Residuated frames for substructural logics, Part I

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Overview

Part I

- Posets
- Join- and meet-semilattices
- Arbitrary joins and complete semilattices
- Lattices and distributive lattices
- Residuated maps
- Polarity frames and Galois lattices
- Residuated lattices, Heyting algebras and Boolean algebras

Part II

- Modal logic and Kripke frames
- Residuated frames
- Lattice expansions and LE-frames
- Gentzen frames
- Introduction to algebraic proof theory
- Display calculi
- Finite model property and finite embeddability property

Some references for this course:

G. Birkhoff: Lattice Theory, 3rd ed, AMS Colloq. Publ., Vol 25 (1967)

R. Wille: Restructuring lattice theory: an approach based on heirachies of concepts, In: Rival I. (ed) Ordered Sets., vol 83. Springer (1982)

P. Jipsen: Categories of Algebraic Contexts Equivalent to Idempotent Semirings and Domain Semirings, in proceedings RAMiCS 2012, LNCS, Vol. 7560, Springer-Verlag (2012), 195–206

N. Galatos, P. Jipsen: Residuated frames with applications to decidability, Trans. of the AMS, 365 (2013), 1219–1249

G. Greco, P. Jipsen, F. Liang, A. Palmigiano, A. Tzimoulis: Algebraic proof theory for LE-logics, arXiv 1808.04642, (2018)

M. A. Moshier: A relational category of formal contexts, preprint

Category theory:

Tom Leinster: Basic category theory, free pdf, 2014

Emily Riehl: Category theory in context, free pdf, 2014

Quote

The aim of theory really is, to a great extent, that of systematically organizing past experience in such a way that the next generation, our students and their students and so on, will be able to absorb the essential aspects in **as painless a way as possible**, and this is the only way in which you can go on cumulatively building up any kind of scientific activity without eventually coming to a dead end.

M. F. Atiyah, “How research is carried out” [Ati74]

Posets

All free variables $x, y, z, \ldots$ in formulas are implicitly universally quantified.

A **poset** $(P, \leq)$ is a set P with a binary relation $\leq$ on P (i.e., $\leq \subseteq P \times P = P^2$) that satisfies

- **reflexivity**: $x \leq x$,
- **antisymmetry**: $x \leq y$ and $y \leq x \implies x = y$ and
- **transitivity**: $x \leq y$ and $y \leq z \implies x \leq z$.

The **dual** of $(P, \leq)$ is $(P, \leq)^\partial = (P, \geq)$, where $x \leq y \iff y \geq x$.

Every poset formula has a **dual** obtained by interchanging $\leq$ and $\geq$.

Elements x, y are **incomparable** if $x \nleq y$ and $y \nleq x$.

Minimal, maximal, bottom and top

An element $m \in P$ is

minimal if $m \geq x \implies m = x$, and

maximal if $m \leq x \implies m = x$.

An element $c \in P$ is a **bottom** if $c \leq x$, and a **top** if $c \geq x$.

Lemma 1: Bottom and top elements, if they exist, are unique.

Prove or disprove: If a poset has a unique minimal element c then c is the bottom.

The **bottom** and **top** element of a poset, if they exists, are denoted by $\bot$ and $\top$.

A **bounded** poset is of the form $(P, \leq, \bot, \top)$ where $\bot, \top$ are the bottom and top element respectively.

Atoms, coatoms, covers, Hasse diagrams

If P has a bottom, then the **set of atoms** of P, denoted by $\mathrm{At}(P)$, is the set of minimal elements of $P \setminus \{\bot\}$.The **set of coatoms** is defined dually.

The **principal filter** and **principal ideal** generated by x in a poset P are defined by $\uparrow_P x = \{y \in P : x \leq y\}$ and $\downarrow_P x = \{y \in P : y \leq x\}$.

The **interval** from x to y is $[x, y]_P = \{z \in P : x \leq z \leq y\}$.

An element y **covers** x in P, denoted $x \prec y$, if $x \neq y$ and $[x, y]_P = \{x, y\}$. We also say x is a **co-cover** of y.

If x has a **unique cover** and/or a **unique co-cover**, they are denoted x^*, x_* respectively.

The **Hasse diagram** for P is the directed graph $(P, \prec)$, drawn so that x is lower than y if $x \prec y$. Arrowheads are usually omitted.

Maps between posets, isomorphisms

A map $f : P \to Q$ is

- **order-preserving** if $x \leq_P y \implies f(x) \leq_Q f(y)$,
- **order-reversing** if $x \leq_P y \implies f(x) \geq_Q f(y)$,
- **order-reflecting** if $f(x) \leq_Q f(y) \implies x \leq_P y$ and
- an **isomorphism** if f is an order-preserving and order-reflecting bijection.

Posets P, Q are **isomorphic** if there exists and isomorphism from P to Q.

Lemma 2: There are 16 posets with 4 elements (up to isomorphism).

Problem: How many posets are there with a top and exactly 5 elements?

Now formulate and prove a general result.

Interval-finite and atomic posets

A poset P is **interval-finite** if every interval has finite cardinality.

The **transitive closure** of a binary relation is the intersection of all transitive binary relations that contain it.

Prove or disprove 3: A poset is interval-finite if and only if the covering relation $\prec$ is the smallest binary relation such that the transitive closure is $\leq$.

A poset with bottom is **atomic** if $x \neq \bot \implies \exists y \in \mathrm{At}(P), y \leq x$.

Prove or disprove 4: every finite poset with bottom is atomic.

Join-semilattices, meet-semilattices, varieties and HSP

A **join-semilattice** $(A, \leq, \vee)$ is a poset $(A, \leq)$ with a binary operation $\vee$ called **join** on A that satisfies $x, y \leq x \vee y$ and

$$x, y \leq z \implies x \vee y \leq z.$$

A **meet-semilattice** is defined dually, and the meet operation is written $\wedge$.

Lemma 5: There are 15 join-semilattices with 5 elements (up to isom.)

An **identity** is a universally quantified atomic formula in a theory with equality as the only relational symbol.

A **variety** is a class of algebras axiomatized by a set of identities.

By Birkhoff's 1935 **HSP theorem** (as formulated by Tarski 1946) a class $\mathscr{V}$ is a variety if and only if $\mathscr{V} = \mathsf{HSP}\mathscr{K}$ for some class $\mathscr{K} \subseteq \mathscr{V}$, where

$\mathsf{H}\mathscr{K} = \{$all homomorphic images of members of $\mathscr{K}\}$,

$\mathsf{S}\mathscr{K} = \{$all subalgebras of members of $\mathscr{K}\}$ and

$\mathsf{P}\mathscr{K} = \{$all direct products of members of $\mathscr{K}\}$.

The variety of join-semilattices; arbitrary meets

Lemma 6: Join-semilattices form a variety, i.e., the operation $\vee$ is

associative: $(x \vee y) \vee z = x \vee (y \vee z)$,

commutative: $x \vee y = y \vee x$ and

idempotent: $x \vee x = x$.

Conversely, if $(A, \vee)$ is an algebra with an associative, commutative, idempotent operation $\vee$ and $x \leq y$ is defined by $x \vee y = y$ then $(A, \leq, \vee)$ is a join-semilattice.

The dual statement holds for meet-semilattices.

For a subset S of a poset P the **arbitrary join** $\bigvee S$ **exists and equals** s, written $\bigvee S = s$, if $x \in S \implies x \leq s$ and $(\forall x \in S, x \leq y) \implies s \leq y$.

Arbitrary meets $\bigwedge$ are defined dually.

Join-irreducibles and completely join-irreducibles

An element x in a join-semilattice A is

join-irreducible if $x = y \vee z \implies x = y$ or $x = z$ and

competely join-irreducible (cji) if $\forall S \subseteq A, x = \bigvee S \implies x \in S$.

$J(A)$ is the set of all join-irreducibles and $J^\infty(A)$ is the set of all cjis.

Meet-irreducible, **cmi**, $M(A)$ and $M^\infty(A)$ are defined dually.

Lemma 7: In a meet-semilattice (a) $\bigvee({\downarrow_P} x \setminus \{x\})$ **always** exists and is either x or x_*.

(b) x is completely join-irreducible if and only if $\bigvee({\downarrow_P} x \setminus \{x\}) = x_*$.

Join-generating sets and atomistic

A subset C of a join-semilattice A is **join-generating** if $A = \{\bigvee S : S \subseteq C \text{ and } \bigvee S \text{ exists}\}$. **Meet-generating** is defined dually.

Lemma 8: If C is join-generating then $J^\infty(A) \subseteq C$.

Prove or disprove: If C is join-generating then $J(A) \subseteq C$.

A join-semilattice A is **perfect** if $J^\infty(A)$ is join-generating.

If A has a bottom, it is **atomistic** if $\mathrm{At}(A)$ is join-generating.

Prove or disprove 8.5: A join-semilattice with bottom is atomistic if and only if it is atomic.

Lattices and 4-crowns

A **4-crown** is a subset $\{a, b, c, d\}$ of 4 distinct elements in a poset such that a, c are incomparable, b, d are incomparable and $a, c \leq b, d$.

Prove or disprove 9: If an interval-finite poset has a top element and no 4-crown then it is a complete join-semilattice.

A **lattice** $(A, \leq, \wedge, \vee)$ is defined as a join-semilattice $(A, \leq, \vee)$ and a meet-semilattice $(A, \leq, \wedge)$ with respect to the **same** order $\leq$.

Lemma 10: Lattices form a variety, axiomatized by: $\vee, \wedge$ are associative and commutative operations that satisfy the **absorption laws** $x \vee (x \wedge y) = x = x \wedge (x \vee y)$.

Complete join- and meet-semilattices; distributivity

A **complete join-semilattice** $(A, \leq, \bigvee)$ is a poset $(A, \leq)$ where arbitrary joins exists for all subsets of A.

A **complete meet-semilattice** $(A, \leq, \bigwedge)$ is defined dually.

A **complete lattice** $(A, \leq, \bigvee, \bigwedge)$ is a complete join- and complete meet-semilattice with respect to the same order.

Lemma 11: Every complete join-semilattice A is a bounded join-semilattice and a complete lattice. For $S \subseteq A$ the arbitrary meet $\bigwedge S = \bigvee\{x \in A : \forall y \in S, x \leq y\}$.

A lattice is **distributive** if it satisfies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Lemma 12: A lattice is distributive if and only if it satisfies one of the following equivalent formulas:

(a) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,

(b) $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$

(c) $x \wedge y = x \wedge z$ and $x \vee y = x \vee z \implies y = z$,

(d) $x \wedge y \leq x \wedge z$ and $x \vee y \leq x \vee z \implies y \leq z$.

Semidistributivity and finite lattices

A lattice is

meet-semidistributive $SD_\wedge$ if $x \wedge y = x \wedge z \implies x \wedge (y \vee z) = x \wedge y$,

join-semidistributive $SD_\vee$ if $x \vee y = x \vee z \implies x \vee (y \wedge z) = x \vee y$.

Lemma 13: There are 15 lattices with 6 elements (up to isomorphism). Formulate a general result relating (interval-)finite join-semilattices and (interval-)finite lattices.

Problem: How many of them are distributive? How many satisfy $SD_\vee$? Find a (possibly bigger) lattice that shows $SD_\vee$ is not equivalent to $SD_\wedge$.

Residuated maps and Galois connections

A **residuated pair** $f \dashv g$ is a pair of maps $f : P \to Q$, $g : Q \to P$ between two posets that satisfy $f(x) \le y \iff x \le g(y)$. The map f is the **left** (or **lower**) **residual** of g, and g is the **right** (or **upper**) **residual** of f.

A **Galois connection** $h \dashv' k$ is a pair of maps $h : P \to Q$, $k : Q \to P$ that satisfy $y \le h(x) \iff x \le k(y)$.

A map $\gamma : P \to P$ is a **closure operation** if it is **extensive**: $x \le \gamma(x)$, **order-preserving** and **idempotent**: $\gamma(\gamma(x)) = \gamma(x)$. The set of **closed elements** of γ is the **image** $\gamma[P] = \{\gamma(x) : x \in P\} = \{x \in P : \gamma(x) = x\}$.

A **basis** for γ is a set $D \subseteq \gamma[P]$ such that $\gamma[P] = \{\bigwedge S : S \subseteq D\}$.

Lem. 14: (a) $f \dashv g \iff f, g$ are order-preserving, $f(g(y)\le y$, $x \le g(f(x)$

(b) $h \dashv' k \iff h, k$ are order-reversing, $x \le k(h(x))$ and $y \le h(k(y))$

(c) f preserves all existing joins, g preserves all existing meets,
h, k map existing meets to joins.

(d) $f(x) = \bigwedge\{y : x \le g(y)\}$, $g(y) = \bigvee\{x : f(x) \le y\}$, $h(x) = \bigvee\{y : x \le k(y)\}$

(e) $f \circ g$ is an interior operation, $g \circ f$, $h \circ k$, $k \circ h$ are closure operations.

Polarity frames and their Galois lattices

A **(polarity) frame** is a triple of sets $\mathbf{W} = (W, W', N)$ s.t. $N \subseteq W \times W'$.

The **powerset** $\mathscr{P}(W) = \{X : X \subseteq W\}$ is a complete lattice with $\subseteq$.

The notation xNY is short for $\forall y \in Y, xNy$, and similarly for XNy, XNY.

Define maps $N^\uparrow : \mathscr{P}(W) \to \mathscr{P}(W')$ by $N^\uparrow X = \{y \in W' : XNy\}$,

$N^\downarrow : \mathscr{P}(W') \to \mathscr{P}(W)$ by $N^\downarrow Y = \{x \in W : xNY\}$,

$\gamma_N : \mathscr{P}(W) \to \mathscr{P}(W)$ by $\gamma_N(X) = N^\downarrow N^\uparrow X$ and $\gamma'_N(Y) = N^\uparrow N^\downarrow Y$.

The **Galois lattice** of $\mathbf{W}$ is $\mathbf{W}^+ = (\gamma_N[\mathscr{P}(W)], \bigcap, \bigvee, W, \gamma_N(\emptyset))$ where $\bigvee_{i\in I} X_i = \gamma({}_N\bigcup_{i\in I} X_i)$.

Lemma 15: (a) The maps $N^\uparrow$ and $N^\downarrow$ form a Galois connection from $\mathscr{P}(W)$ to $\mathscr{P}(W')$, and γ_N, γ'_N are closure operations on $\mathscr{P}(W)$, $\mathscr{P}(W')$.

(b) The Galois lattice of a frame is a complete lattice.

(c) If γ is a closure op. on $\mathscr{P}(W)$, then $\exists W'$, $\exists N \subseteq W \times W'$ s.t. $\gamma = \gamma_N$.

(d) $D = \{N^\downarrow\{y\} : y \in W'\}$ is a basis for γ_N and meet-generating for $\mathbf{W}^+$.

(e) The set $C = \{\gamma_N(\{x\}) : x \in W\}$ is a join-generating set for $\mathbf{W}^+$.

Examples of frames

For a poset $(P, \leq)$ the **Dedekind-MacNeille frame $\mathbf{W}_P$** is the frame $(P, P, \leq)$.

A **Boolean frame** is a frame where $W = W'$ and $N = \neq$.

For a set S, the **partition frame** of S is $\Pi_S = (W, W', N)$ where $W = \{\{x, y\} : x \neq y \in S\}$, $W' = \{\{X, S \setminus X\} : \emptyset \neq X \subsetneq S\}$ and $\{x, y\} N \{X, Y\} \iff \{x, y\} \subseteq X$ or $\{x, y\} \subseteq Y$.

Lemma 16: (a) $\mathbf{W}_P^+$ is the Dedekind-MacNeille completion of P, i.e., the smallest complete lattice in which P is embedded.

(b) The Galois algebra of a Boolean frame $(W, W, \neq)$ is $\mathscr{P}(W)$.

(c) The Galois algebra of a partition frame Π_S is the partition lattice on S.

Residuated lattices

$\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \backslash, /)$ is a **residuated lattice** if $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid (i.e., $\cdot$ is associative and $1x = x = x1$)and the **residuation property holds**:

$$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$

The operation $\cdot$ binds stronger than the **residuals** $\backslash, /$ and they bind stronger than $\wedge, \vee$. Residuated lattices are algebraic models of substructural logics, with $\cdot$ as *fusion* (noncommutative resource conscious dynamic conjunction) and implications $\backslash, /$.

Lemma 17: In a residuated lattice $1 \leq x\backslash x$, $x(x\backslash y) \leq y$, $x \leq y\backslash yx$, $xy\backslash z = y\backslash(x\backslash z)$, $x(y \vee z) = xy \vee xz$, $(x \vee y)z = xz \vee yz$, $(x \vee y)\backslash z = x\backslash z \wedge y\backslash z$, $x\backslash(y \wedge z) = x\backslash y \wedge x\backslash z$ and $(x\backslash y)/z = x\backslash(y/z)$.

The variety RL and bounded integral RLs

Lemma 18: The residuation property can be expressed by 4 identities hence the class RL of residuated lattices is a variety.

A residuated lattice is **integral** if 1 is the top element (i.e., $x \le 1$ holds).

Lemma 19: $xy \le x \wedge y$ and $x\backslash x = 1$ hold in every integral residuated lattice.

$\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \bot, \backslash, /)$ is a **bounded residuated lattice** if $(A, \wedge, \vee, \cdot, 1, \backslash, /)$ is a residuated lattice and $\bot$ is the bottom element.

Lemma 20: $\bot x = \bot = x\bot$ and $x \le \bot\backslash\bot$, hence $\bot\backslash\bot$ is the top element $\top$.

Heyting algebras and Boolean algebras as RLs

A residuated lattice is **commutative** if it satisfies the identity $xy = yx$.

Lemma 21: Commutativity is equivalent to $x\backslash y = y/x$. In this case we define $x \to y = x\backslash y$.

A **Heyting algebra** is a bounded residuated lattice that satisfies $xy = x \wedge y$.

Lemma 22: For a residuated lattice, $xy = x \wedge y$ is equivalent to $xx = x \leq 1$.

$\mathbf{A} = (A, \wedge, \vee, \neg, 0, 1)$ is a **Boolean algebra** (BA) if $(A, \wedge, \vee)$ is a bounded distributive lattice and $x \wedge \neg x = 0$ and $x \vee \neg x = 1$ for all $x \in A$. Define $x \to y = \neg x \vee y$.

Lemma 23: If $(A, \wedge, \vee, \neg, 0, 1)$ is a BA then $(A, \wedge, \vee, \wedge, 1, 0, \to)$ is a Heyting algebra.

Homework

a) Prove **ALL** the lemmas for which you don't already know the proof.

b) Go to `http://math.chapman.edu/~jipsen/js/`, select the **Random matrix** program and use it to generate 4x4, 4x5 and 5x5 01-matrices (use random_matrix(4,4,0,1) etc.).

Find some matrices with no repeated rows or columns and use them as N relation, then construct the Galois lattice $L = \mathbf{W}^+$ for such matrices.

c) Extract the reduced frame $L_+ = (J(L), M(L), \leq)$. Note that ${L_+}^+ \cong L$ even though N and L^+ may be different.

d) Go to `http://www1.chapman.edu/~jipsen/FCA/`, figure out how to enter your frame and let your browser draw the lattice.

Thanks! **There will be a test on the last day...**