Residuated frames for substructural logics, Part II

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Overview

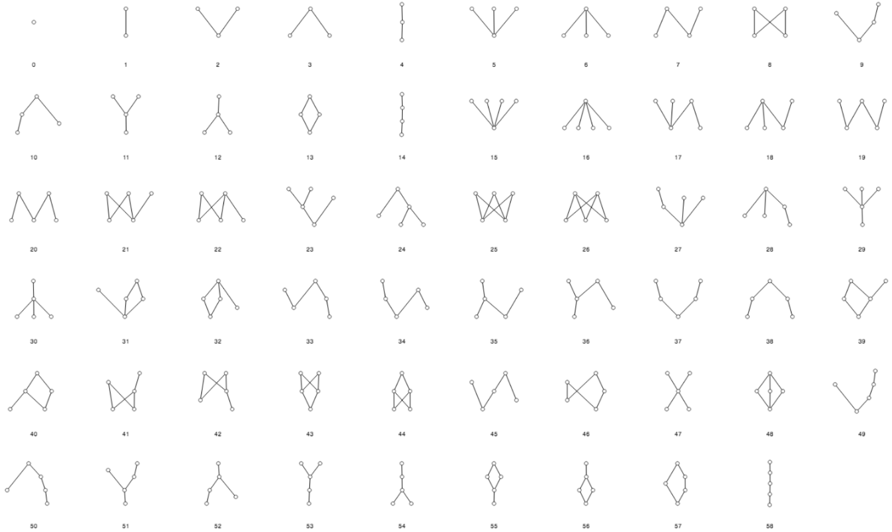
Part I

- Posets
- Join- and meet-semilattices
- Arbitrary joins and complete semilattices
- Lattices and distributive lattices
- Residuated maps
- Polarity frames and Galois lattices
- Residuated lattices, Heyting algebras and Boolean algebras

Part II

- Modal algebras and Kripke frames
- Residuated frames
- Sequent systems
- Gentzen frames
- Introduction to algebraic proof theory
- Decidability of equational theory of RL
- Finite model property and finite embeddability property

The 59 connected posets of size ≤ 5



`http://math.chapman.edu/~jipsen/posets/posets59.html`

Modal algebras and Kripke frames

A **modal algebra** is of the form $\mathbf{A} = (\mathbf{A}_0, \Diamond)$ where $\mathbf{A}_0 = (A, \wedge, \vee, \neg, \bot, \top)$ is a Boolean algebra and $\Diamond : A \to A$ is a strict ($\Diamond\bot = \bot$) join-preserving ($\Diamond(x \vee y) = \Diamond x \vee \Diamond y$) unary operation.

A **Kripke frame** is a binary relational structure $\mathbf{W} = (W, R)$, i.e., $R \subseteq W^2$.

From $\mathbf{W}$ we define the **complex algebra** $\mathbf{W}^+ = (\mathscr{P}(W), \cap, \cup, -, \emptyset, W, \Diamond)$, where $\Diamond(X) = R^{-1}[X]$.

Lemma 24: The complex algebra of a Kripke frame is a modal algebra.

From any modal algebra one can define a Kripke frame using ultrafilters.

Here we first consider the simpler setting of **atomic** modal algebras.

Lemma 25: For Boolean algebras atomic and atomistic are equivalent.

Kripke frames of complete and atomic modal algebras

A modal algebra $\mathbf{A}$ is **atomic** if its Boolean reduct $\mathbf{A}_0$ is atomic.

The operation $\Diamond$ is uniquely determined by its restriction to $W_{\mathbf{A}} = \mathrm{At}(\mathbf{A}_0)$.

$\mathbf{A}_+ = (W_{\mathbf{A}}, R)$, where $aRb \iff a \le \Diamond b$

Lemma 26: (a) An atomic modal algebra $\mathbf{A}$ embeds into ${\mathbf{A}_+}^+$ by $x \mapsto W_{\mathbf{A}} \cap \downarrow x$.

(b) If $\mathbf{A}$ is complete and atomic and $\Diamond$ preserves arbitrary joins then this embedding is an isomorphism.

(c) Any Kripke frame $\mathbf{W}$ is isomorphic to ${\mathbf{W}^+}_+$.

This is the basis for the **duality** between **Kripke frames** and **complete and atomic modal algebras**.

Extends to Kripke frames with **many** binary operations R_i $(i \in I)$ and complete and atomic multi-modal algebras with **many** unary $\Diamond_i$ $(i \in I)$

Extends further to **Kripke frames of an arbitrary first-order signature** and **complete and atomic Boolean algebras with operators** BAOs (sometimes called polymodal algebras).

Extending the duality beyond Boolean algebras

Jónsson-Tarski [AJoM 1951, '52] developed the theory of **BAOs** and their canonical extensions, motivated by **relation algebras**.

Their results provide a **duality in algebraic form**, that predates Kripke's work by more than a decade

Residuated lattices are generalizations of relation algebras, so Nick Galatos and I wanted to have similar technology for **lattices with strict join-preserving operations**.

Goldblatt [APAL 1989], **Gehrke-Jónsson** [MJapn 1994, '00, MScan '04] developed the theory of **distributive lattices with operations**, duality and canonical extensions.

Birkhoff [1940], **Markowski** [1973], **Urquhart** [AU 1978], **Hartung** [AU 1992], **Hartonas and Dunn** [AU 1997], **Gehrke and Harding** [JAlg 2000] developed dualities and canonical extensions for **lattices**.

Goldblatt [BLMS 1975], **Harding** [Tatra 1998] provided dualities and canonical extensions for ortholattices.

Extending the duality to residuated lattices

Dunn [1990, JSL '93], **Dunn, Gehrke and Palmigiano** [JSL 2005], **Gehrke** [SL 2006] developed dualities for substructural logics based on polarity frames.

Girard [TCS 1987], **Okada and Terui** [JSL 1999] use phase spaces to give semantics for linear logics. Aligned well with proof theory, allowing them to give semantic proofs of cut-elimination and decidability.

Galatos and J. [2008, TAMS '13] Residuated frames with applications to decidability: links **polarity frames** with **proof theory** and **residuated lattices** in a flexible way.

Gives algebraic semantics to syntactic **sequent calculi**.

Extends **correspondence theory** from modal logic to the **nondistributive setting**.

Applies to **all reducts** of lattice expansions with **fully residuated** operations.

For simplicity we consider mostly **residuated lattices** and **tense lattices**.

Residuated frames

A **residuated frame** is a structure $\mathbf{W} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/)$ where W and W' are sets, $N \subseteq W \times W'$, $\circ \subseteq W^3$, $\backslash\!\!\backslash \subseteq W \times W' \times W$ and $/\!\!/ \subseteq W' \times W \times W$ such that for all $x, y \in W$, $w \in W'$

$$\text{(Nuc)} \qquad (x \circ y)\, N\, w \;\Leftrightarrow\; y\, N\, (x \backslash\!\!\backslash w) \;\Leftrightarrow\; x\, N\, (w /\!\!/ y)$$

Here $x \circ y = \{z : (x, y, z) \in \circ\}$ and similarly for $\backslash\!\!\backslash$, $/\!\!/$

We again use $X\, N\, y$ to abbreviate $\forall x \in X,\ x\, N\, y$ and likewise for $x\, N\, Y$

W is said to be **associative** if it satisfies $N^{\uparrow}((x \circ y) \circ z) = N^{\uparrow}(x \circ (y \circ z))$, i.e., if it satisfies the following equivalence for all $x, y, z \in W, w' \in W'$

$$\forall w(\exists u(u \in x \circ y \text{ and } w \in u \circ z) \implies wNw') \iff \forall w(\exists v(w \in x \circ v \text{ and } v \in$$

It is said to **have a unit** $E \subseteq W$ if $N^{\uparrow}(x \circ E) = N^{\uparrow}\{x\} = N^{\uparrow}(E \circ x)$, i.e., if for all $x \in W, w' \in W'$

$$\forall w \forall e \in E(w \in x \circ e \implies wNw') \iff xNw' \iff \forall w \forall e \in E(w \in$$

Nuclei

A **nucleus** γ on a residuated lattice **L** is a closure operator on L such that $\gamma(x)\gamma(y) \leq \gamma(xy)$ (or $\gamma(\gamma(x)\gamma(y)) = \gamma(xy)$).

Lemma 27. Given a residuated lattice $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ and a nucleus on **L**, the algebra $\mathbf{L}_\gamma = (\gamma[L], \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1))$, is a residuated lattice, where $x \cdot_\gamma y = \gamma(x \cdot y)$, $x \vee_\gamma y = \gamma(x \vee y)$.

Lemma 28. For a frame **W**, γ_N is a nucleus on $(\mathscr{P}(W), \cap, \cup, \circ, \backslash, /)$.

Corollary 29. If **W** is an associative unital residuated frame then the **Galois algebra** $\mathbf{W}^+ = (\gamma_N[\mathscr{P}(W)], \cap, \cup_{\gamma_N}, \circ_{\gamma_N}, \backslash, /, \gamma_N(E))$ is a complete residuated lattice.

Lemma 30. If **L** is a residuated lattice then $\mathbf{W_L} = (L, L, \leq, \cdot, \backslash, /, 1)$ is an associative unital residuated frame.

We will see later that for $\mathbf{W_L}$, $x \mapsto N^{\downarrow}\{x\}$ is an embedding.

Frames of complete perfect lattices

Recall that a lattice **L** is **perfect** if every element is a join of elements of $J^\infty(\mathbf{L})$ and a meet of elements of $M^\infty(\mathbf{L})$.

Lemma 31: A Boolean algebra is perfect if and only if it is atomic.

For a perfect residuated lattice **A**, let $\mathbf{A}_+ = (J^\infty(\mathbf{A}), M^\infty(\mathbf{A}), \leq, \circ, \backslash\!\!\backslash, /\!\!/, E)$ where $x \circ y = \{z \in J^\infty(\mathbf{A}) : z \leq xy\}$ and $E = \{z \in J^\infty(\mathbf{A}) : z \leq 1\}$

Lemma 32: $\mathbf{A}_+$ is a residuated frame and if **A** is complete and perfect and $\cdot$ preserves arbitrary joins then $(\mathbf{A}_+)^+ \cong \mathbf{A}$.

In particular, any finite lattice is complete and perfect.

A residuated frame **W** is **reduced** if $\gamma_N(\gamma_N(x) - \{x\}) \neq \gamma_N(x)$ for all $x \in W$, and the same holds for γ'_N and all $x \in W'$.

Lemma 33: A frame **W** is reduced $\iff$ $\gamma(x) = \gamma(z) \implies x = z$ and $\gamma(x)$ is completely join-irreducible in the Galois lattice of **W**.

Lemma 34: For a reduced residuated frame $\mathbf{W}^+{}_+ \cong \mathbf{W}$

So for finite residuated lattices, residuated frames give a compact dual representation analogous to Kripke frames for modal algebras.

Connections with proof theory

Let Fm be the set of all terms in the language $\{\vee, \wedge, \cdot, \backslash, /, 1\}$.

Let $\circ$ be a binary symbol, ε a constant symbol and define $(W, \circ, \varepsilon)$ to be the free monoid with unit ε generated by Fm.

A **single conclusion sequent** is a pair $(x, a) \in W \times Fm$, usually written $x \Rightarrow a$.

A **sequent rule** is a pair $(\{s_1, \ldots, s_n\}, s_0)$ where $s_0, \ldots, s_n$ are sequents.

Rules are presented: $\dfrac{s_1 \quad s_2 \quad \ldots \quad s_n}{s_0}$ (name), axioms: $\overline{s_0}$ (ax)

A **sequent proof** is a rooted tree labelled by sequents where each node and its children are labelled by uniform substitution instances of sequent rules.

RLsq: Sequent rules for residuated lattices

$$\frac{x \Rightarrow a \quad y \circ a \circ z \Rightarrow c}{y \circ x \circ z \Rightarrow c}\ (\text{cut}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id})$$

$$\frac{y \circ a \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \mathrm{L}\ell) \quad \frac{y \circ b \circ z \Rightarrow c}{y \circ a \wedge b \circ z \Rightarrow c}\ (\wedge \mathrm{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \mathrm{R})$$

$$\frac{y \circ a \circ z \Rightarrow c \quad y \circ b \circ z \Rightarrow c}{y \circ a \vee b \circ z \Rightarrow c}\ (\vee \mathrm{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \mathrm{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \mathrm{R}r)$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ x \circ (a \backslash b) \circ z \Rightarrow c}\ (\backslash \mathrm{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \mathrm{R})$$

$$\frac{x \Rightarrow a \quad y \circ b \circ z \Rightarrow c}{y \circ (b/a) \circ x \circ z \Rightarrow c}\ (/\mathrm{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathrm{R})$$

$$\frac{y \circ a \circ b \circ z \Rightarrow c}{y \circ a \cdot b \circ z \Rightarrow c}\ (\cdot \mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot \mathrm{R})$$

$$\frac{y \circ z \Rightarrow a}{y \circ 1 \circ z \Rightarrow a}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathrm{R})$$

where $a, b, c \in Fm$, $x, y, z \in W$.

RLsq versus equational logic

Example of a *cut-free* **RL** proof $\dfrac{\dfrac{\dfrac{\dfrac{z\Rightarrow z \quad x\Rightarrow x}{z\circ(z\backslash x)\Rightarrow x}}{z\circ(z\backslash x\wedge z\backslash y)\Rightarrow x} \quad \dfrac{\dfrac{z\Rightarrow z \quad y\Rightarrow y}{z\circ(z\backslash y)\Rightarrow y}}{z\circ(z\backslash x\wedge z\backslash y)\Rightarrow y}}{z\circ(z\backslash x\wedge z\backslash y)\Rightarrow x\wedge y}}{z\backslash x\wedge z\backslash y\Rightarrow z\backslash(x\wedge y)}$

Much simpler than equational proofs since **replacement** is not allowed.

A toy implementation: `http://www1.chapman.edu/~jipsen/reslat/`

Exercise 35: Find an equational proof of
$(z\backslash x\wedge z\backslash y)\vee(z\backslash(x\wedge y))=z\backslash(x\wedge y)$ from the identities of RL:

$$
\begin{array}{ccc}
(x\vee y)\vee z=x\vee(y\vee z) & (xy)z=x(yz) & x(x\backslash z\wedge y)\vee z=z\\
(x\wedge y)\wedge z=x\wedge(y\wedge z) & x1=x=x1 & x\backslash(xz\vee y)\wedge z=z\\
x\vee y=y\vee x & x\vee(x\wedge y)=x & (y\wedge z/x)x\vee z=z\\
x\wedge y=y\wedge x & x\wedge(x\vee y)=x & (y\vee zx)/x\wedge z=z
\end{array}
$$

RLsq with context notation for nonassociative case

$$\frac{x \Rightarrow a \quad u[a] \Rightarrow c}{u[x] \Rightarrow c}\ (\text{cut}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id})$$

$$\frac{u[a] \Rightarrow c}{u[a \wedge b] \Rightarrow c}\ (\wedge \text{L}\ell) \quad \frac{u[b] \Rightarrow c}{u[a \wedge b] \Rightarrow c}\ (\wedge \text{L}r) \quad \frac{x \Rightarrow a}{x \Rightarrow a \wedge b}\ (\wedge \text{R})$$

$$\frac{u[a] \Rightarrow c}{u[a \vee b] \Rightarrow c}\ (\vee \text{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \text{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \text{R}r)$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[x \circ (a \backslash b)] \Rightarrow c}\ (\backslash \text{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a \backslash b}\ (\backslash \text{R})$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[(b/a) \circ x] \Rightarrow c}\ (/\text{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\text{R})$$

$$\frac{u[a \circ b] \Rightarrow c}{u[a \cdot b] \Rightarrow c}\ (\cdot \text{L}) \qquad \frac{x \Rightarrow a}{x \circ y \Rightarrow a \cdot b}\ (\cdot \text{R})$$

$$\frac{u[\varepsilon] \Rightarrow a}{u[1] \Rightarrow a}\ (1\text{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\text{R})$$

Basic substructural logics

If the sequent s is provable in **RLsq** from the set of *sequents* S, we write $S \vdash_{\mathbf{RL}} s$.

$$\frac{u[x \circ y] \Rightarrow c}{u[y \circ x] \Rightarrow c}\ (e) \qquad \text{(exchange)} \qquad xy \le yx$$

$$\frac{u[x \circ x] \Rightarrow c}{u[x] \Rightarrow c}\ (c) \qquad \text{(contraction)} \qquad x \le x^2$$

$$\frac{u[\varepsilon] \Rightarrow c}{u[x] \Rightarrow c}\ (i) \qquad \text{(integrality)} \qquad x \le 1$$

We write $\mathbf{RL_{ec}}$ for $\mathbf{RL} + (e) + (c)$.

Example of frame based on a sequent system

Consider the Gentzen system **RLsq**.

We define the frame $\mathbf{W_{RL}}$, where

$(W, \circ, \varepsilon)$ is the free monoid over the set Fm of all formulas.

$W' = S_W \times Fm$, where S_W is the set of all *unary linear polynomials* $u[x] = y{\circ}x{\circ}z$ of W, and

$x\, N\, (u, a)$ iff $\vdash_{\mathbf{RL}} u[x] \Rightarrow a$.

For $(u, a) /\!\!/ x = \{(u[\_ \circ x], a)\}$ and $x \backslash\!\!\backslash (u, a) = \{(u[x \circ \_], a)\}$, we have

$$\begin{aligned} x \circ y N(u, a) \quad & \text{iff} \ \vdash_{\mathbf{RL}} u[x \circ y] \Rightarrow a \\ & \text{iff} \ \vdash_{\mathbf{RL}} u[x{\circ}y] \Rightarrow a \\ & \text{iff} \ \ xN(u[\_ \circ y], a) \\ & \text{iff} \ \ yN(u[x \circ \_], a). \end{aligned}$$

Example of partial subalgebra frame

Let $\mathbf{A}$ be a residuated lattice and $\mathbf{B}$ a partial subalgebra of $\mathbf{A}$.

We define the frame $\mathbf{W}_{\mathbf{A},\mathbf{B}}$, where

$(W, \cdot, 1)$ to be the submonoid of $\mathbf{A}$ generated by B,

$W' = S_B \times B$, where S_W is the set of all *unary linear polynomials* $u[x] = y \circ x \circ z$ of $(W, \cdot, 1)$, and

$x\,N\,(u, b)$ by $u[x] \leq_{\mathbf{A}} b$.

For $(u, a) \mathbin{/\!\!/} x = \{(u[\_ \cdot x], a)\}$ and $x \mathbin{\backslash\!\!\backslash} (u, a) = \{(u[x \cdot \_], a)\}$, we have

$$\begin{aligned} x \cdot y N(u, a) \quad & \text{iff } u[x \cdot y] \leq a \\ & \text{iff } x N(u[\_ \cdot y], a) \\ & \text{iff } y N(u[x \cdot \_], a). \end{aligned}$$

The sequent calculus **GN**

$$\frac{xNa \quad aNz}{xNz}\ (\text{CUT}) \qquad \frac{}{aNa}\ (\text{Id})$$

$$\frac{xNa \quad bNz}{x \circ (a\backslash b)Nz}\ (\backslash\text{L}) \qquad \frac{a \circ xNb}{xNa\backslash b}\ (\backslash\text{R})$$

$$\frac{xNa \quad bNz}{(b/a) \circ xNz}\ (/\text{L}) \qquad \frac{x \circ aNb}{xNb/a}\ (/\text{R})$$

$$\frac{a \circ bNz}{a \cdot bNz}\ (\cdot\text{L}) \qquad \frac{xNa}{x \circ yNa \cdot b}\ (\cdot\text{R})$$

$$\frac{aNz}{a \wedge bNz}\ (\wedge\text{L}\ell) \qquad \frac{bNz}{a \wedge bNz}\ (\wedge\text{L}r) \qquad \frac{xNa}{xNa \wedge b}\ (\wedge\text{R})$$

$$\frac{aNz}{a \vee bNz}\ (\vee\text{L}) \qquad \frac{xNa}{xNa \vee b}\ (\vee\text{R}\ell) \qquad \frac{xNb}{xNa \vee b}\ (\vee\text{R}r)$$

$$\frac{\varepsilon Nz}{1Nz}\ (1\text{L}) \qquad \frac{}{\varepsilon N1}\ (1\text{R})$$

Gentzen frames

The following properties hold for $\mathbf{W_L}$, $\mathbf{W_{RL}}$ and $\mathbf{W_{A,B}}$:

$\mathbf{W}$ is a residuated frame

$\mathbf{B}$ is a (partial) algebra of the same type, $(\mathbf{B} = \mathbf{L}, \mathbf{Fm}, \mathbf{B})$

B generates $(W, \circ, \varepsilon)$ (as a monoid)

W' contains a copy of B $(b \leftrightarrow (id, b))$

N satisfies **GN**, for all $a, b \in B$, $x, y \in W$, $z \in W'$.

We call such pairs $(\mathbf{W}, \mathbf{B})$ *Gentzen frames*.

A *cut-free Gentzen frame* is not assumed to satisfy the (CUT)-rule.

Theorem. Given a Gentzen frame $(\mathbf{W}, \mathbf{B})$, the map $N^{\downarrow}\{\} : \mathbf{B} \to \mathbf{W}^{+}$, $b \mapsto N^{\downarrow}\{b\}$ is a (partial) homomorphism.

(Namely, if $a, b \in B$ and $a \bullet b \in B$ ($\bullet$ is a connective) then $N^{\downarrow}\{a \bullet_{\mathbf{B}} b\} = N^{\downarrow}\{a\} \bullet_{\mathbf{W}^+} N^{\downarrow}\{b\}$).

Proof

Key Lemma. Let $(\mathbf{W}, \mathbf{B})$ be a cut-free Gentzen frame. For all $a, b \in B$, $X, Y \in \mathbf{W}^+$ and for every connective $\bullet$, if $a \bullet b \in B$, $a \in X \subseteq N^{\downarrow}\{a\}$ and $b \in Y \subseteq N^{\downarrow}\{b\}$, then

$a \bullet_{\mathbf{B}} b \in X \bullet_{\mathbf{W}^+} Y \subseteq N^{\downarrow}\{a \bullet_{\mathbf{B}} b\}$ $(1_{\mathbf{B}} \in 1_{\mathbf{W}^+} \subseteq N^{\downarrow}\{1_{\mathbf{B}}\})$

In particular, $a \bullet_{\mathbf{B}} b \in N^{\downarrow}\{a\} \bullet_{\mathbf{W}^+} N^{\downarrow}\{b\} \subseteq N^{\downarrow}\{a \bullet_{\mathbf{B}} b\}$.

Furthermore, given (CUT), we have equality.

Proof Let $\bullet = \vee$. If $x \in X$, then $x \in N^{\downarrow}\{a\}$; so xNa and $xNa \vee b$, by $(\vee R\ell)$; hence $x \in N^{\downarrow}\{a \vee b\}$ and $X \subseteq N^{\downarrow}\{a \vee b\}$. Likewise $Y \subseteq N^{\downarrow}\{a \vee b\}$, so $X \cup Y \subseteq N^{\downarrow}\{a \vee b\}$ and $X \vee Y = \gamma(X \cup Y) \subseteq N^{\downarrow}\{a \vee b\}$.

On the other hand, let $X \vee Y \subseteq N^{\downarrow}\{z\}$, for some $z \in W$. Then, $a \in X \subseteq X \vee Y \subseteq N^{\downarrow}\{z\}$, so aNz. Similarly, bNz, so $a \vee bNz$ by $(\vee L)$, hence $a \vee b \in N^{\downarrow}\{z\}$. Thus, $a \vee b \in X \vee Y$.

We used that every closed set is an intersection of *basic closed sets* $N^{\downarrow}\{z\}$, for $z \in W$.