Residuated frames for substructural logics, Part IV

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Overview

Part III

- Introduction to algebraic proof theory
- Okada lemma
- Decidability of equational theory of RL
- Finite model property
- Finite embeddability property

Part IV

- **End of summer school test**
- Categorical duality for complete lattices
- Duality for join-semilattices and algebraic frames
- Correspondence theory briefly
- Bunched implication algebras and their frames
- Some open problems

Final Exam for **30000** points

Problem 1. [**10000** points] (a) Given reduced polarity frames $\mathbf{V},\mathbf{W}$ how does one construct $(\mathbf{V}^+ \times \mathbf{W}^+)_+$?

Hint: consider, e.g., the Boolean algebras $\mathbf{V} = \mathbf{2}$ and $\mathbf{W} = \mathbf{2}^2$.

(b) If $\mathbf{V},\mathbf{W}$ are residuated frames, how are $\circ_{\mathbf{V}}$ and $\circ_{\mathbf{W}}$ combined to get the ternary relation for $(\mathbf{V}^+ \times \mathbf{W}^+)_+$?

Problem 2. [**20000** points] The **coalesced ordinal sum** $L \oplus M$ of two bounded lattices L, M is given by the disjoint union $(L \setminus \{\top\}) \uplus M$ with the partial order given by $\leq_{L\setminus\{\top\}} \uplus \leq_M \uplus (L \setminus \{\top\}) \times M$.

(a) Draw the ordinal sum when $L = \mathbf{2}^\mathbf{2}$ and $M = \mathbf{2}^\mathbf{3}$.

(b) Find the polarity frame of this 11-element lattice.

(c) For any bounded perfect lattices L, M how is $(L \oplus M)_+$ constructed from L_+ and M_+?

(d) If L, M are integral residuated lattices, define $\circ$, E on $(L \oplus M)_+$ in a natural way to get a residuated frame. Is integrality needed?

1000 bonus points for any mistake you report in the notes.

Frame morphisms

Complete lattices with complete homomorphisms form a category

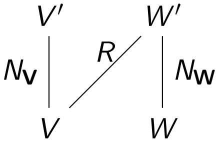
What are the **appropriate** morphisms for polarity frames?

For a frame $\mathbf{W} = (W, W', N)$ the relation N is an identity morphism that induces the identity map $\gamma = N^{\downarrow}N^{\uparrow}$ on the closed sets

A **frame morphism** $R : \mathbf{V} \to \mathbf{W} = (V, W', R)$ is a relation $R \subseteq V \times W'$ such that $N_{\mathbf{V}}^{\downarrow}N_{\mathbf{V}}^{\uparrow}R^{\downarrow} = R^{\downarrow} = R^{\downarrow}N_{\mathbf{W}}^{\uparrow}N_{\mathbf{W}}^{\downarrow}$

or equivalently $R^{\uparrow}N_{\mathbf{V}}^{\downarrow}N_{\mathbf{V}}^{\uparrow} = R^{\uparrow} = N_{\mathbf{W}}^{\uparrow}N_{\mathbf{W}}^{\downarrow}R^{\uparrow}$

In either case we say that R is **compatible**



Compatible morphisms $\equiv^\partial$ meet-semilattice homomorphisms

Lemma. *If R is compatible then $R^{\downarrow}N^{\uparrow}_{\mathbf{W}} : \mathbf{W}^+ \to \mathbf{V}^+$ preserves $\bigcap$*

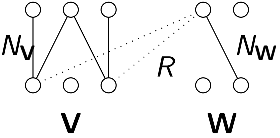
Proof: Let $\{A_k : k \in K\}$ be a family of Galois closed sets of W

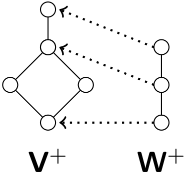
Since $R^{\downarrow}N^{\uparrow}_{\mathbf{W}}N^{\downarrow}_{\mathbf{W}} = R^{\downarrow}$,

$$R^{\downarrow}N^{\uparrow}_{\mathbf{W}} \bigcap_{k\in K} A_k = R^{\downarrow} \bigvee_{k\in K} N^{\uparrow}_{\mathbf{W}}A_k = R^{\downarrow}N^{\uparrow}_{\mathbf{W}}N^{\downarrow}_{\mathbf{W}} \bigcup_{k\in K} N^{\uparrow}_{\mathbf{W}}A_k = R^{\downarrow} \bigcup_{k\in K} N^{\uparrow}_{\mathbf{W}}A_k$$

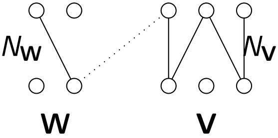
$= \bigcap_{k\in K} R^{\downarrow}N^{\uparrow}_{\mathbf{W}}A_k$ because $R^{\downarrow}\bigcup_{k\in K} B_k = \bigcap_{k\in K} R^{\downarrow}B_k$ always

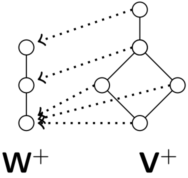
The result is in $\mathbf{V}^+$ since $R^{\downarrow} = N^{\downarrow}_{\mathbf{V}}N^{\uparrow}_{\mathbf{V}}R^{\downarrow}$ $\square$



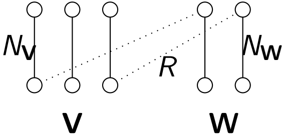


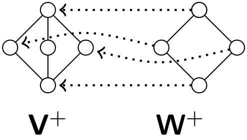
More examples





Urquhart and Hartung only define duals of surjective lattice homomorphims. Here is the dual of a non-surjective homomorphism:

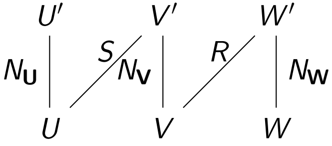




The category of polarity frames

Theorem. [Moshier 2012] (i) The collection **PFrm** of all **frames with compatible relations** as morphisms is a category

If $S : \mathbf{U} \to \mathbf{V}$ and $R : \mathbf{V} \to \mathbf{W}$ then **composition** is given by



$xS\underline{;}Ry$ iff $x \in S^{\downarrow} N_{\mathbf{V}}^{\uparrow} R^{\downarrow}\{y\}$

(ii) The category **PFrm** is dual to the category **INF** of **complete semilattices with completely meet-preserving homomorphisms**

The adjoint functors are $^{+} : \mathbf{PFrm} \to \mathbf{INF}$ and $\mathbf{W}_{(-)} : \mathbf{INF} \to \mathbf{PFrm}$

On morphisms, $R^{+} = R^{\downarrow} N_{\mathbf{W}}^{\uparrow} : \mathbf{W}^{+} \to \mathbf{V}^{+}$ and for an

INF morphism $h : L \to M$, we have $\mathbf{W}_h = \{(x, y) \in M \times L : x \leq h(y)\}$

Lattice compatible morphisms

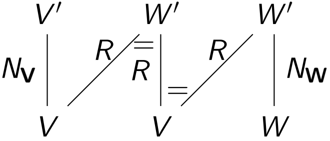
Lemma: $R^{\downarrow}N_{\mathbf{W}}^{\uparrow}$ preserves $\bigvee$ iff there exists a compatible relation $R_*: W \to V'$ such that $R^{\downarrow}N_{\mathbf{W}}^{\uparrow} = N_{\mathbf{V}}^{\downarrow}R_*^{\uparrow}$ (call R lattice compatible)

Theorem: The category **LPFrm** of all frames with lattice compatible relations as morphisms is dual to the category **CLat** of complete lattices with complete lattice morphisms.

Lemma: (i) $R : \mathbf{V} \to \mathbf{W}$ is a monomorphism in **PFrm** iff $R^{\downarrow}R^{\uparrow} = N_{\mathbf{V}}^{\downarrow}N_{\mathbf{V}}^{\uparrow}$

(ii) $R : \mathbf{V} \to \mathbf{W}$ is a epimorphism in **PFrm** iff $R^{\uparrow}R^{\downarrow} = N_{\mathbf{W}}^{\uparrow}N_{\mathbf{W}}^{\downarrow}$

Note that every morphism has **itself** as epi-mono factorization



Reduced frames

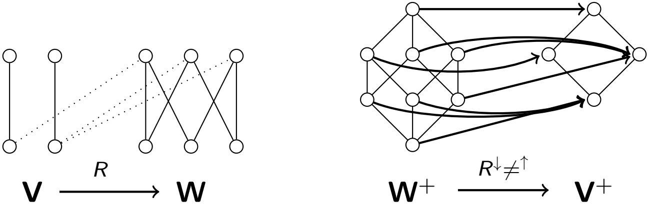
LPFrm is "much larger" than **CLat** since many different (but isomorphic) frames represent the same lattice

For a finite lattice L the **reduced frame** $(J(L), M(L), \leq)$ is isomorphic to $\mathbf{W}_L$

For finite L the reduced frame can be **logarithmic** in size of L

E.g. a finite BA B has $(At(B), coAt(B), \leq) \cong (At(B), At(B), \neq)$ as reduced frame

Morphisms between Boolean polarity frames



On complete and atomic Boolean algebras, any relation is a **PFrm** morphism

So there are $2^{2\cdot 3} = 64$ meet-homomorphisms from **V** to **W**

For the example above, only six relations are **LPFrm** morphisms

Overview of Dualities

Algebras w. homs		Spaces w. "cont" maps
CABool	$\equiv^\partial$	**Sets**
Bool	$\equiv^\partial$	**Stone**
CPerfDLat	$\equiv^\partial$	**Poset**
BDLat	$\equiv^\partial$	**Priestley** or **Spectral**
INF or **SUP**	$\equiv^\partial$	**PFrm** (Moshier)
CLat	$\equiv^\partial$	**LPFrm**
$\mathbf{JSLat}_\perp$	$\equiv^\partial$	**AlgPFrm**
Lat w. surj. homs	$\equiv^\partial$	Urquhart 78/Hartung 92

Algebraic Frames

$\mathbf{W} = (W, W', N)$ is an **algebraic polarity frame** if
$N^{\downarrow}N^{\uparrow}A = \bigcup\{N^{\downarrow}N^{\uparrow}F : F \subseteq_\omega A\}$ for all $A \subseteq W$

$\iff$ $N^{\downarrow}N^{\uparrow}$ preserves directed unions

AlgPFrm is the class of all **algebraic polarity frames**

$R : \mathbf{V} \to \mathbf{W}$ is an **AlgPFrm morphism** if $R^{\downarrow}N^{\uparrow}_{\mathbf{W}}$ preserves directed unions

The **compact sets** are $\mathscr{K}(\mathbf{W}) = \{N^{\downarrow}N^{\uparrow}F : F \subseteq_\omega W\}$

Theorem *The category* $\mathbf{JSLat}_\perp$ *is dual to* **AlgPFrm**. *The adjoint functors are* $\mathscr{K} : \mathbf{ACxt} \to \mathbf{JSLat}_\perp$ *and* $\mathbf{W}_\mathscr{I} : \mathbf{JSLat}_\perp \to \mathbf{ACxt}$, *where* $\mathbf{W}_{\mathscr{I},L} = (L, \mathscr{I}L, N)$ *and* $N = \{(a, D) : a \in D\}$.

On morphisms, $\mathscr{K}(R) = R^{\downarrow}N^{\uparrow} : \mathscr{K}(Y) \to \mathscr{K}(X)$ *and for a* $\mathbf{JSLat}_\perp$ *morph.* $h : L \to M$, $\mathbf{W}_{\mathscr{I},h} = \{(a, D) \in M \times \mathscr{I}L : h^{-1}[a] \cap D \neq \emptyset\}$

Frames of idempotent Join-semirings

Let **W** be an algebraic frame. A relation $\circ \subseteq W^3$ is called **algebraic** if for all $A, B \in \mathscr{K}(W)$ the set $N^{\downarrow}N^{\uparrow}(A \circ B)$ is also in $\mathscr{K}(W)$.

An ***idempotent semiring frame*** is of the form $(W, W', N, \circ, E)$ such that

(W, W', N) is an algebraic polarity frame,

$\circ, E$ are an algebraic ternary and unary relation on W,

the closure operator γ_N is a nucleus, i.e., $\gamma_N(X) \circ \gamma_N(Y) \subseteq \gamma_N(X \circ Y)$,

and for all $x, y, z \in W$ we have

$$N^{\uparrow}((x \circ y) \circ z) = N^{\uparrow}(x \circ (y \circ z)) \text{ and}$$

$$N^{\uparrow}(x \circ E) = N^{\uparrow}\{x\} = N^{\uparrow}(E \circ x).$$

Idempotent Matrix Semirings

Given a semiring L, let $M_n(L)$ be the semiring of all $n \times n$ matrices with entries from L.

This object has $|L|^{n^2}$ many elements, but for idempotent semirings the frame $\mathbf{W}$ of $M_n(L)$ is much smaller since it can be constructed from n^2 disjoint copies of the idempotent semiring frame $\mathbf{V} = \mathbf{W}_{\mathscr{I},L}$.

Let $W = \{(i,j,a) : a \in V, i,j = 1,\ldots,n\}$ and
$W' = \{(i,j,a) : a \in V', i,j = 1,\ldots,n\}$

Define $(i,j,a)N_{\mathbf{W}}(i',j',a') \iff i \neq i'$ or $j \neq j'$ or $aN_{\mathbf{V}}a'$

$E = \{(i,i,a) : a \in E, i = 1,\ldots,n\}$, and

$(i,j,a) \circ (k,l,b) = \{(i,l,c) : j = k$ and $c \in a \circ b\}$

Then $(W, W', N_{\mathbf{W}}, \circ, E)$ is the frame of the matrix semiring $M_n(L)$

Counting finite reduced separating frames

A frame is **reduced** if $\gamma(\gamma(x) - \{x\}) \neq \gamma(x)$ and same for γ'

For any perfect lattice $(J(L), M(L), \leq)$ is always **reduced**

How many reduced frames are there with $|W| = m$ and $|W'| = n$?

$m = 0$, $n = 0$: $\mathbf{W}_1 = (\emptyset, \emptyset, \emptyset)$, $\mathbf{W}^+$ = trivial lattice

$m = 0$, $n = 1$: no reduced frame

$m = 1$, $n = 1$: $\mathbf{W}_2 = (\{0\}, \{0\}, \emptyset)$, $\mathbf{W}^+$ = 2-element lattice

$m = 1$, $n = 2$: no reduced frame

$m = 2$, $n = 2$: $\mathbf{W}_3 = (\{0, 1\}, \{0, 1\}, \{(0, 0)\})$, $\mathbf{W}^+$ = 3-elt lattice

$\mathbf{W}_4 = (\{0, 1\}, \{0, 1\}, \{(0, 0), (1, 1)\})$, $\mathbf{W}^+$ = 4-element lattice

Counting finite reduced polarity frames

There are **no** reduced frames for $m=2$, $n=3$

Number of reduced separating frames (up to isomorphism)

	$n=3$	4	5	6	7	8	9
$m=3$	7	2	0	0	0	0	0
4	2	45	50	25	4	0	0
5	0	50	717	2241	3670	3598	2181
6	0	25	2241	37535	266178	?	?
7	0	4	3670	266178	?	?	?
8	0	0	3598	?	?	?	?
9	0	0	2181	?	?	?	?

(semi)distributive and selfdual lattices lie along the diagonal

Residuated frames

Recall: a **residuated lattice** is an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1)$ such that $(A, \vee, \wedge)$ is a **lattice**, $(A, \cdot, 1)$ is a **monoid** and for all $x, y, z \in A \qquad xy \le z \iff y \le x\backslash z \iff x \le z/y$.

A **residuated frame** is a structure $\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!/, E)$ such that (W, W', N) is a polarity frame, $\circ \subseteq W^3$, $\backslash\!\backslash \subseteq W \times W' \times W$, $/\!/ \subseteq W' \times W^2$ and for all $x, y, z \in W$ and $w \in W'$

1. $(x \circ y) N w \iff y N (x \backslash\!\backslash w) \iff x N (w /\!/ y) \qquad$ ($\circ$ is **nuclear**).
2. $N^{\uparrow}((x \circ y) \circ z) = N^{\uparrow}(x \circ (y \circ z))$
3. $N^{\uparrow}(E \circ x) = N^{\uparrow}\{x\} = N^{\uparrow}(x \circ E)$

Note: $x \circ y = \{z : \circ(x, y, z)\}$, $X \circ Y = \{z : \circ(x, y, z), x \in X, y \in Y\}$

Note that 1.-3. can be written as 1st-order formulas on $\mathbf{W}$

1. **corresponds** to $\circ_\gamma$ being a residuated operation on $\mathbf{W}^+$

Morphisms for residuated frames

Let $\mathbf{V}, \mathbf{W}$ be residuated frames. A relation $R \subseteq W \times W'$ is a morphism $R : \mathbf{V} \to \mathbf{W}$ if R is **lattice compatible** and

1. $R^{\downarrow} N^{\uparrow}(x \circ_{\mathbf{W}} y) = (R^{\downarrow} N^{\uparrow}\{x\}) \circ_{\mathbf{V}} (R^{\downarrow} N^{\uparrow}\{y\})$
2. $R^{\downarrow} N^{\uparrow}(x \backslash_{\mathbf{W}} N^{\downarrow} y) = (R^{\downarrow} N^{\uparrow}\{x\}) \backslash_{\mathbf{V}} (R^{\downarrow}\{y\})$
3. $R^{\downarrow} N^{\uparrow}(N^{\downarrow} x /_{\mathbf{W}} y) = (R^{\downarrow}\{x\}) /_{\mathbf{V}} (R^{\downarrow} N^{\uparrow}\{y\})$
4. $R^{\downarrow} N^{\uparrow}(E_{\mathbf{W}}) = \gamma(E_{\mathbf{V}})$

Theorem. The category of reduced separating residuated frames and morphisms is dually equivalent to the category of complete perfect residuated lattices and homomorphisms.

Correspondence examples

Recall:

A is **integral** if it satisfies $x \leq 1$

A is **commutative** if it satisfies $xy = yx$

A is **modular** if it satisfies $x \leq z \implies (x \vee y) \wedge z = x \vee (y \wedge z)$

Let **W** be a residuated frame. Then $\mathbf{W}^+$ is **integral** iff

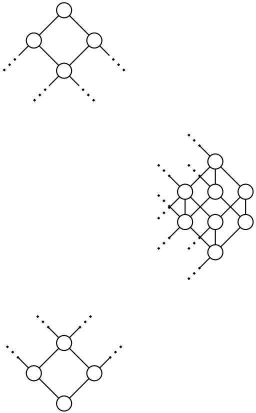
$\gamma(E_{\mathbf{W}}) = W$ iff $\forall x \in W, y \in W' (\forall z \in E_{\mathbf{W}}(zNy) \Rightarrow xNy)$

$\mathbf{W}^+$ is **commutative** iff $N^{\uparrow}(x \circ y) = N^{\uparrow}(y \circ x)$. Translate this to a first-order formula on **W**.

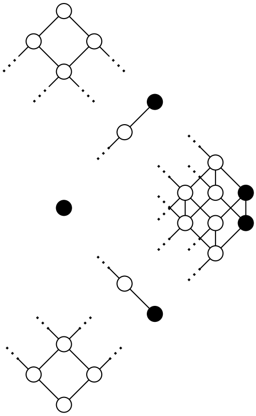
Modularity is not canonical

First look at how the MacNeille completion of a distributive lattice can fail to be modular [N. Funayama 1944]

Let L be the following sublattice of $(\mathbb{N} \oplus \mathbb{N}^{\partial})^2 \times \mathbf{2}$

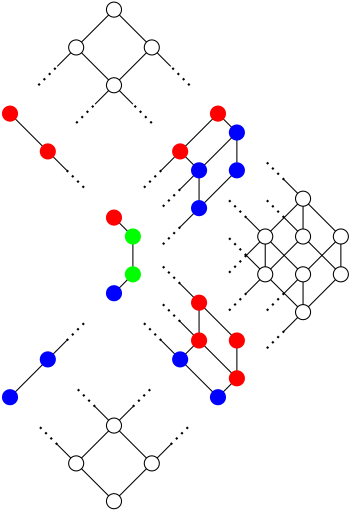


L



$\overline{L}$

The canonical extension of L



L^δ

Modularity is not canonical

Can we find a pentagon in M^δ for some modular lattice M?
No simple proof known; here is **John Harding**'s 1998 argument

Let M be the **modular** lattice of **finite and cofinite dimensional subspaces** of an infinite dimensional Hilbert space.

[**Von Neumann** 1936] A **continuous geometry** is a complete modular lattice L with a function $D : L \to [0,1]$ that has finite range or is surjective and
if $a < b$ then $D(a) < D(b)$ $\quad$ $D(a \vee b) + D(a \wedge b) = D(a) + D(b)$
$D(a) = 0$ if and only if $a = 0$, $\quad$ $D(a) = 1$ if and only if $a = 1$,
and D agrees on elements with a common complement.

[**Kaplansky** 1955] Any orthocomplemented complete modular lattice is a **continuous geometry**

So **if** M^δ is modular, it must be a continuous geometry

But M^δ has infinitely many atoms, which leads to a contradiction.

Generalized bunched implication algebras

A **generalized bunched implication algebra** or GBI-algebra $(A, \wedge, \vee, \to, \top, \bot, \cdot, 1, \backslash, /)$ is a **residuated lattice** $(A, \wedge, \vee, \cdot, 1, \backslash, /)$ such that $(A, \wedge, \vee, \to, \top, \bot)$ is a **Heyting algebra**, i.e., $\top, \bot$ are top and bottom elements and

$$x \wedge y \leq z \iff y \leq x \to z$$

or equivalently the following 2 identities hold

$$x \leq y \to ((x \wedge y) \vee z) \qquad x \wedge (x \to y) \leq y$$

Intuitionistic negation is $\neg x = x \to \bot$, **linear negation** is $\sim x = x \backslash \neg 1$.

Relation algebras are a subvariety of **GBI**:

RA $=$ **GBI+** $\sim x = \neg 1/x, \sim\sim x = x, \neg\neg x = x, \neg\sim(xy) = (\neg\sim y)(\neg\sim x))$

Bunched implication algebras or BI-algebras are commutative GBI-algebras

Distributive residuated frames

A **distributive residuated frame** is of the form

$\mathbf{W} = (W, W', N, \circ, \bbslash, \sslash, \curlywedge, \leftthreetimes, \rightthreetimes)$, where $\circ, \curlywedge \subseteq W^3$,

$N \subseteq W \times W'$ is a $\circ$-nuclear relation with respect to $\bbslash, \sslash$ and

distributively $\curlywedge$**-nuclear** with respect to $\leftthreetimes, \rightthreetimes$, which means:

$$x \curlywedge y \, N \, z \quad \text{iff} \quad y \, N \, x \leftthreetimes z \quad \text{iff} \quad x \, N \, z \rightthreetimes y$$

and it satisfies the following implications,

$$\begin{array}{c} x \curlywedge (y \curlywedge w) N z \\ \hline\hline (x \curlywedge y) \curlywedge w N z \end{array} \ [\curlywedge a] \qquad \frac{x \curlywedge y N z}{y \curlywedge x N z} \ [\curlywedge e]$$

$$\frac{x N z}{x \curlywedge y N z} \ [\curlywedge i] \qquad \frac{x \curlywedge x N z}{x N z} \ [\curlywedge c]$$

Distributive Gentzen frames

A **distributive Gentzen frame**, is a pair $(\mathbf{W}, \mathbf{B})$ where

- $\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!/, \{\varepsilon\}, \curlywedge, \leftthreetimes, \rightthreetimes)$ is a distributive frame, with $\circ$ and $\curlywedge$ binary **operations**,
- $\mathbf{B}$ is a partial algebra,
- $(W, \circ, \varepsilon, \curlywedge)$ is an associative bi-groupoid with unit for $\circ$ generated by $B \subseteq W$,
- there is an injection of B into W' (under which B is identified with a subset of W') and
- N satisfies "the standard list of Gentzen rules"

Theorem

[Galatos and J. 2017] ***Distributive residuated lattices*** *and* ***GBI-algebras*** *have cut-free Gentzen systems, and the same holds for subvarieties defined by $1, \circ, \vee$-rules.*
Distributive residuated lattices and subvarieties defined by $1, \circ, \vee$-rules that do not increase complexity have the finite model property.

Open Problems

Give an elementary proof that modularity is not canonical.

Axiomatize the variety generated by canonical extensions of MV-algebras.

Is join-semidistributivity canonical?

Find a sequent calculus that decides equations for GBI-algebras by proof search.

Lattice-ordered pregroups are ℓ-monoids $(A, \vee, \wedge, \cdot, 1)$ with two more unary operations ${}^{\ell}$, ^{r} that satisfy

$x^{\ell}x \leq 1 \leq xx^{\ell}$ and $xx^{r} \leq 1 \leq x^{r}x$. Are the lattice reducts distributive? (They are known to be semidistributive.)

Thanks!