $B_A := \{\min F \mid F \in B^*_A\}$ is directed. Hence $x := \bigsqcup^{\uparrow} B_A$ exists. Obviously, x is a lower bound of A. If x' is another lower bound of A, then $B_{x'} \subseteq B_A$. So $\bigsqcup^{\uparrow} B_{x'} \sqsubseteq x$. But since $\mathsf{KOF}(X)$ is a basis of the topology, $x' \sqsubseteq \bigsqcup^{\uparrow} B_{x'}$.

If A is compact and $A \subseteq F$ for an open filter F, then by compactness there is some $G \in \mathsf{KOF}(X)$ for which $A \subseteq G \subseteq F$. Thus $\mathsf{fsat}(A) = \bigcap B^*_A = \uparrow \bigsqcap A$. □

$\mathsf{FSat}(X)$ has a bit more concrete structure. In particular, suppose $\mathcal{D}$ is a directed set of open filters. Then the union is also an open filter. Hence this union is F-saturated. In other words, in $\mathsf{FSat}(X)$ a directed join of open filters is simply a union.

We now consider what conditions on an *HMS* space are necessary and sufficient for $\mathsf{KOF}(X)$ to form a lattice, not just a semilattice.

Theorem 3.2. *For an HMS space, the following are equivalent.*

(1) $\mathsf{OF}(X)$ *forms a sublattice of* $\mathsf{FSat}(X)$*;*
(2) $\mathsf{KOF}(X)$ *forms a sublattice of* $\mathsf{FSat}(X)$*;*
(3) $\mathsf{fsat}(U)$ *is again open for any open* U*.*

Proof. Suppose (1) holds. For compact open filters F and G, the join in $\mathsf{FSat}(X)$ is $\mathsf{fsat}(F \cup G)$. But $F \cup G$ is compact, hence by Lemma 3.1 so is $\mathsf{fsat}(F \cup G)$. Likewise, $\mathsf{fsat}(\varnothing)$ is the least element of $\mathsf{Fsat}(X)$ and is compact.

Suppose (2) holds. Consider an open set U. Since X is a (complete) meet semilattice, U generates a filter F. That is, $x \in F$ if and only if for some $y_0, \dots, y_m \in U$, $y_0 \sqcap \dots \sqcap y_m \sqsubseteq x$. Evidently, it suffices to show that F is open, for then $F = \mathsf{fsat}(U)$. For $x \in F$, pick $y_0, \dots, y_m \in U$ that meet below it. According to Lemma 3.1, $y_i = \bigsqcup^{\uparrow} B_{y_i}$. But U is open. So we may choose an element of $a_i \in B_{y_i} \cap U$ in place of y_i. Now, $\uparrow a_i$ is a compact open filter, so (2) tells us that $\mathsf{fsat}(\uparrow a_0 \cup \dots \cup \uparrow a_m) = \uparrow(a_0 \sqcap \dots \sqcap a_m) \subseteq F$ is a compact open filter that contains x. Hence $F = \mathsf{fsat}(U)$ is open.

Suppose (3) holds. Then for any two open filters, $\mathsf{fsat}(F \cup G)$ is open. It is a filter because it is not empty. Likewise, $\mathsf{fsat}(\varnothing)$ is open and non-empty. □

We refer to the spaces satisfying the conditions of the theorem as *BL* spaces (*BL* abbreviating "bounded lattice").

The next task is to show that every semilattice and every lattice occurs isomorphically as $\mathsf{KOF}(X)$ for some *HMS* space and some *BL* space, respectively. The basic construction is the same in both cases, and establishes that *HMS* and *BL* spaces are simply algebraic and arithmetic lattices with their Scott topologies.

We know that *HMS* spaces and *BL* spaces are complete lattices with respect to specialization. But in fact, they are more structured than that.

A complete lattice C is said to be *algebraic* if and only if it is isomorphic to $\mathsf{Idl}(J)$ for some join semilattice J. Here $\mathsf{Idl}(J)$ simply refers to the lattice of ideals of J, i.e., subsets that are closed under $\downarrow$ and finite joins. Of course, there is nothing preventing us from thinking of $\mathsf{Idl}(J)$ as being the lattice of filters of J^{∂} instead, but the tradition is to characterize algebraicity in terms of ideals. The reader familiar with classical universal algebra will recall that algebraic lattices are precisely those lattices that occur as congruence lattices of algebraic structures (where the 'ideal formulation' is natural). A complete lattice C is said to be *arithmetic* if it is isomorphic to $\mathsf{Idl}(L)$ for some lattice L. Again, this can just as well be defined in terms of filters. Of course, there are several other useful internal characterizations of algebraic and arithmetic lattices (details can be found in [5]), but this characterization is easiest to use for our purposes.

For a meet semilattice M, let $\mathsf{Filt}(M)$ be the space of filters in M with the Scott topology. Since filters of M correspond to ideals of M^∂, every algebraic lattice occurs as $\mathsf{Filt}(M)$. This topology can be captured by a canonical basis. Namely, for $a \in M$, let

$$\varphi_a := \{F \in \mathsf{Filt}(M) \mid a \in F\}.$$

Lemma 3.3. *Let P be a partially ordered set. Then the opens of $\mathsf{Filt}(P)$ are order isomorphic with the collection of lower sets of P.*

Proof. Suppose $D \subseteq P$ is a lower set. Define $\mathcal{U}_D := \{F \in \mathsf{Filt}(P) \mid D \cap F \neq \varnothing\}$. Clearly, $\mathcal{U}_D$ is an upper set of filters. Moreover, if $\mathcal{D}$ is a directed set of filters, and $\bigcup \mathcal{D} \in \mathcal{U}_D$ then for some $F \in \mathcal{D}$, $F \in \mathcal{U}_D$. So $\mathcal{U}_D$ is Scott open.

Suppose $\mathcal{U}$ is a Scott open set of filters, define $D_{\mathcal{U}} := \{a \in P \mid {\uparrow}a \in \mathcal{U}\}$. Since $\mathcal{U}$ is an upper set, this is a lower set. Because any filter F is the directed union of principal filters contained in it, $F \in \mathcal{U}$ if and only if there exists $a \in F$ such that $a \in D_{\mathcal{U}}$. Likewise, for a lower set D, $a \in D$ if and only if ${\uparrow}a \in \mathcal{U}_D$. So the constructions $D_{\mathcal{U}}$ and $\mathcal{U}_D$ are order-preserving bijections. □

For a meet semilattice M, let $\mathsf{DL}(M)$ denote the free distributive lattice over M. That is, $\mathsf{DL}(M)$ is concretely built as the collection of finite unions of principal lower sets in M. Join is union and meet is computed in general by extension of ${\downarrow}a \cap {\downarrow}b = {\downarrow}(a \wedge b)$. The map $a \mapsto {\downarrow}a$ is the semilattice embedding $M \to \mathsf{DL}(M)$.

Lemma 3.4. *For every meet semilattice M, $\mathsf{Filt}(M)$ is homeomorphic to $\mathsf{spec}(\mathsf{DL}(M))$.*

Proof. A filter F in M determines a filter basis $\{{\downarrow}a \mid a \in F\}$ in $\mathsf{DL}(M)$, which evidently generates a prime filter. A prime filter $P \subseteq \mathsf{DL}(M)$ determines a filter $\{a \in M \mid {\downarrow}a \in P\}$ in M. These are easily checked to be inverses of one another. It is also routine, using Lemma 3.3, to check that these two maps are continuous. □

In the case that L is a lattice, $\mathsf{Filt}(L)$ has additional structure. We collect various useful facts in the following.

Lemma 3.5. *Let L be a lattice. In $\mathsf{Filt}(L)$ the following hold.*

(1) *An open $\mathcal{U}_D$ is a filter if and only if D is an ideal in L.*
(2) *Finite joins in $\mathsf{FSat}(\mathsf{Filt}(L))$ of compact open filters are given by joins in L. That is, $\mathsf{fsat}(\mathcal{U}_{{\downarrow}a} \cup \mathcal{U}_{{\downarrow}b}) = \mathcal{U}_{{\downarrow}(a\vee b)}$ and similarly, $\mathsf{fsat}(\varnothing) = \mathcal{U}_{\{0\}}$.*
(3) *The way below relation is given by $\mathcal{U}_D \ll \mathcal{U}_E$ if and only if for some finite $\{a_1, \ldots, a_n\} \subseteq E$, D is a subset of $\bigcup_{i=1}^n {\downarrow}a_i$.*

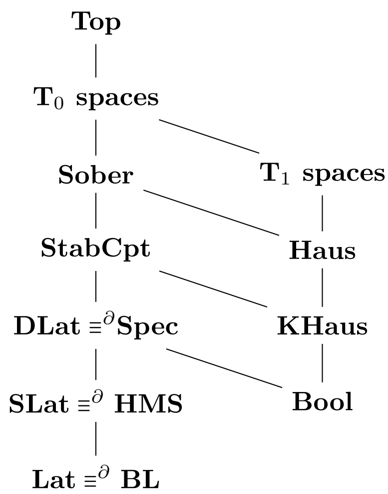


Figure 2. Inclusion relation among some subcategories of Top

Proof. For (1), suppose D is an ideal in L, and $a \in F \cap D$ and $b \in G \cap D$. So $a \vee b \in D$, and $x \vee y \in F \cap G$. So $\mathcal{U}_D$ is a filter of filters. Conversely, suppose $\mathcal{U}_D$ is a filter of filters, and $a, b \in D$. Then $\uparrow a \in \mathcal{U}_D$ and $\uparrow b \in \mathcal{U}_D$. So $\uparrow a \cap \uparrow b = \uparrow(a \vee b) \in \mathcal{U}_D$. That is, $a \vee b \in D$.

For (2), $\mathcal{U}_{\downarrow(a\vee b)}$ is an F-saturated set containing $\mathcal{U}_{\downarrow a} \cup \mathcal{U}_{\downarrow b}$. If $\mathcal{U}_I$ contains $\mathcal{U}_{\downarrow a} \cup \mathcal{U}_{\downarrow b}$, then in particular, $a, b \in I$. So $a \vee b \in I$. Evidently, $\mathcal{U}_{\{0\}} = \{L\}$, which is the smallest F-saturated set of filters of L.

The characterization of $\ll$ in (3) is a standard fact about the Scott topology of an algebraic dcpo [5]. □

Lemma 3.6. *For a meet semilattice M, $\mathsf{Filt}(M)$ is an HMS space. For a lattice L, $\mathsf{Filt}(L)$ is a BL space.*

Proof. For the first claim, because $\{1\}$ and M are the least and greatest elements, it remains to check that the open filters form a basis. But the sets φ_a for $a \in M$ form a basis, and these clearly are filters.

For a lattice L, it remains to check that $\mathsf{fsat}(\mathcal{U}_D)$ is open when D is a lower set in L. The open filters containing $\mathcal{U}_D$ are bijective with the ideals containing D. So let I be the smallest ideal containing D. Then $\mathcal{U}_I$ is evidently equal to $\mathsf{fsat}(\mathcal{U}_D)$. □

Putting all these facts together we obtain the following.

Theorem 3.7. *Any meet semilattice M is isomorphic to $\mathsf{KOF}(\mathsf{Filt}(M))$. Any HMS space X is homeomorphic to $\mathsf{Filt}(\mathsf{KOF}(X))$. These constructions restrict to lattices and BL spaces.*

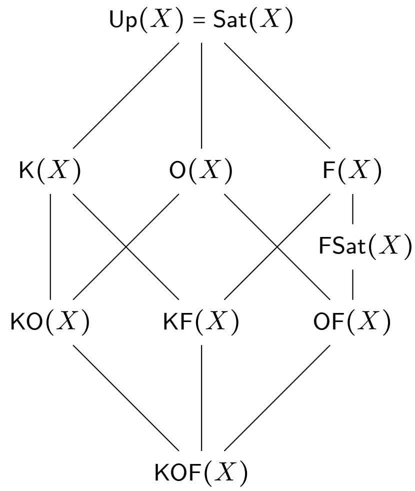


FIGURE 3. Position of the canonical extension for a *BL* space X

Proof. The map sending $a \in M$ to $\varphi_a \coloneqq \{f \in \mathsf{Filt}(M) \mid a \in f\}$ is the isomorphism. Similarly, the map sending $x \in X$ to $\theta_x \coloneqq \{F \in \mathsf{KOF}(X) \mid x \in F\}$ is the homeomorphism. □

Notice that these results also tell us that the *HMS* spaces are exactly the algebraic lattices and the *BL* spaces are exactly the arithmetic lattices, both with their Scott topologies.

4. Canonical Extension

Jónsson and Tarski [10] introduced canonical extensions of Boolean algebras to provide an algebraic setting for relational semantics of Boolean algebras with operators. Canonical extensions of bounded lattices are due to Gehrke and Harding [3], and they have applications to bounded residuated lattices, positive (even non-distributive) logics, linear logics and various other non-classical logics. As mentioned in Section 2, canonical extensions are characterized abstractly as completions that are lattice dense and lattice compact. In the current section we show that canonical extension arise naturally from BL spaces.

Theorem 4.1. *For every BL space X, $\mathsf{FSat}(X)$ is a canonical extension of $\mathsf{KOF}(X)$.*

Proof. One half of lattice density is almost trivial. Consider an open filter $F = \bigcup\{\uparrow a \mid a \in F \cap \mathsf{Fin}(X)\}$. From Lemma 3.1, this union is directed, so it is the join in $\mathsf{FSat}(X)$. Hence any $S \in \mathsf{FSat}(X)$ takes the form

$$\begin{aligned} S &= \bigcap\{F \in \mathsf{OF}(X) \mid S \subseteq F\} \\ &= \bigcap\Big\{\bigcup\{\uparrow a \mid a \in F \cap \mathsf{Fin}(X)\} \mid F \in \mathsf{OF}(X) \text{ and } S \subseteq F\Big\} \\ &= \bigcap\Big\{\bigvee\{\uparrow a \mid a \in F \cap \mathsf{Fin}(X)\} \mid F \in \mathsf{OF}(X) \text{ and } S \subseteq F\Big\}. \end{aligned}$$

For the other half of density, consider $S \in \mathsf{FSat}(X)$. Then $S = \bigcap_{i\in I} F_i$ for some family of open filters $\{F_i\}$. Each F_i is a directed join, hence union, of compact open filters $\{\uparrow a_{ij}\}_{j\in J_i}$. So

$$\begin{aligned} S &= \bigcup_{\gamma \in \prod_{i\in I} J_i} \bigcap_{i\in I} \uparrow a_{i,\gamma(i)} \\ &\subseteq \mathsf{fsat}\Big(\bigcup_{\gamma \in \prod_{i\in I} J_i} \bigcap_{i\in I} \uparrow a_{i,\gamma(i)}\Big) \\ &= \bigvee_{\gamma \in \prod_{i\in I} J_i} \bigcap_{i\in I} \uparrow a_{i,\gamma(i)} \\ &\subseteq S. \end{aligned}$$

For lattice compactness, it suffices to show that when $\{F_i\}_i$ is a downward directed family of compact open filters and $\{G_j\}_j$ is an upward directed family of compact open filters, if $\bigcap_i F_i \subseteq \bigcup_j G_j$, then for some i and j, $F_i \subseteq G_j$. Each F_i is a principal filter, so let $a_i = \min F_i$. Because the family $\{F_i\}_i$ is downward directed, the set of these generators $\{a_i\}_i$ is directed. By sobriety of X, this directed set has a least upper bound, say x. Evidently, $x \in \bigcap_i F_i$, and every open neighborhood of x includes some a_i. In particular, $\bigcup_j G_j$ is such a neighborhood. So for some i, $a_i \in \bigcup_j G_j$. Hence for some j, $a_i \in G_j$. ☐

Corollary 4.2. *Every lattice has a canonical extension, unique up to isomorphism.*

Proof. For a lattice L, let $X = \mathsf{Filt}(L)$. By the preceding theorem and Theorem 3.7, $\mathsf{FSat}(X)$ is a canonical extension of $\mathsf{KOF}(X)$ which is isomorphic to L.

Suppose C is also a canonical extension of L. To simplify notation, assume that L is a sublattice of C.

Define maps $j\colon \mathsf{OF}(X) \to \mathsf{Joins}_C(L)$ and $m\colon X \to \mathsf{Meets}_C(L)$ by

$$\begin{aligned} j(F) &= \bigvee_C \{a \in L \mid \uparrow a \subseteq F\} \\ m(x) &= \bigwedge_C \{a \in L \mid a \in x\} \end{aligned}$$

By lattice compactness of the completion C, $m(x) \leq j(F)$ holds if and only if $x \in F$. Also by lattice density, for every $\gamma \in C$,

$$\bigvee_C \{m(x) \mid m(x) \leq \gamma\} = \gamma = \bigwedge_C \{j(F) \mid \gamma \leq j(F)\}.$$

Now define four maps $f^*, g_*\colon \mathsf{FSat}(X) \to C$ and $g^*, f_*\colon C \to \mathsf{FSat}(X)$ as follows:

$$f^*(S) = \bigvee_C \{m(x) \mid x \in S\} \qquad f_*(\gamma) = \bigcap\{F \mid \gamma \le j(F)\}$$

$$g^*(\gamma) = \mathsf{fsat}(\{x \mid m(x) \le \gamma\}) \qquad g_*(S) = \bigwedge_C \{j(F) \mid S \subseteq F\}$$

Variables appearing here and for the remainder of this proof are implicitly typed as $x \in X$, $F \in \mathsf{OF}(X)$, $S \in \mathsf{FSat}(X)$ and $\gamma \in C$.

Now we check four facts: (i) f^* is left adjoint to f_*, (ii) g^* is left adjoint to g_*, (iii) $f^* = g_*$ and (iv) $f_* = g^*$. These suffice to establish that f^* and f_* (equivalently, g_* and g^*) are the desired isomorphisms.

Fact (i): $f^*(S) \le \gamma$ if and only if $x \in S$ implies $m(x) \le \gamma$. And $S \subseteq f_*(\gamma)$ if and only if $x \in S$ implies $x \in F$ for every F such that $\gamma \le j(F)$. Lattice compactness and density mean that $m(x) \le \gamma$ is equivalent to $x \in F$ for every F such that $\gamma \le j(F)$.

Fact (ii): $g^*(\gamma) \subseteq S$ if and only if $m(x) \le \gamma$ implies $x \in S$, because S is f-saturated. And $\gamma \le g_*(S)$ if and only if $\gamma \le j(F)$ for every $F \supseteq S$. Again lattice compactness and density mean these conditions are equivalent.

Fact (iii): By density, $f^*(S) = \bigwedge_C\{j(F) \mid f^*(S) \le j(F)\}$. But the condition $f^*(S) \le j(F)$ holds if and only if $m(x) \le j(F)$ for all $x \in S$. So by compactness, this is equivalent to $S \subseteq F$.

Fact (iv): $g^*(\gamma) = \bigcap\{F \mid \{x \mid m(x) \le \gamma\} \subseteq F\}$ as an f-saturation. But $\{x \mid m(x) \le \gamma\} \subseteq F$ is equivalent to $\gamma \le j(F)$. □

For comparison, we note that for Boolean algebras and distributive lattices the canonical extension can also be obtained from the dual space by taking the powerset of the Stone space and all upsets of the Priestley space respectively.

5. Morphisms

Clearly, the next thing to do is to extend Theorem 3.7 to a duality of categories. We do this by first characterizing those (continuous) functions between *HMS* spaces that correspond to meet semilattice homomorphisms. Subsequently we cut this down to lattice homomorphisms.

Lemma 5.1. *For a function $f\colon X \to Y$ between HMS spaces, the following are equivalent.*

(1) *f^{-1} restricted to $\mathsf{KOF}(Y)$ co-restricts to $\mathsf{KOF}(X)$.*
(2) *f is spectral and f^{-1} restricted to $\mathsf{OF}(Y)$ co-restricts to $\mathsf{OF}(X)$.*
(3) *f is spectral and $\mathsf{fsat}(f^{-1}(B)) \subseteq f^{-1}(\mathsf{fsat}(B))$ for all $B \subseteq Y$.*
(4) *f is spectral and $\mathsf{fsat}(f^{-1}(U)) \subseteq f^{-1}(\mathsf{fsat}(U))$ for all opens $U \subseteq Y$.*

Proof. Suppose (1). Then immediately f is continuous. Also $U \ll V$ holds if and only if there is a compact open K so that $U \subseteq K \subseteq V$. But K is simply a finite union of compact open filters $F_1 \cup \dots \cup F_m$. So $f^{-1}(U) \subseteq$

$f^{-1}(F_1) \cup \cdots \cup f^{-1}(F_m) \subseteq f^{-1}(V)$. The middle set is a finite union of compact open filters. Therefore f is spectral. Suppose $F \in \mathsf{OF}(Y)$. Then F is a directed union of compact open filters. So $f^{-1}(F)$ is a directed union of compact open filters, hence is an open filter.

Suppose (2), and consider $B \subseteq Y$. Since $\mathsf{fsat}(B) = \bigcap\{F \in \mathsf{OF}(X) \mid B \subseteq F\}$ and $f^{-1}(F) \in \mathsf{OF}(X)$ for any $F \in \mathsf{OF}(Y)$, $f^{-1}(\mathsf{fsat}(B))$ is an intersection of fewer open filters than $\mathsf{fsat}(f^{-1}(B))$.

Trivially, (3) implies (4).

Suppose (4). In particular, $\mathsf{fsat}(f^{-1}(\uparrow a)) \subseteq f^{-1}(\uparrow a)$ because $\uparrow a$ is already F-saturated. Obviously, $f^{-1}(\uparrow a) \subseteq \mathsf{fsat}(f^{-1}(\uparrow a))$. Because f is spectral, $f^{-1}(\uparrow a)$ is compact open. Because X is a *HMS* space (hence has a top element), $\mathsf{fsat}(f^{-1}(\uparrow a)) = f^{-1}(\uparrow a)$ is a compact open filter. □

A function f is called *F-continuous* if it satisfies the equivalent conditions of the lemma. This leads to our first duality theorem.

Theorem 5.2. *The category of semilattices and meet-preserving functions is dually equivalent to the category of HMS spaces and F-continuous functions. This cuts down to the full subcategory of lattices and meet-preserving functions and the full subcategory of BL spaces and F-continuous functions.*

Proof. Lemma 5.1 clearly indicates that KOF extends to a functor into the category of meet semilattices via the restriction of $\mathsf{KOF}(f) = f^{-1}$ to compact open filters.

Likewise for $h\colon L \to M$, $\mathsf{Filt}(h) = h^{-1}$ sends filters to filters because h preserves meets. Moreover, for a compact open filter $\varphi_a \in \mathsf{KOF}(\mathsf{Filt}(L))$, we have $F \in \mathsf{Filt}(h)^{-1}(\varphi_a)$ if and only if $h(a) \in F$ if and only if $F \in \varphi_{h(a)}$. Therefore, $\mathsf{Filt}(h)$ is F-continuous.

Evidently, the isomorphism and homeomorphism of Theorem 3.7 are natural in these functors.

□

To cut this duality down to lattice homomorphisms, we recall that $\mathsf{OF}(X)$ is the *BL* space dual to $\mathsf{Fin}(X)$. So an F-continuous map from $\mathsf{OF}(X)$ to $\mathsf{OF}(Y)$ corresponds dually to a *join*-preserving map between $\mathsf{KOF}(Y)$ and $\mathsf{KOF}(X)$.

Lemma 5.3. *For an F-continuous map $f\colon X \to Y$ between BL spaces, the following are equivalent.*

(1) f^{-1} *preserves finite joins of compact open filters.*
(2) f^{-1} *preserves finite joins of open filters.*
(3) f^{-1} *preserves all joins of open filters.*
(4) $f^{-1}(\mathsf{fsat}(U)) \subseteq \mathsf{fsat}(f^{-1}(U))$ *for any open $U \subseteq Y$.*

Proof. Obviously, (3) implies (2) and (2) implies (1).

Suppose (4) holds, and let F_i be an indexed family of open filters. Then

$$\begin{aligned} f^{-1}(\bigvee_i F_i) &= f^{-1}(\mathsf{fsat}(\bigcup_i F_i)) \\ &= \mathsf{fsat}(f^{-1}(\bigcup_i F_i)) \\ &= \mathsf{fsat}(\bigcup_i f^{-1}(F_i)) \\ &= \bigvee_i f^{-1}(F_i). \end{aligned}$$

Finally, suppose (1) holds. Consider an open $U \subseteq Y$ and $F \in \mathsf{OF}(X)$ so that $f^{-1}(U) \subseteq F$. Per Theorem 3.2,

$$\mathsf{fsat}(U) = \bigcup\{\uparrow(a_1 \sqcap \ldots \sqcap a_m) \mid a_1, \ldots, a_m \in U \cap \mathsf{Fin}(Y)\}.$$

Since f^{-1} preserves finite joins, we also have $f^{-1}(\mathsf{fsat}(U)) \subseteq F$. □

Say that a spectral function is *F-stable* if

$$f^{-1}(\mathsf{fsat}(U)) = \mathsf{fsat}(f^{-1}(U))$$

for any open U.

Theorem 5.4. *The category of lattices and lattice homomorphisms is dually equivalent to the category of BL spaces and F-stable functions.*

Proof. Evidently the dual equivalence of Theorem 5.2 cuts down to this, so long as the functor Filt cuts down. That is, we need to check that $\mathsf{Filt}(h)$ is F-stable when $h: L \to M$ is a lattice homomorphism. We already know it is F-continuous.

Suppose L and M are lattices and $h: L \to M$ is a lattice homomorphism. Consider two compact open filters $\varphi_a, \varphi_b \in \mathsf{KOF}(\mathsf{Filt}(L))$, and observe that $\mathsf{Filt}(h)^{-1}(\varphi_a) = \varphi_{h(a)}$. The join of φ_a and φ_b in $\mathsf{KOF}(\mathsf{Filt}(L))$ is $\mathsf{fsat}(\varphi_a \cup \varphi_b) = \varphi_{a\vee b}$. And of course $\mathsf{Filt}(h)^{-1}(\varphi_{a\vee b}) = \varphi_{h(a)\vee h(b)}$ is the join of $\varphi_{h(a)}$ and $\varphi_{h(b)}$. □

In a finite lattice, $\mathsf{Filt}(L)$ is isomorphic to L^∂, and since all upper sets in $\mathsf{Filt}(L)$ are open, $\mathsf{KOF}(\mathsf{Filt}(L))$ is isomorphic to $\mathsf{Filt}(L)^\partial$. That is, the natural isomorphism from a finite lattice L to $\mathsf{KOF}(\mathsf{Filt}(L))$ is rather trivial. For a non-trivial example, consider the lattice consisting of two copies of $[0,1]$ with 0's and 1's identified. We can write x for elements of one copy and x' for the corresponding elements of the other copy. So there are two types of filter:

(1) $\uparrow x$ – the principal filter generated by any $x \in L$;
(2) $\Uparrow x = \uparrow x \setminus \{x\}$ – the 'round' filter of elements strictly above x by any $x \in L \setminus \{1\}$. In the special case of 0, we make a distinction between $\Uparrow 0$ and $\Uparrow 0'$ according to which copy of $[0,1]$ is used.

Figure 4 illustrates L and $\mathsf{Filt}(L)$.

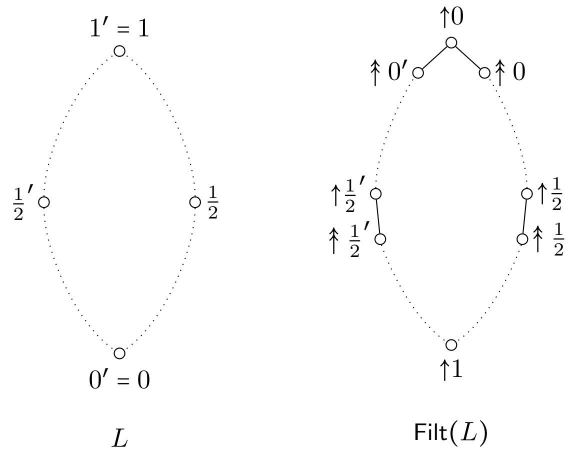


FIGURE 4. A non-distributive lattice and its filters

In $\mathsf{Filt}(L)$ (specialization order being inclusion), we have three types of filters,

$$\mathcal{H}_x := \{F \in \mathsf{Filt}(L) \mid \Uparrow x \subseteq F\} \qquad [x \neq 1]$$
$$\mathcal{G}_x := \{F \in \mathsf{Filt}(L) \mid \uparrow x \subseteq F\}$$
$$\mathcal{F}_x := \bigcup_{y<x} \mathcal{H}_y \qquad [x \neq 0]$$

Clearly, $\mathcal{F}_x \subseteq \mathcal{G}_x \subseteq \mathcal{H}_x$ when these exist. The filters $\mathcal{H}_x$ and $\mathcal{G}_x$ are compact. The filters $\mathcal{G}_x$ and $\mathcal{F}_x$ are open. So the filters $\mathcal{G}_x$ constitute $\mathsf{KOF}(X)$. Figure 5 illustrates $\mathsf{F}(\mathsf{Filt}(L))$, $\mathsf{KF}(\mathsf{Filt}(L))$, $\mathsf{OF}(\mathsf{Filt}(L))$ and $\mathsf{KOF}(\mathsf{Filt}(L))$. In this example, every member of $\mathsf{F}(\mathsf{Filt}(L))$ is saturated, hence the canonical extension of L is isomorphic to $\mathsf{F}(\mathsf{Filt}(L))$.

Note that the dual space $X = \mathsf{Filt}(L)$ of a lattice L is at least as large as L, and can be considerably larger (up to $2^{|L|}$ in the case of an infinite Boolean algebra). This agrees with the duality of Hartonas, but is in contrast to so-called reduced dualities such as Urquhart's L-spaces and Hartung's topological contexts which can be logarithmically smaller than the size of L. However the description of morphisms for such dualities is less intuitive since one has to resort to certain (pairs of) relations or consider only duals of surjective lattice morphisms. Moreover, the reduction in size makes use of the axiom of choice, whereas dualities that use all filters are constructive.

6. Conclusion

We have established a dual equivalence between **Lat** and the category of BL spaces and F-stable maps, an easily described subcategory of **Top**. In addition,

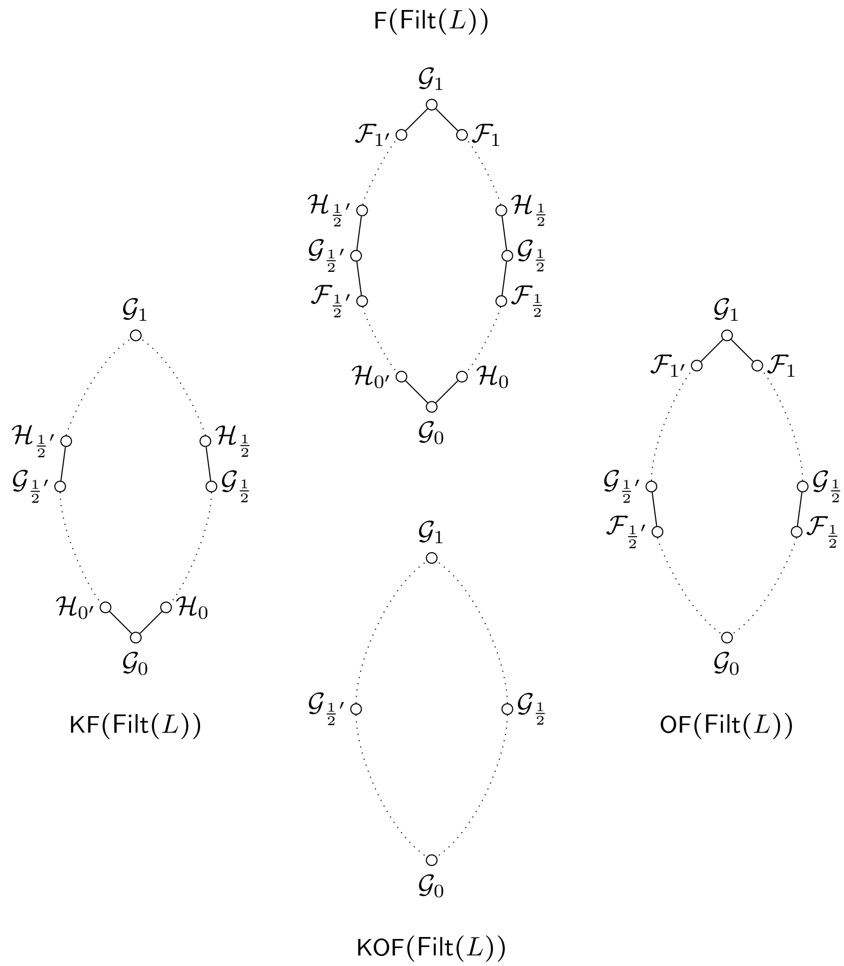


Figure 5. Sets of filters of $\mathsf{Filt}(L)$ ordered by inclusion

in a BL space X, the very natural construction of the complete lattice of F-saturated subsets produces the canonical extension of $\mathsf{KOF}(X)$. Along the way, we also have established a dual equivalence between the category of semilattice reducts of lattices and the category of HMS spaces and F-continuous maps.

In the sequel paper, we extend the topological duality for lattices to handle n-ary operations that are join-reversing or meet-preserving in each argument, or dually that are meet-reversing or join-preserving in each argument. Such operations are called quasioperators, and we consider several examples to illustrate the general case. Similar extensions have been discussed by Hartonas, but our topological duality for the underlying lattices simplifies the description of morphisms in the dual category.

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