An Elementary Humanomics Approach to Boundedly Rational Quadratic Models

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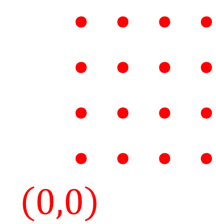
Preliminary

- Potential Games [Monderer and Shapley] have been shown to be isomorphic to "congestion games"
  - Computer scientists model internet congestion and network analysis using rational potential game theory (c.f., Wolpert, and others).
- Analysis of certain "boundedly rational" potential games *leads to* statistical mechanics (SM)*
  - SM has been deemed a viable model to address some important questions in economics (c.f., Brock, Durlauf, etc.) such as emergence, scaling, etc.

* In fact Anderson, Goeree, and Holt (2004) showed the Quantal Response Equilibrium results from a partially averaged form of dynamics studied here. They didn't realize that was a "mean-field" version of the unaveraged dynamics which leads to the Gibbs measure of equilibrium statistical mechanics.

Game Theory

- Consider a finite number of "agents", i.e., a fictitious decision maker ("agent") in our model
  - For example, someone who can buy or sell a good
- We can (*but don't need to*) imagine that each agent is located on a two-dimensional grid of integer-coordinate points



  - The set of all points on our grid is denoted G
  - A specific "agent" is denoted by their point of location "i", for some $i \in G$ (e.g., $i = (1,2)$)
  - *Not necessarily "spatial"*; local connectedness in "space" may or *may not* exist
    - e.g., it could represent agents who bid on the same contract, etc.

- At any moment in time, agent i can select an action or strategy $x_i \in A$
  - A is the set of decisions an agent can make
    - ► e.g., agents could buy or sell an item and we could take

$$A = \{-1,1\} \ ; \ -1 = \text{buy}, \ 1 = \text{sell}$$

    - ► e.g., the number of goods produced by a group of companies:

$$A = \text{interval of real numbers} = [\text{low, high}]$$

  - x_i is agent i's *decision variable*; e.g., $x_i = -1$ (buy)
- A *configuration* $\vec{x} = (x_1, x_2, \dots, x_N)$ is any possible state of the system
  - e.g., $\vec{x} = (-1, -1, \dots, -1)$ everyone is buying,
  - e.g., $(1, -1, -1, 1, \dots)$ some buying, some selling, etc.

- The set of all possible configurations is Ω ($= \prod_i A$), which is called *(pure) state space*.

- Each agent has a *payoff function* $\pi_{i(x_1, \ldots, x_N)}$

  - it gives the payoff (real number) agent i gets for a given state of the system

  - e.g., $\pi_4(1,1,\ldots,1)$ gives agent 4's payoff if everyone is selling

- Each agent has an *action function* $a_{i}(x_1, \dots, x_N)$
  - it allows agents to control their behavior
    - ► can enact rewards and punishments towards other agents
  - e.g., $a_7(-1, -1, \dots, -1)$ gives agent 7's action if everyone is buying
- We only consider "potential games", which means we have a function $V(x_1, \dots, x_N)$ such that every agent's action a_i function satisfies

POTENTIAL GAME CONDITION

$$\frac{\partial a_i}{\partial x_i} = \frac{\partial V}{\partial x_i}$$

Dynamics for the Game

- At any point in time, agents will change their decisions to try to maximize their action
- A simple (myopic) model would be

AGENTS FOLLOW MAX ACTION DIRECTION

$$dx_i = \frac{\partial a_i}{\partial x_i}\, dt$$

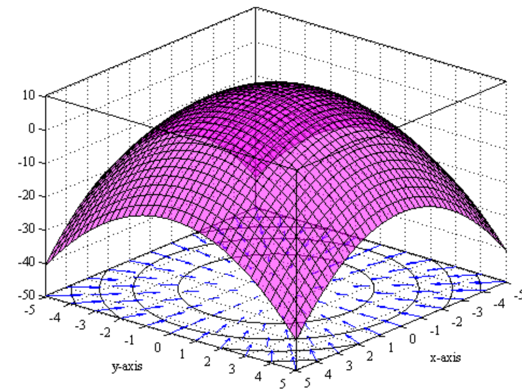
  - agent i changes their decision variable x_i by a small amount dx_i over a small amount of time dt in proportion to the derivative of their action function
  - if the derivative is positive (negative), then the action will increase if agent i increases (decreases) their decision variable x_i

- Since we have a potential, we can

  replace $\partial a_i / \partial x_i$ with $\partial V / \partial x_i$

- Collecting all the differential equations together gives

  $$(dx_1, dx_2, \dots) = \left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots\right) dt$$

  or more compactly,

  RATIONAL DYNAMICS

  $$\boxed{d\vec{x} = \nabla V \, dt}$$

  where the gradient $\nabla V = (\partial V / \partial x_1, \partial V / \partial x_2, \dots)$ is a vector pointing in the direction of the greatest rate of increase of the potential

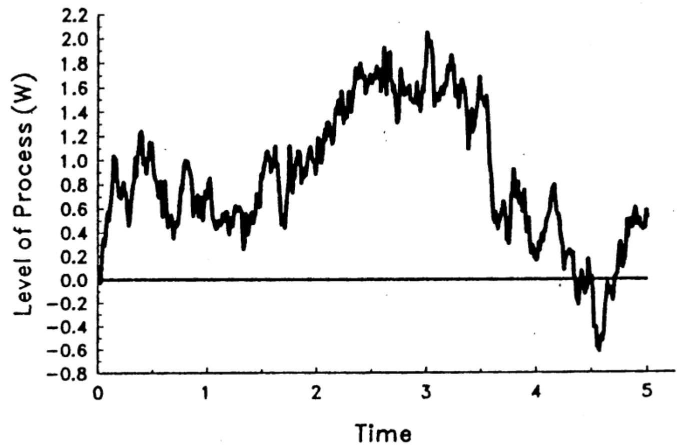
- Everyone is then moving (myopically) in the direction of greatest increase of their action functions

Bounded Rationality

- **Agents don't always have exact information, due to random error**
  - economists *recently* found justification from a behavioral explanation of the intrinsic randomness needed to justify the long-used mixed-strategy Nash equilibrium (Kuhn's Theorem)
  - boundedly-rational errors are failures to choose the most optimal payoff and are intrinsic to the agents
  - examples are experimentation, mistakes in judgment, lack of complete information, maintaining the reputation of a product, etc.

- We use a Wiener process $w_i(t)$ to model error
  - for each t, $w_i(t)$ is a random variable
  - $w_i(0) = 0$
  - increments are independent
    - e.g., $0 \leq r < s < t$, then $w_i(t) - w_i(s)$ and $w_i(s) - w_i(r)$ are independent
  - $w_i(s) - w_i(r)$ is normally distributed with mean 0 and variance $s - r$
  - the function $w_i(t)$ is (almost everywhere) a continuous function of t

- An example of a specific Wiener process is shown below



  - we can see that at most times in this example, a random positive value would be added to the rational part of the agent’s decision
  - for times around 4.5, negative values would be added to the rational part

The Boundedly Rational Quenched Model for Economic Behavior

- Now we combine the rational and non-rational parts of our agents' decision-making (drift-diffusion) model with decision variables x_i

STOCHASTIC DYNAMICAL MODEL

$$\boxed{d\vec{x} = \nabla V\, dt + \upsilon\, d\vec{w}(t) + \mathbf{r}(\vec{x})d\vec{z}(t)}$$

- Here, $d\vec{w} = (dw_1, dw_2, \ldots, dw_N)$ is a Gaussian White Noise process
- each $dw_i(t)$ is an increment over infinitesimal time dt (Itô sense) ,i.e.,

$$dw_i(t) = w_i(t + dt) - w_i(t)$$

  - υ is a *fluctuation* variable that allows us to adjust how much influence the random part has
- The process $d\vec{z}$ only changes on the boundary for reflection and **r** is the reflection matrix for normal reflection on the boundary
  - "reflecting boundary conditions" are used so that the agents' decisions stay within the high and low bounds of the decision variables (e.g., can't produce less than 0 goods)

Joint Distribution of Decisions

- Since the agents' decisions have a random component, their decision at any point in time will be determined by a "joint density"
  - this is a probability density function $f(\vec{x}, t)$ on decision space Ω at time t
  - $f(\vec{x}, t)$ gives the probability that agents make decisions represented by $\vec{x}$ at time t
  - f changes over time, but will reach a fixed function in the long run called the equilibrium measure

Equilibrium Measure

- The stationary joint distribution function satisfies the Itô / Fokker-Planck equation

$$\frac{\partial f(\vec{x},t)}{\partial t} = 0 = -\boldsymbol{\nabla}\cdot\left[\,\boldsymbol{\nabla} V\big(\vec{x}(t)\big)\, f(\vec{x},t)\right] + \frac{\nu^2}{2}\boldsymbol{\nabla}^2 f(\vec{x},t)$$

- The solution to the equation is the Gibbs state

$$f(\vec{x},t) = f_{eq}(\vec{x}) = \frac{\exp\left(\frac{2}{\nu^2}\,V(\vec{x})\right)}{\int_\Omega \exp\left(\frac{2}{\nu^2}\,V(\vec{u})\right) d\vec{u}}$$

  - **The time variable t is for the time scale of economic interactions**

* W Kang, K Ramanan: "On the Submartingale Problem for Reflected Diffusions in Domains with Piecewise Smooth Boundaries" 2014

- In statistical mechanics, the Gibbs measure has the same form, with

$$\frac{2}{\upsilon^2} V(\vec{x}) \quad \text{replaced by} \quad -\frac{1}{kT} E(\vec{x})$$

  - where k is Boltzmann's constant
  - T is temperature: $T = \upsilon^2/(2k)$ Fluctuation-Dissipation Thm
  - $E(\vec{x}) = -V(\vec{x})$ is the energy of configuration $\vec{x}$

- The analogy of a boundedly-rational potential game to statistical mechanics (physics) is
  - the influence of non-rationality υ^2 is proportional to "temperature" (fluctuation-dissipation theorem)
  - the potential V is the negative "energy" of the system

Humanomics Modeling

- Agents: two or more; $i = 2$ to N
- Gratitude Configuration γ
  - γ_{ij} gratitude/resentment i has for j
  - visualize as bond between sites i and j on graph
  - e.g., $\gamma_{ij} = 1 = \gamma_{ji}$ mutual gratitude of i and j,
    $\gamma_{ij} = -1 = \gamma_{ji}$ is mutual resentment
- Two timescales are used
  - economic equilibrium (i.e., stationary state)
  - feelings of gratitude/resentment

- Strategy Variables
  - agent i's strategy is $x_i \in [\, \underline{x} \,, \overline{x} \,]$
  - a configuration of decisions is

$$\vec{x} = (x_1, x_2, \ldots, x_N)$$

- Payoff Functions
  - $\pi_i(\vec{x})$ is agent i's payoff function
    - ► captures transfers from i to j and returns from j to i in accordance with i's benefit from j's action and i's reward to j
- Action Functions
  - $a_i(\vec{x})$ is agent i's action function
    - ► allows agents to reward/punish other agents when prompted by gratitude/resentment
    - ► can reflect self-interested behavior

- Quadratic Payoffs
  - We consider

$$\pi_i(\vec{x}) = \sum_{k \geq j = 1}^{N} J_{jk}^{(i)} x_j x_k + \sum_{j=1}^{N} h_j^{(i)} x_j + C_i$$

    - ► adding or subtracting quadratic payoffs results in a quadratic function (closure)
  - Quantity/Price-type payoffs

$$\pi_k(\vec{x}) = x_k \left(\sum_{m \leq k} J_{mk}^{(k)} x_m + \sum_{m > k} J_{km}^{(k)} x_m + h_k^{(k)} \right)$$

    - ► these types appear in Cournot and speculator/hedging models

- Implement Action*

$$a_i(\vec{x}) = \sum_{j=1}^{N} \gamma_{ij}\, \pi_j(\vec{x})$$

- Fundamental Premise: Axiom 3 *Humanomics*
  - ▶ gratitude/resentment prompts reward/punishment
- Model is consistent with this
  - ▶ $a_i = \pi_i$ reflects that i is only self-interested and maximizes their payoffs
  - ▶ if $\gamma_{ij} = \pm 1$ indicates gratitude/resentment of i to j, then the action function for i

    $$a_i = \pi_i \pm \pi_j$$

    results in higher/lower expected payoff for agent j
  - ▶ we assume mutual gratitude/resentment: $\gamma_{ij} = \gamma_{ji}$ thus if above a_i,

    $$a_j = \pi_j \pm \pi_i$$

    - this is a condition for the existence of a *potential*

* This form is for "aligning" (ferromagnetic) interactions. It was shown that for "opposing" (antiferromagnetic) interactions, the addition/subtraction of payoffs can be punishing/rewarding. Therefore the form of the action depends on the interactions of the game.

Two-Person Aligning Game

- Payoffs
  - $\pi_1(x_1, x_2) = x_1(Jx_2 + h_1)$
  - $\pi_2(x_1, x_2) = x_2(Jx_1 + h_2)$
    - ► $J > 0$ (aligning), $h_1 > 0$, $h_2 > 0$

- Gratitude configuration is single variable

$$\gamma = \gamma_{12} = \gamma_{21} \in \{-1,1\}$$

  - mutual gratitude (1) / resentment (-1)

- The same J in π_1 and π_2, along with mutual gratitude/resentment gives a potential

$$V(\vec{x},\gamma) = (1+\gamma)\,Jx_1x_2 + h_1x_1 + h_2x_2$$

- Timescale of gratitude/resentment is much slower than economic equilibrium
  - "Quenched" model with Quenched PDF

$$\Phi(\gamma) = \begin{cases} p & \gamma = 1 \quad \text{(gratitude)} \\ 1-p & \gamma = -1 \quad \text{(resentment)} \end{cases}$$

    - p is "empirical frequency of mutual gratitude"
    - i.e., p $(1-p)$ is the empirical frequency of opportunity for i to take action for the benefit (hurt) of other that invokes mutual gratitude (resentment)

Gibbs Equilibrium ($T > 0$)

- Partition Function for N agents at inverse temperature $\beta = T^{-1}$

$$Z_{N,\gamma} := \int e^{\beta V} d\vec{x}$$

  - gratitude configuration variable γ is **quenched** (frozen)
- Finite-Agent Free Potential

$$F_\gamma(\beta, J, h_1, h_2) := \frac{1}{\beta N} \ln Z_{N,\gamma}$$

  - used to find Gibbs equilibrium values when non-rational behavior influences decisions (i.e., $T > 0$)
  - generating function for expected value of decision variables y_i and correlations among them

Quenched Mean

- Recall that the mean of the *observable* g, with respect to the Gibbs equilibrium, is

$$\langle g \rangle_{N,\gamma} = Z_{N,\gamma}^{-1} \cdot \int g(\vec{x})\, e^{\beta V(\vec{x};\gamma)} d\vec{x}$$

- The Quenched mean is then

$$\mathrm{E}_{\gamma}\left[\langle g \rangle_{N,\gamma}\right]$$

  - the mean here is taken with respect to the pdf for the gratitude configuration random variable

Expected Values of Payoffs

- Resentment (mean) payoffs ($\gamma = -1$)

$$\langle \pi_1 \rangle_{\gamma=-1}$$
$$= \left(\bar{x} \coth(\beta h_1 \bar{x}) - \frac{1}{\beta h_1} \right) \left[J \left(\bar{x} \coth(\beta h_2 \bar{x}) - \frac{1}{\beta h_2} \right) + h_1 \right]$$

$$\langle \pi_2 \rangle_{\gamma=-1}$$
$$= \left(\bar{x} \coth(\beta h_2 \bar{x}) - \frac{1}{\beta h_2} \right) \left[J \left(\bar{x} \coth(\beta h_1 \bar{x}) - \frac{1}{\beta h_1} \right) + h_2 \right]$$

- $\gamma > -1$ payoffs computed from below. Note: $\mathrm{Ei(x)} := \int_{-\infty}^{x} e^x/x$

$$\langle x_1 x_2\rangle_{\gamma>-1} = \frac{1}{\beta(1+\gamma)}\left.\frac{\partial}{\partial J}\right|_{\gamma>-1} \ln \mathcal{Z}_\gamma$$

$$= \frac{\beta h_2 \alpha_\gamma - 1}{\beta(1+\gamma)J} + \frac{1}{\beta(1+\gamma)} \frac{\left\{\begin{array}{l} \frac{\mathrm{e}^{(\alpha_\gamma+\bar{x})\bar{u}}}{(\alpha_\gamma+\bar{x})\bar{u}}\left[\frac{-\beta h_1 h_2}{(1+\gamma)J^2} + \beta(1+\gamma)\bar{x}^2\right] - \frac{\mathrm{e}^{(\alpha_\gamma-\bar{x})\bar{u}}}{(\alpha_\gamma-\bar{x})\bar{u}}\left[\frac{-\beta h_1 h_2}{(1+\gamma)J^2} - \beta(1+\gamma)\bar{x}^2\right] \\ \quad + \frac{\mathrm{e}^{(\alpha_\gamma+\bar{x})\underline{u}}}{(\alpha_\gamma+\bar{x})\underline{u}}\left[\frac{-\beta h_1 h_2}{(1+\gamma)J^2} - \beta(1+\gamma)\bar{x}^2\right] - \frac{\mathrm{e}^{(\alpha_\gamma-\bar{x})\underline{u}}}{(\alpha_\gamma-\bar{x})\underline{u}}\left[\frac{-\beta h_1 h_2}{(1+\gamma)J^2} + \beta(1+\gamma)\bar{x}^2\right] \end{array}\right\}}{\left\{\mathrm{Ei}([\alpha_\gamma+\bar{x}]\bar{u}) - \mathrm{Ei}([\alpha_\gamma-\bar{x}]\bar{u})\right\} - \left\{\mathrm{Ei}([\alpha_\gamma+\bar{x}]\underline{u}) - \mathrm{Ei}([\alpha_\gamma-\bar{x}]\underline{u})\right\}}$$

$$\langle x_1\rangle_{\gamma>-1} = \frac{1}{\beta}\left.\frac{\partial}{\partial h_1}\right|_{\gamma>-1} \ln \mathcal{Z}_\gamma$$

$$= \frac{-h_2}{(1+\gamma)J} + \frac{1}{\beta} \frac{\left\{\begin{array}{l} \frac{\mathrm{e}^{(\alpha_\gamma+\bar{x})\bar{u}}}{(\alpha_\gamma+\bar{x})\bar{u}}\left[\frac{\beta h_2}{(1+\gamma)J} + \beta\bar{x}\right] - \frac{\mathrm{e}^{(\alpha_\gamma-\bar{x})\bar{u}}}{(\alpha_\gamma-\bar{x})\bar{u}}\left[\frac{\beta h_2}{(1+\gamma)J} + \beta\bar{x}\right] \\ \quad + \frac{\mathrm{e}^{(\alpha_\gamma+\bar{x})\underline{u}}}{(\alpha_\gamma+\bar{x})\underline{u}}\left[\frac{\beta h_2}{(1+\gamma)J} - \beta\bar{x}\right] - \frac{\mathrm{e}^{(\alpha_\gamma-\bar{x})\underline{u}}}{(\alpha_\gamma-\bar{x})\underline{u}}\left[\frac{\beta h_2}{(1+\gamma)J} - \beta\bar{x}\right] \end{array}\right\}}{\left\{\mathrm{Ei}([\alpha_\gamma+\bar{x}]\bar{u}) - \mathrm{Ei}([\alpha_\gamma-\bar{x}]\bar{u})\right\} - \left\{\mathrm{Ei}([\alpha_\gamma+\bar{x}]\underline{u}) - \mathrm{Ei}([\alpha_\gamma-\bar{x}]\underline{u})\right\}}$$

$$\langle x_2\rangle_{\gamma>-1} = \frac{1}{\beta}\left.\frac{\partial}{\partial h_2}\right|_{\gamma>-1} \ln \mathcal{Z}_\gamma$$

$$= \frac{-h_1}{(1+\gamma)J} + \frac{1}{\beta} \frac{\left\{\begin{array}{l} \frac{\mathrm{e}^{(\alpha_\gamma+\bar{x})\bar{u}}}{(\alpha_\gamma+\bar{x})\bar{u}}\left[\frac{\beta h_1}{(1+\gamma)J} + \beta\bar{x}\right] - \frac{\mathrm{e}^{(\alpha_\gamma-\bar{x})\bar{u}}}{(\alpha_\gamma-\bar{x})\bar{u}}\left[\frac{\beta h_1}{(1+\gamma)J} - \beta\bar{x}\right] \\ \quad + \frac{\mathrm{e}^{(\alpha_\gamma+\bar{x})\underline{u}}}{(\alpha_\gamma+\bar{x})\underline{u}}\left[\frac{\beta h_1}{(1+\gamma)J} + \beta\bar{x}\right] - \frac{\mathrm{e}^{(\alpha_\gamma-\bar{x})\underline{u}}}{(\alpha_\gamma-\bar{x})\underline{u}}\left[\frac{\beta h_1}{(1+\gamma)J} - \beta\bar{x}\right] \end{array}\right\}}{\left\{\mathrm{Ei}([\alpha_\gamma+\bar{x}]\bar{u}) - \mathrm{Ei}([\alpha_\gamma-\bar{x}]\bar{u})\right\} - \left\{\mathrm{Ei}([\alpha_\gamma+\bar{x}]\underline{u}) - \mathrm{Ei}([\alpha_\gamma-\bar{x}]\underline{u})\right\}}$$

- Expected Gratitude Payoffs

$$\langle \pi_1 \rangle_{\gamma > -1} = J \langle x_1 x_2 \rangle_{\gamma > -1} + h_1 \langle x_1 \rangle_{\gamma > -1}$$

$$\langle \pi_2 \rangle_{\gamma > -1} = J \langle x_1 x_2 \rangle_{\gamma > -1} + h_2 \langle x_2 \rangle_{\gamma > -1}$$

- Quenched payoffs: consider effects of mutual gratitude (w_g) /resentment (w_r)

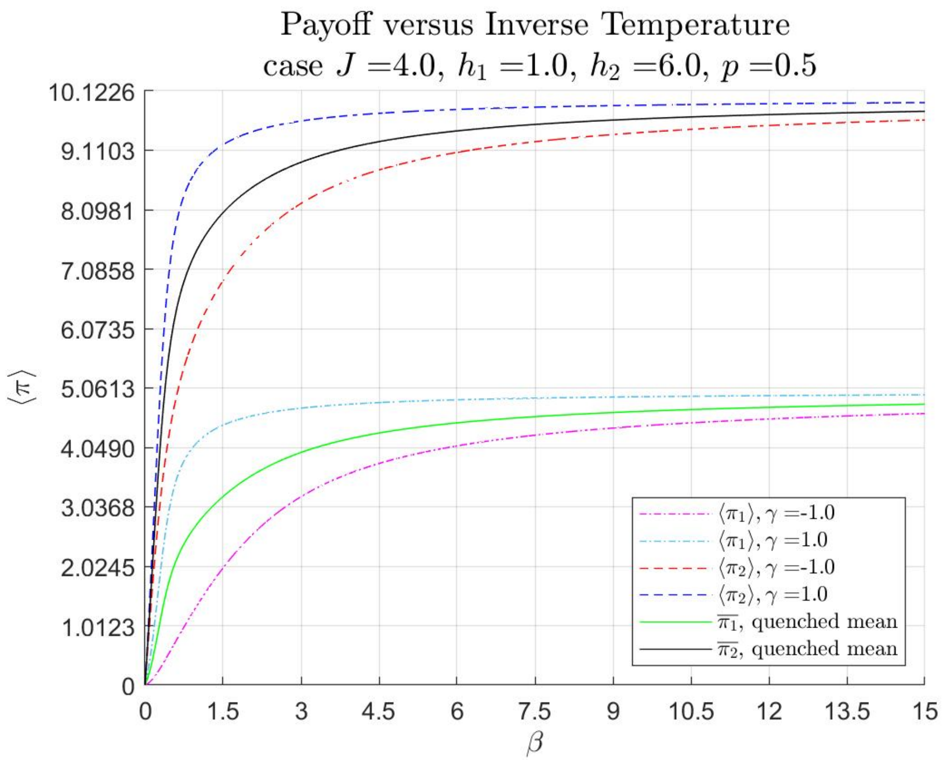
$$\overline{\pi_1} \coloneqq \mathrm{E}_\gamma\left[\langle \pi_1 \rangle_\gamma\right] = \langle \pi_1 \rangle_{\gamma = w_r} \Phi(w_r) + \langle \pi_1 \rangle_{\gamma = w_g} \Phi\left(w_g\right)$$

$$\overline{\pi_2} \coloneqq \mathrm{E}_\gamma\left[\langle \pi_2 \rangle_\gamma\right] = \langle \pi_2 \rangle_{\gamma = w_r} \Phi(w_r) + \langle \pi_2 \rangle_{\gamma = w_g} \Phi\left(w_g\right)$$

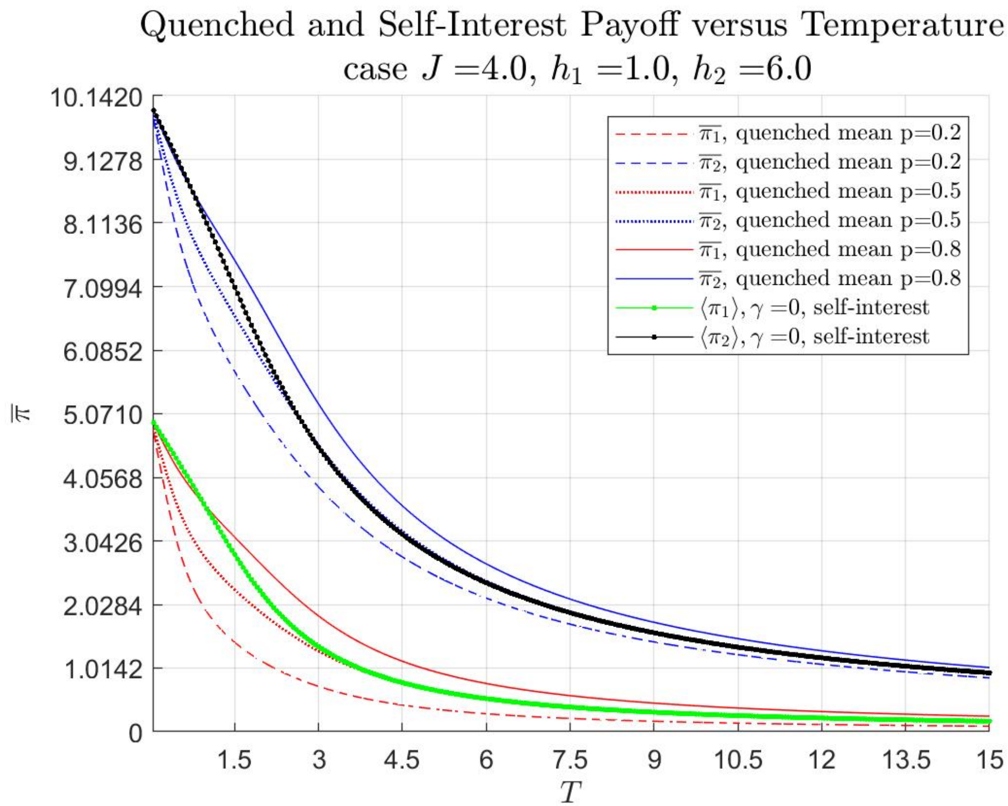
  - we take the probability for mutual gratitude to be $\Phi(w_g) = p,\ 0 \le p \le 1$
  - the probability for mutual resentment is then $\Phi(w_r) = 1 - p$

Numerical Results

- We look at the model above with parameter values
  - $J = 4,$
  - $h_1 = 1,$
  - $h_2 = 6,$
  - $\bar{x} = 1,$
  - $w_g = 1, \quad w_r = -1$



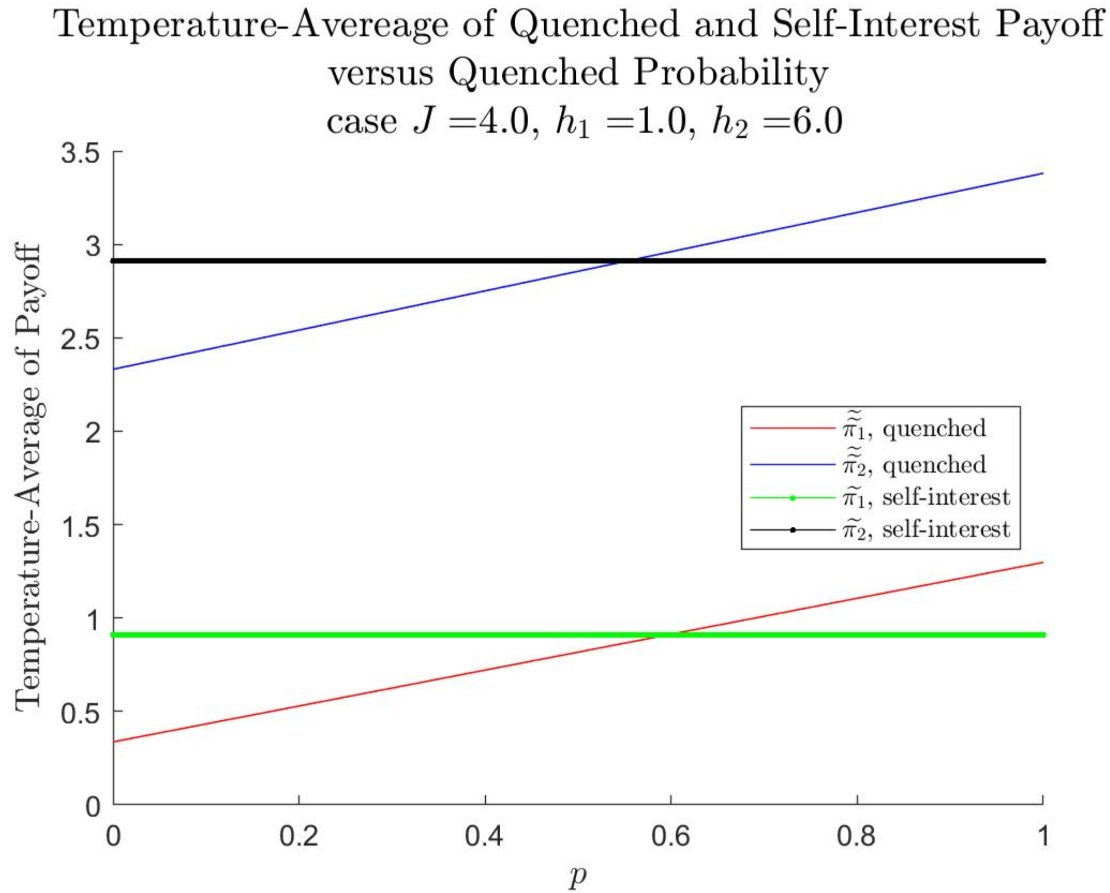
- Two-person quenched game with quench probability of mutual gratitude $p = 0.5$ and mutual resentment $1 - p = 0.5$.
- At $\beta = 0$, behavior is purely random (Gibbs measure is uniform) hence all payoffs are zero.
- As $\beta \to \infty$, the payoffs approach the Nash equilibrium, and the gratitude and resentment payoffs are equal there (see example 7 in paper)
  - Corollary: Nash equilibrium has no predictive value in this case
- Mutual gratitude results in higher payoffs than mutual resentment
- Mean payoffs are increasing in β as a result of correlation inequalities (Appendix A)
- Learning curve – concavity → decreasing marginal gains as knowledge increases



- Two-person quenched game with self-interested payoffs and various quenched payoffs.
- For any fixed temperature, the quenched payoff will strictly increase (linearly) with p, the quenched probability weight on mutual gratitude.
  - Therefore the quenched payoff will be larger than the self-interested payoff when the quenched probability weight is larger than a critical value
  - Critical quenched probability depends on agent and temperature
- Above this critical value
  - agent is motivated to change from acting self-interestedly and to engage the other agent in actions that can result in gratitude/resentment

Average Behavior over Temperature

- Assume agents behave with different levels of rationality
  - uniformly spread out over a range of temperatures
- Then consider the mean payoffs taken over a range of temperatures
  - $\widetilde{\widetilde{\pi_{i;p}}} \coloneqq \frac{1}{T_m} \int_0^{T_m} \overline{\pi_{i;p}} \, dT$
    - ▶ temperature-mean of quenched payoffs
  - $\widetilde{\pi}_i \coloneqq \frac{1}{T_m} \int_0^{T_m} \langle \pi_i \rangle_{\gamma=0} \, dT$
    - ▶ temperature-mean of self-interested payoffs



- uniform distribution of non-rationality over $0 \leq T \leq 15$, for illustration
- Quenched payoff greater than self-interest payoff
  - Agent 1: when $p > p_1^* \approx 0.597$ (bottom graph)
  - Agent 2: when $p > p_2^* \approx 0.553$ (top graph)
- Above critical probabilities p_i^*, agents motivated to engage in actions that result in gratitude/resentment in other agent
- That $p_i^* > 0.5$ shows agents are averse to resentment (Axiom 4 *Humanomics*)
  - Agent 2, who receives lower payoff, is more averse to resent than agent 1
- Effort costs to switch from SI to interactive behavior shift the intersection right

Conclusion

- We introduced simple, tractable models
  - implemented fundamental elements of *Humanomics*
    - mutual gratitude/resentment with reward/punishment in form of higher/lower payoff
  - bounded rationality
  - timescales for economic equilibrium & feelings of gratitude/resentment
- Quenched model (faster economic equilibrium)
  - new insight into critical quenched probabilities
    - agents are resentment-averse, consistent with Axiom 4
    - Nash equilibrium does not have any predictive power for this model
  - infinite-agent homogeneous interaction model is spin glass
    - discrete quenched probability distribution
    - disorder comes from random (high-temperature) behavior, *not* frustration

Future Research

- Create model where timescales of economic equilibrium and feelings of gratitude/resentment are the same
  - economic variables and gratitude/resentment variables would interact
- Spin-Glass Infinite-agent homogeneous interaction quenched model
  - not a traditional spin glass – random, non-frustrated disorder
  - how would we interpret the (likely) phase transition?
- Anti-aligning models ($J < 0$)
- Aligning model with $N = 3,4,5,6, \ldots$ agents
  - do similar results hold as $N = 2$ in this paper?
- Homogeneous gratitude configurations
  - gratitude/resentment need not be mutual; i.e., can be one-sided