Residuated frames for substructural logics, Part III

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Overview

Part III

- Introduction to algebraic proof theory
- Okada lemma
- Decidability of equational theory of RL
- Finite model property
- Finite embeddability property

Part IV

- Categorical duality for complete lattices
- Duality for join-semilattices and algebraic frames
- Correspondence theory briefly
- De Morgan lattices, ortholattices and involutive frames
- Involutive FL-algebras and involutive residuated frames
- Bunched implication algebras and their frames
- Open problems and projects

The sequent calculus **GN**

$$\frac{xNa \quad aNz}{xNz}\ (\mathrm{CUT}) \qquad \frac{}{aNa}\ (\mathrm{Id})$$

$$\frac{xNa \quad bNz}{x\circ(a\backslash b)Nz}\ (\backslash\mathrm{L}) \qquad \frac{a\circ xNb}{xNa\backslash b}\ (\backslash\mathrm{R})$$

$$\frac{xNa \quad bNz}{(b/a)\circ xNz}\ (/\mathrm{L}) \qquad \frac{x\circ aNb}{xNb/a}\ (/\mathrm{R})$$

$$\frac{a\circ bNz}{a\cdot bNz}\ (\cdot\mathrm{L}) \qquad \frac{xNa \quad yNb}{x\circ yNa\cdot b}\ (\cdot\mathrm{R})$$

$$\frac{aNz}{a\wedge bNz}\ (\wedge\mathrm{L}\ell) \qquad \frac{bNz}{a\wedge bNz}\ (\wedge\mathrm{L}r) \qquad \frac{xNa \quad xNb}{xNa\wedge b}\ (\wedge\mathrm{R})$$

$$\frac{aNz \quad bNz}{a\vee bNz}\ (\vee\mathrm{L}) \qquad \frac{xNa}{xNa\vee b}\ (\vee\mathrm{R}\ell) \qquad \frac{xNb}{xNa\vee b}\ (\vee\mathrm{R}r)$$

$$\frac{\varepsilon Nz}{1Nz}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon N1}\ (1\mathrm{R})$$

$a, b \in Fm$ or B, $x, y \in W$, $z \in W'$, xNa is short for $xN(id, a)$ meaning $x \Rightarrow a$.

Semantics of sequent calculi: Residuated frames

Let $\mathbf{GN}_{cf}$ be the sequent calculus **GN without** the **cut rule**

Define a binary relation $N \subseteq W \times W' = F_{\mathrm{Mon}(\circ,\varepsilon)}(Fm) \times (S_W \times Fm)$ by

$$wN(u,a) \iff u[w] \Rightarrow a \quad \text{is provable in } \mathbf{GN}_{cf}$$

Define the **accessibility** relations $\circ \subseteq W^3$, $\backslash\!\!\backslash, /\!\!/$ by

$$\circ(v_1, v_2, w) \iff v_1 \circ v_2 = w$$

$$\backslash\!\!\backslash = \{((u,a), x, (u[\_ \circ x], a)) : u \in W, a \in Fm, x \in W\}$$

$$/\!\!/ = \{(x, (u,a), (u(x \circ \_), a)) : u \in W, a \in Fm, x \in W\}$$

$$E = \{\varepsilon\}$$

Lemma 36: $\mathbf{W_{RL}} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/, E)$ is a **residuated frame**

Algebraic cut-admissibility

Theorem (Okada, Terui 1999, Galatos, J. 2013)

The following are equivalent:

1. $t \Rightarrow a$ *is provable in* $\boldsymbol{GN}$
2. $t \le a$ *holds in* $\boldsymbol{RL}$
3. $t \Rightarrow a$ *is provable in* $\boldsymbol{GN}_{cf}$

Proof (outline): (3⇒1) is obvious. (1⇒2) Assume $t \Rightarrow a$ is provable **with cut**. Show that **all sequent rules** hold as quasiequations in **RL** (where $\Rightarrow, \circ$ are **replaced by** $\le, \cdot$)

(2⇒3) Assume $t \le a$ holds in **RL** and define the algebra $\mathbf{W}^+_{\mathbf{RL}} = (\gamma_N[\mathscr{P}(W)], \cup_{\gamma_N}, \cap, \cdot, 1, \backslash, /)$ using the **closed sets** $\gamma_N(X)$ of the **polarity frame** (W, W', N) and $1 = \gamma_N(\{\varepsilon\})$

$X \cdot Y = \gamma_N(\{w : \circ(u, v, w)$ for some $u \in X, v \in Y\})$

$X \backslash Y = \{w \in W : X \cdot \{w\} \subseteq Y\}$ $\qquad$ $Y/X = \{w \in W : \{w\} \cdot X \subseteq Y\}$.

Proof outline (continued)

Then $\mathbf{W}_{\mathbf{RL}}^{+}$ is a residuated lattice, hence satisfies $t \le a$.

Let $f : Fm \to \mathbf{B} = Fm$ be any **homomorphism**.

Extend to $\bar{f} : W \to \mathbf{W}_{\mathbf{RL}}^{+}$ by $\bar{f}(p) = N^{\downarrow}\{f(p)\}$, so $t \le a$ implies $\bar{f}(t) \subseteq \bar{f}(a)$

Recall, for all $b \in Fm$, $N^{\downarrow}\{b\} = \{w \in W : wN(x_0, b)\}$.

Prove by **induction** that $f(b) \in \bar{f}(b) \subseteq N^{\downarrow}\{f(b)\}$ for all $b \in Fm$.

Then $f(t) \in \bar{f}(t) \subseteq \bar{f}(a) \subseteq N^{\downarrow}\{f(a)\}$, hence $f(t)N(x_0, f(a))$.

Using $f = id$, it follows that $\quad t \Rightarrow a \quad$ holds in $\mathbf{GN}_{cf}$. $\square$

Gentzen frames

The following properties hold for $\mathbf{W_L}$, $\mathbf{W_{RL}}$ and $\mathbf{W_{A,B}}$:

$\mathbf{W}$ is a residuated frame

$\mathbf{B}$ is a (partial) algebra of the same type, $(\mathbf{B} = \mathbf{L}, \mathbf{Fm}, \mathbf{B})$

B generates $(W, \circ, \varepsilon)$ (as a monoid)

W' contains a copy of B $(b \leftrightarrow (id, b))$

N satisfies **GN**, for all $a, b \in B$, $x, y \in W$, $z \in W'$.

We call such pairs $(\mathbf{W}, \mathbf{B})$ *Gentzen frames*.

A *cut-free Gentzen frame* is not assumed to satisfy the (CUT)-rule.

Theorem. Given a Gentzen frame $(\mathbf{W}, \mathbf{B})$, the map
$N^{\downarrow}\{\} : \mathbf{B} \to \mathbf{W}^{+}$, $b \mapsto N^{\downarrow}\{b\}$ is a (partial) homomorphism.

(Namely, if $a, b \in B$ and $a \bullet b \in B$ ($\bullet$ is a connective) then
$N^{\downarrow}\{a \bullet_{\mathbf{B}} b\} = N^{\downarrow}\{a\} \bullet_{\mathbf{W}^+} N^{\downarrow}\{b\}$).

Proof

Key Lemma. Let $(\mathbf{W}, \mathbf{B})$ be a cut-free Gentzen frame. For all $a, b \in B$, $X, Y \in \mathbf{W}^+$ and for every connective $\bullet$, if $a \bullet b \in B$, $a \in X \subseteq N^{\downarrow}\{a\}$ and $b \in Y \subseteq N^{\downarrow}\{b\}$, then

$a \bullet_{\mathbf{B}} b \in X \bullet_{\mathbf{W}^+} Y \subseteq N^{\downarrow}\{a \bullet_{\mathbf{B}} b\}$ (and $1_{\mathbf{B}} \in 1_{\mathbf{W}^+} \subseteq N^{\downarrow}\{1_{\mathbf{B}}\}$)

In particular, $a \bullet_{\mathbf{B}} b \in N^{\downarrow}\{a\} \bullet_{\mathbf{W}^+} N^{\downarrow}\{b\} \subseteq N^{\downarrow}\{a \bullet_{\mathbf{B}} b\}$.

Furthermore, given (CUT), we have equality.

Proof Let $\bullet = \vee$. If $x \in X$, then $x \in N^{\downarrow}\{a\}$; so xNa and $xNa \vee b$, by $(\vee R\ell)$; hence $x \in N^{\downarrow}\{a \vee b\}$ and $X \subseteq N^{\downarrow}\{a \vee b\}$. Likewise $Y \subseteq N^{\downarrow}\{a \vee b\}$, so $X \cup Y \subseteq N^{\downarrow}\{a \vee b\}$ and $X \vee Y = \gamma(X \cup Y) \subseteq N^{\downarrow}\{a \vee b\}$.

On the other hand, let $X \vee Y \subseteq N^{\downarrow}\{z\}$, for some $z \in W$. Then, $a \in X \subseteq X \vee Y \subseteq N^{\downarrow}\{z\}$, so aNz. Similarly, bNz, so $a \vee bNz$ by $(\vee L)$, hence $a \vee b \in N^{\downarrow}\{z\}$. Thus, $a \vee b \in X \vee Y$.

We used that every closed set is an intersection of *basic closed sets* $N^{\downarrow}\{z\}$, for $z \in W$.

Applications of frames: MacNeille-completion

For a residuated lattice $\mathbf{L}$, we associated the Gentzen frame $(\mathbf{W_L}, \mathbf{L})$.

The underlying poset of $\mathbf{W_L^+}$ is the *MacNeille completion* of the underlying poset reduct of $\mathbf{L}$.

Theorem. The map $x \mapsto N^{\downarrow}\{x\}$ is an embedding of $\mathbf{L}$ into $\mathbf{W_L^+}$.

(A general residuated frame is)

Completeness - Cut elimination

Corollary. **RLsq** is complete with respect to $\mathbf{W}^+_{\mathbf{RL}}$.

Corollary. The algebra $\mathbf{W}^+_{\mathbf{RL}}$ generates the variety RL.

The frame $\mathbf{W}_{\mathbf{RL_{cf}}}$ corresponds to cut-free $\mathbf{RLsq_{cf}}$.

Corollary (Cut Elimination). **RLsq** and $\mathbf{RLsq_{cf}}$ prove the same sequents.

Corollary. The equational theory of RL is decidable.

Finite model property, FMP

For $\mathbf{W}_{\mathbf{RLsq}}$, given $(x, z) \in W \times W'$ (if $z = (u, c)$, then $u(x) \Rightarrow c$ is a sequent), we define $(x, z)^{\uparrow}$ as the smallest subset of $W \times W'$ that contains (x, z) and is closed upwards with respect to the rules of $\mathbf{RLsq_{cf}}$. Note that $(x, z)^{\uparrow}$ is finite.

A new frame $\mathbf{W}^{\uparrow}$ associated with $N' = N \cup ((x, z)^{\uparrow})^c$ is residuated and Gentzen.

$(N')^c$ is finite, so has **finite** domain $Dom((N')^c)$ and codomain $Cod((N')^c)$

For every $y \notin Cod((N')^c)$, $N'^{\downarrow}\{y\} = W$. So, $\{N'^{\downarrow}\{y\} : y \in W\}$ is finite and a basis for $\gamma_{N'}$. So $\mathbf{W}^{\uparrow +}$ is finite.

Moreover, if $u(x) \Rightarrow c$ is not provable in **RLsq**, then it is not valid in $\mathbf{W}^{\uparrow +}$.

Corollary . The system **RLsq** has the finite model property.

Corollary. The variety **RL** is generated by its finite members.

Finite embedding property, FEP

A class of algebras $\mathscr{K}$ has the **finite embeddability property (FEP)** if for every $\mathbf{A} \in \mathscr{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathscr{K}$.

The corresponding logic has the *strong finite model property*:

if $\Phi \not\models \psi$, for finite Φ, then there is a finite counter-model, namely there is $\mathbf{D} \in \mathscr{K}$ and a homomorphism $f : \mathbf{Fm} \to \mathbf{D}$, such that $f(\phi) = 1$, for all $\phi \in \Phi$, but $f(\psi) \neq 1$.

For logics with finitely many axioms and rules, this implies that the deducibility relation is decidable

For a finitely axiomatized variety, FEP implies the decidability of the universal theory

FEP for integral RLs with $\{\vee, \cdot, 1\}$-equations

Blok and van Alten 2002 proved FEP for integral RLs, and extended it to residuated groupoids (2005)

Theorem. Every variety of integral RL's axiomatized by equations over $\{\vee, \cdot, 1\}$ has the FEP.

$\mathbf{B}$ embeds in $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ via $N^{\downarrow}\{\_\} : \mathbf{B} \to \mathbf{W}^+_{\mathbf{A},\mathbf{B}}$

$\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ is finite

$\mathbf{W}^+_{\mathbf{A},\mathbf{B}} \in \mathscr{V}$

Corollary. These varieties are generated as quasivarieties by their finite members.

Corollary. The corresponding logics have the *strong finite model property*

Finiteness

Idea for finiteness: Every element in $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ is an intersection of basic elements. So it suffices to prove that there are only finitely many such elements.

Replace the frame $\mathbf{W}_{\mathbf{A},\mathbf{B}}$ by one $\mathbf{W}^{\mathbf{M}}_{\mathbf{A},\mathbf{B}}$, where it is easier to work.

Let $\mathbf{M}$ be the free monoid with unit over the set B and $f : M \to W$ the extension of the identity map.

$$M \xrightarrow{f} W \overset{N}{\text{---}} W'.$$

$\mathbf{M}$ is partially ordered by $s \leq^{\mathbf{M}} t$ if s is a subsequence of t.

Blok and van Alten show that this order has no infinite antichains and no infinite ascending chains, so by Higman's lemma it is dually well-ordered.

Integrality ($x \leq 1$) is used to show that any downset in $\mathbf{M}$ is $\downarrow M_0$ for some finite set $M_0 \subseteq M$.

Equations 1

Idea: Express equations over $\{\vee, \cdot, 1\}$ at the frame level.

For an equation ε over $\{\vee, \cdot, 1\}$ we distribute products over joins to get $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$. s_i, t_j: monoid terms.

$s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$.

The first is equivalent to: $\&(s_j \leq t_1 \vee \cdots \vee t_n)$.

We proceed by example: $x^2 y \leq xy \vee yx$

$(x_1 \vee x_2)^2 y \leq (x_1 \vee x_2)y \vee y(x_1 \vee x_2)$

$x_1^2 y \vee x_1 x_2 y \vee x_2 x_1 y \vee x_2^2 y \leq x_1 y \vee x_2 y \vee y x_1 \vee y x_2$

$x_1 x_2 y \leq x_1 y \vee x_2 y \vee y x_1 \vee y x_2$

$$\frac{x_1 y \leq v \quad x_2 y \leq v \quad y x_1 \leq v \quad y x_2 \leq v}{x_1 x_2 y \leq v}$$

$$\frac{x_1 \circ y \; N \; z \quad x_2 \circ y \; N \; z \quad y \circ x_1 \; N \; z \quad y \circ x_2 \; N \; z}{x_1 \circ x_2 \circ y \; N \; z} \; R(\varepsilon)$$

Equations 2

Theorem. If $(\mathbf{W}, \mathbf{B})$ is a Gentzen frame and ε an equation over $\{\vee, \cdot, 1\}$, then $(\mathbf{W}, \mathbf{B})$ satisfies $\mathrm{R}(\varepsilon)$ iff $\mathbf{W}^+$ satisfies ε.

(The linearity of the denominator of $R(\varepsilon)$ plays an important role in the proof.)

Corollary. If an equation over $\{\vee, \cdot, 1\}$ is valid in $\mathbf{A}$, then it is also valid in $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$, for every partial subalgebra $\mathbf{B}$ of $\mathbf{A}$.

Consequently, $\mathbf{W}^+_{\mathbf{A},\mathbf{B}} \in \mathscr{V}$.

Structural rules

Given an equation ε of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\cdot, 1\}$-terms we construct the rule $R(\varepsilon)$

$$\frac{u[t_1] \Rightarrow a}{u[t_0] \Rightarrow a} \;(R(\varepsilon))$$

where the t_i's are evaluated in $(W, \circ, \varepsilon)$. Such a rule is called *linear* if all variables in t_0 are distinct.

Theorem. Every system obtained from **RLsq** by adding linear rules has the cut elimination property.

A set of rules of the form $R(\varepsilon)$ is called *reducing* if there is a complexity measure that decreases with upward applications of the rules (and the rules of **RLsq**).

Theorem. Every system obtained from **RLsq** by adding linear reducing rules is decidable. The subvariety of residuated lattices axiomatized by the corresponding equations has decidable equational theory.

Applications

- Cut-elimination (CE) and finite model property (FMP) for **FL** and (cyclic) **InFL**. Generation by finite members for RL, InFL
- M. Kozak 2008 proved distributive **FL** has the FMP, and using our approach the same result holds for any extension of **DFL** with linear reducing structural rules
- The finite embeddability property (FEP) for integral RL with $\{\vee, \cdot, 1\}$-axioms
- The above extend to the non-associative case, also with the addition of suitable structural rules

Recent papers using residuated frames

Terui 2018 MacNeille completion and Buchholz' Omega rule for parameter-free second order logics, arXiv:1804.11066v1

Galatos and Prenosil 2018 On an equivalence between integral and involutive residuated structures, preprint

Buszkowski 2017 Involutive nonassociative Lambek calculus, sequent systems and complexity, Bull. Sec. of Logic, 46, 75–91

Cardona and Galatos 2017 The FEP for some varieties of fully distributive knotted residuated lattices, Alg. Univ., 78, 363–376

Ciabattoni, Galatos and Terui 2017 Algebraic proof theory: hypersequents and hypercompletions, Ann. of Pure and App. Logic, 168

Baldi and Terui 2016 Densification of FL chains via residuated frames, Alg. Univ., 75(2), 169–195

Horčík 2015 Word problem for knotted residuated lattices, J. of Pure and App. Alg., 219, 1548–1563

Ciabattoni, Galatos and Terui 2012 Algebraic proof theory for substructural logics: cut-elimination and completions, Ann. of Pure and App. Logic, 163, 266–290

Filter-ideal frames and canonical extensions

Let L be a (bounded) lattice.

Recall that the **Dedekind-MacNeille frame** is $\mathbf{W}_L = (L, L, \leq)$.

$\mathbf{W}_L^+$ is the **MacNeille completion** of L.

A **filter** $F \subseteq L$ is a meet-closed upset: $z \geq x, y \in F \implies x \wedge y, z \in F$.

Ideals of L are defined dually.

Let $\mathscr{F} L = \{$nonempty filters of $L\}$, $\mathscr{I} L = \{$nonempty ideals of $L\}$

Define $N \subseteq \mathscr{F} L \times \mathscr{I} L$ by $\quad\quad F\,N\,D$ iff $F \cap D \neq \emptyset$

The **filter-ideal frame** is $FI_L = (\mathscr{F} L, \mathscr{I} L, N)$

The **canonical extension** of L is $L^\delta = FI_L^+$

Completions

A **completion** of a lattice L is an embedding $e : L \to \bar{L}$ where $\bar{L}$ is a complete lattice

The MacNeille completion is **join-dense** and **meet-dense** (every $x \in \mathbf{W}_L^+$ is a **join** of elements of L and a **meet** of elements of L)

$\implies$ all existing joins and meets of L are **preserved**

The **canonical extension** is **dense** (every $x \in L^\delta$ is a **join of a meet** of elements of L and a **meet of a join** of elements of L)

and **compact** (if $\bigwedge A \le \bigvee B$ for $A, B \subseteq L$ then $a_1 \wedge \cdots \wedge a_m \le b_1 \vee \cdots \vee b_m$ for some $a_i \in A, b_j \in B$).

$\implies$ all infinite joins and meets of L are destroyed

So these polarity frames are **two extremes** of lattice completions

Polarity frames in general are a good way to study **dense completions**