Lindenbaum-style proof of completeness for infinitary logics

Part I

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We are interested in ...

- logics given as (possibly) infinitary consequence relations of shape $\Gamma \vdash \varphi$, in a finitary language,
- proving irreducible theories form a basis of the closure system of all theories,
- applying the above to prove strong completeness of such logics.

We will consider ...

- logics given as (possibly) infinitary consequence relations of shape $\Gamma \vdash \varphi$, in a finitary language,
  - allowing for a strong disjunction
  - having a countable axiomatic presentation
- proving irreducible theories form a basis of the closure system of all theories,
  - proving separation by prime theories
  - using a pair-extension lemma
- applying the above to prove strong completeness of such logics.

Infinitary many-valued logics

Łukasiewicz logic Ł$_\infty$ in the language with $\to, \neg$ given semantically over the real interval $[0,1]$:

$$\neg x = 1 - x \qquad x \to y = \min(1, 1 - x + y)$$

and

$$\Gamma \models \varphi \text{ iff } (\forall e\colon Fm \to [0,1])(e[\Gamma] \subseteq \{1\} \Rightarrow e(\varphi) = 1)$$

Then Ł$_\infty$ is not finitary:

$$\{\neg\varphi \to \varphi \& \overset{n}{\ldots} \& \varphi \mid n \geq 0\} \models \varphi,$$

where $\varphi \,\&\, \psi = \neg(\varphi \to \neg\psi)$.

Infinitary classical modal logics

There are interesting examples of noncompact modal logics, that are thus not strongly complete, e.g.

- In PDL:

$$\{[\alpha;\beta^n]\varphi \mid n \in \mathbb{N}\} \models [\alpha;\beta^*]\varphi$$

- In logics of common knowledge:

$$\{E^{n+1}\varphi \mid n \in \mathbb{N}\} \models C\varphi$$

Question: if infinitary rules (as a.g. the above) are allowed, can we obtain a strongly complete axiomatization?

Strong completeness ← canonical model construction ← Lindenbaum Lemma

L.L. in infinitary classical modal logics — some known results

- 1977 Sundholm: strong completeness of Von Wrights temporal logic
- 1984 Goldblatt: a general result about the existence of maximally consistent theories satisfying certain closure conditions,
  1993: a general approach to prove Lindenbaum lemma in an infinitary setting.
- 1994, Segerberg: a general method of strong completeness proof for noncompact modal logics, using saturated sets of formulas (in many cases coincide with maximally consistent theories).
- 2008 Lavalette, Kooi, and Verbrugge: Lindenbaum lemma and strong completeness of infinitary axiomatization of PDL and some related non-compact modal logics (such as epistemic logics with common knowledge modality).

What is a logic

Var: a countable infinite set of propositional variables

$\mathcal{L}$: an at most countable propositional language

Fm: a set of formulas in variables Var and a language $\mathcal{L}$

A logic $\vdash$ is a relation between sets of formulas and formulas s.t.:

- $\{\varphi\} \vdash \varphi$ (Reflexivity)
- If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$ (Monotonicity)
- If $\Delta \vdash \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$ (Cut)
- If $\Gamma \vdash \varphi$, then $\sigma[\Gamma] \vdash \sigma(\varphi)$ for each substitution σ (Structurality)

A logic is finitary if: $\Gamma \vdash \varphi$ implies there is a finite $\Gamma' \subseteq \Gamma$ s.t. $\Gamma' \vdash \varphi$.

Theories

$T \subseteq Fm$ is a theory: if $T \vdash \varphi$, then $\varphi \in T$.

A theory T is prime if it is not an intersection of two strictly bigger theories.

Theorem (Lindenbaum lemma)

Let $\vdash$ be a finitary logic. If $\Gamma \nvdash \varphi$, then there is a prime theory $T \supseteq \Gamma$ such that $\varphi \notin T$.

Axiomatization

Proofs are well-founded trees, i.e., trees with no infinitely-long branch.

A logic is countably axiomatizable if it has an axiomatic system with countably many instances of rules.

Note: each finitary logic is countably axiomatizable.

Not conversely: let $Ł_\infty$ be the extension of Łukasiewicz logic $Ł$ by the rule

$$\{\neg\varphi \to \varphi^n \mid n \geq 0\} \rhd \varphi.$$

We can show that $Ł_\infty$ is not finitary but clearly it is countably axiomatizable.

Strong disjunction

A connective $\vee$ (primitive of defined) is called strong disjunction in $\vdash$ if:

$$\varphi \vdash \varphi \vee \psi \qquad\qquad \psi \vdash \varphi \vee \psi \tag{PD}$$

$$\frac{\Gamma, \Phi \vdash \chi \qquad\qquad \Gamma, \Psi \vdash \chi}{\Gamma \cup \{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\} \vdash \chi} \tag{sPCP}$$

If $\vee$ is a strong disjunction, then a theory T is prime iff for each φ and ψ:
if $\varphi \vee \psi \in T$, then $\varphi \in T$ or $\psi \in T$.

Logic $Ł_\infty$ is a non-finitary logic with a strong disjunction.

The main result

Theorem (Lindenbaum Lemma for certain infinitary logics)

Let $\vdash$ be a countably axiomatizable logic with a strong disjunction. If $\Gamma \nvdash \varphi$, then there is a prime theory $T \supseteq \Gamma$ such that $\varphi \notin T$.

Some notes, … before we show the proof

1. the lattice connective $\vee$ need not satisfy sPCP

but some other connective could

In global S4 it would entail $\varphi \vee \neg\varphi \vdash^{g}_{\mathrm{S4}} \Box\varphi \vee \neg\varphi$, i.e.,

$$\vdash^{g}_{\mathrm{S4}} \varphi \to \Box\varphi$$

which can be easily refuted

On the other hand we can show that:

$$\frac{\Gamma, \varphi \vdash^{g}_{\mathrm{S4}} \chi \qquad \Gamma, \psi \vdash^{g}_{\mathrm{S4}} \chi}{\Gamma \cup \{\Box\varphi \vee \Box\psi\} \vdash^{g}_{\mathrm{S4}} \chi}$$

Some notes, … before we show the proof

1. the lattice connective $\vee$ need not satisfy sPCP

but some other connective could

2. the condition of countable axiomatizability cannot be omitted

Consider language with $\vee$, and a constant $\mathbf{i}$ for each $i \in \omega$. Let L be the expansion of the disjunction-fragment of classical logic by:

$$\{\mathbf{i} \vee \chi \mid i \in C\} \rhd \chi$$

for each infinite set $C \subseteq \omega$.

Then $\vee$ is a strong disjunction in L but Lindenbaum Lemma fails:

$\{2\mathbf{i} \vee 2\mathbf{i} + \mathbf{1} \mid i \in \omega\} \nvdash 0$, but each prime theory extending it does.

Some notes, … before we show the proof

1. the lattice connective $\vee$ need not satisfy sPCP

but some other connective could

2. the condition of countable axiomatizability cannot be omitted

3. the condition of having strong disjunction cannot be omitted

Consider the logic L with unary operation $\Box$ given by rules (for $n \in \omega$):

$$\{\Box^m \varphi \mid m > n\} \rhd \varphi$$

Clearly L is countably axiomatizable and

$$\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \text{iff} \qquad \chi = \varphi \text{ or } \Gamma \vdash_{\mathrm{L}} \chi$$

Thus if T is a theory, so is $T \cup \{\psi\}$ and so only Fm is a prime theory

Finally note that there are non-trivial theories (i.e., $\emptyset$)

A small reformulation, … before we show the proof

For each logic $\vdash_{\mathrm{L}}$ with a strong disjunction $\vee$ we define a relation $\Vdash_{\mathrm{L}}$:

$$\Gamma \Vdash_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{there is a finite non-empty } \Delta' \subseteq \Delta \text{ and } \Gamma \vdash_{\mathrm{L}} \bigvee \Delta'.$$

A tuple $\langle \Gamma, \Delta \rangle$ is a pair if $\Gamma \nVdash \Delta$ and it is a full pair if $\Gamma \cup \Delta = Fm$

Claim: observe that if $\langle \Gamma, \Delta \rangle$ is a full pair, then Γ is prime theory and if Γ is a prime theory, then $\langle \Gamma, Fm \setminus \Gamma \rangle$ is full pair

Proposition

A logic $\vdash_{\mathrm{L}}$ enjoys the Lindenbaum lemma iff each pair $\langle \Gamma, \Delta \rangle$ where Δ is finite can be extended into a full pair.

A pair $\langle \Gamma', \Delta' \rangle$ extends $\langle \Gamma, \Delta \rangle$ if $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$

A final ingredient, . . . before we show the proof

If $\vee$ is strong disjunction, then $\Vdash_{\mathrm{L}}$ enjoys the Strong-Cut for finite Δs:

$$\frac{\{\Gamma \Vdash_{\mathrm{L}} \Delta \cup \{\varphi\} \mid \varphi \in \Phi\} \qquad \Gamma \cup \Phi \Vdash_{\mathrm{L}} \Delta}{\Gamma \Vdash_{\mathrm{L}} \Delta}.$$

Let us set $\chi = \bigvee \Delta$ then clearly:

$$\frac{\{\Gamma \vdash_{\mathrm{L}} \chi \vee \varphi\} \mid \varphi \in \Phi\} \qquad \dfrac{\Gamma \cup \Phi \vdash_{\mathrm{L}} \chi \qquad \Gamma \cup \{\chi\} \vdash_{\mathrm{L}} \chi}{\Gamma \cup \{\chi \vee \varphi \mid \varphi \in \Phi\} \vdash_{\mathrm{L}} \chi}}{\Gamma \vdash_{\mathrm{L}} \chi}.$$

So all is fine if we prove that (a bit more):

If $\Vdash_{\mathrm{L}}$ enjoys the Strong-Cut (for finite Δs), then each pair $\langle \Gamma, \Delta \rangle$ (where Δ is finite) can be extended into a full pair.

And now finally the proof

Enumerate all rules $\Lambda_i \rhd \varphi_i$.

Define an increasing sequence of pairs $\langle \Gamma_i, \Delta_i \rangle$ starting with $\langle \Gamma_0, \Delta_0 \rangle = \langle \Gamma, \Delta \rangle$.

The induction step. We distinguish two cases:

- If $\langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle$ is a pair, then $\langle \Gamma_{i+1}, \Delta_{i+1} \rangle = \langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle$.
- If $\langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle$ is not a pair, then there has to be $\chi_i \in \Lambda_i$ such that $\langle \Gamma_i, \Delta_i \cup \{\chi_i\} \rangle$ is a pair so we set $\langle \Gamma_{i+1}, \Delta_{i+1} \rangle = \langle \Gamma_i, \Delta_i \cup \{\chi_i\} \rangle$.

  Why there is such χ_i?

$$\dfrac{\dfrac{\{\Gamma_i \Vdash \Delta_i \cup \{\varphi_i\} \cup \{\chi_i\} \mid \chi_i \in \Lambda_i\} \qquad \Gamma_i \cup \Lambda_i \Vdash \Delta_i \cup \{\varphi_i\}}{\Gamma_i \Vdash \Delta_i \cup \{\varphi_i\}} \qquad \Gamma_i \cup \{\varphi_i\} \Vdash \Delta_i}{\Gamma_i \Vdash \Delta_i}$$

Assume that we have a ‘dummy’ rule $\psi \rhd \psi$, thus each ψ is in some Γ_i or Δ_i

Proof (cont.) define $\Gamma' = \bigcup \Gamma_i$ and $\Delta' = \bigcup \Delta_i$

Claim: for each ψ we have: if $\Gamma' \vdash \psi$ than $\psi \in \Gamma_j$ for some j.

Proof of the Claim: let us fix a proof of ψ from Γ'; we prove it for each formula labeling some of its nodes.

If the node is a leaf the claim is trivial.

Consider a node obtained using rule $\Lambda_i \rhd \varphi_i$

If we proceed by the first case in our induction step we have $\varphi_i \in \Gamma_{i+1}$.

Assume we proceed by the second case: then $\chi_i \in \Lambda_i \cap \Delta_{i+1}$.

As $\Gamma' \vdash \chi_i$ (it labels a node preceding φ_i), then by IP: $\Gamma_j \vdash \chi_i$ for some j.

Thus $\Gamma_{\max\{i+1,j\}} \Vdash_{\mathrm{L}} \Delta_{\max\{i+1,j\}}$, a contradiction.

Proof (cont.) define $\Gamma' = \bigcup \Gamma_i$ and $\Delta' = \bigcup \Delta_i$

Claim: for each ψ we have: if $\Gamma' \vdash \psi$ than $\psi \in \Gamma_j$ for some j.

The conclusion of the proof: we prove that $\langle \Gamma', \Delta' \rangle$ is a pair.

Assume that $\Gamma' \vdash \bigvee \Delta''$ for some finite $\Delta'' \subseteq \Delta'$.

Thus by the Claim: $\Gamma_j \vdash \bigvee \Delta''$ for some j

As $\Delta'' \subseteq \Delta_i$ for some i we have:

$$\Gamma_{\max\{i,j\}} \Vdash_{\mathrm{L}} \Delta_{\max\{i,j\}},$$

a contradiction.

So we have proved that ...

Let L be countably axiomatizable logics with a strong disjunction $\vee$. Then

1. $\Vdash_{\mathrm{L}}$ has the Pair Extension Property for finite Δs.
2. $\Vdash_{\mathrm{L}}$ enjoys the Strong-Cut for finite Δs.
3. $\vdash_{\mathrm{L}}$ enjoys the Lindenbaum lemma

When we can extend all pairs?

Let L be countably axiomatizable logics with a strong disjunction $\vee$. TFAE

1. $\Vdash_{\mathrm{L}}$ has the Pair Extension Property.
2. $\Vdash_{\mathrm{L}}$ enjoys the Strong-Cut.
3. L is finitary.

PART II

(tbc by Petr tomorrow)

Remarks for completeness of infinitary logics:

- Pair Extension Property for finite Δs suffices to obtain a separation by prime theories,
- restriction to finite Δs might limit canonical model construction (valuation lemma for normal diamond-like operators):

$$\Diamond\alpha \in \Gamma \text{ implies } \langle\{\alpha\}, \{\beta \mid \Diamond\beta \notin \Gamma\}\rangle \text{ is a pair,}$$

but can we extend it to a full one to create a prime theory Σ?

- one can get around it if a suitable negation is available:
  - the classical negation and deduction theorem allows one to extend $\{\alpha\} \cup \{\neg\beta \mid \Diamond\beta \notin \Gamma\}$ to obtain a MCS Σ,
  - the de Morgan involutive negation allows one to extend $\langle\{\neg\beta \mid \Diamond\beta \notin \Gamma\}, \{\neg\alpha\}\rangle$ and obtain $*\Sigma$ (Σ is than recovered as the complement of $\neg * \Sigma$).

Infinitary logic PDL_ω

A countable axiomatization of PDL_ω, ensuring the disjunction is a strong disjunction, can be given with rules Modus Ponens and the infinitary rule:

$$\{[\alpha;\beta^n]\varphi \mid n \in \mathbb{N}\} \rhd [\alpha;\beta^*]\varphi, \qquad (\mathrm{Inf}^*)$$

plus all the box-forms of (the instances of) the rule:

$$[\alpha]\Gamma \rhd [\alpha]\varphi, \text{ for each } \alpha \text{ and } \Gamma \rhd \varphi$$

We obtain Lindenbaum Lemma for PDL_ω. This suffices to prove strong canonical completeness of PDL_ω.

cf. de Lavalette, G.R., Kooi, B., Verbrugge, R.: Strong completeness and limited canonicity for PDL. Journal of Logic, Language and Information 7(1), 6987 (2008).

A similar approach applies to

I. Sedlár: Propositional dynamic logic with Belnapian truth values. In: AiML, volume 11, pp. 503-519, 2016.

Common knowledge or belief (classical)

A countable axiomatization, ensuring the disjunction is a strong disjunction, based on modal axioms for each K_a, and:

$$E\varphi \leftrightarrow \bigwedge_{a \in G} K_a\varphi, \qquad C\varphi \leftrightarrow E(\varphi \wedge C\varphi)$$

and the infinitary rule (all instances for all boxes $\circ$):

$$\{\circ E^{n+1}\varphi \mid n \in \mathbb{N}\} \rhd \circ C\varphi$$

(Boxes are all syntactically derived modalities of a box type, i.e. monotone and $\wedge$ preserving.)

Again, we obtain Lindenbaum Lemma. This suffices to prove strong canonical completeness.

Common belief (based on Belnap-Dunn logic)

The syntax given by:

$$\phi ::= p \mid t \mid f \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \Diamond_i\phi \mid C\phi$$

Frames for BD are based on involutive posets $(X, \leq, *)$, equipped with monotone relations $\{S_i \mid i \in I\}$

$$S_i : X^{op} \times X \to 2$$

Valuation of atoms by uppersets in X are extended in the obvious way to constants and $\wedge, \vee$.

$$\begin{array}{lcl} x \Vdash \neg\alpha & \equiv & *x \nVdash \alpha \\ x \Vdash \Diamond_i\alpha & \equiv & \exists s(sS_ix \wedge s \Vdash \alpha) \\ x \Vdash \neg\Diamond_i\neg\alpha & \equiv & \forall s(*sS_i * x \to s \Vdash \alpha) \end{array}$$

Common belief is intended to be the greatest fixed point

$$C\phi \ \equiv \nu x. \bigwedge_{i\in I} \Diamond_i(\phi \wedge x)$$

Semantically

$$||C\phi|| = \bigcup\{Y \in UX \mid Y \subseteq ||\phi||_{x:Y}\}$$

Modal axioms

The idea is to extend a suitable axiomatics of BD with axioms and rules:

$$\Diamond_i(p \vee q) \dashv\vdash (\Diamond_i p \vee \Diamond_i q) \qquad \Diamond_i f \vee p \vdash p \qquad \emptyset \vdash \neg\Diamond_i f$$

$$Cp \vdash \bigwedge_{i \in I} \Diamond_i(p \wedge Cp)$$

and ensure the resulting $\vdash$ is closed under (meta)rules:

$$\frac{\alpha \vdash \beta}{\Diamond_i\alpha \vdash \Diamond_i\beta} \qquad \frac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha} \qquad \frac{\alpha \vdash \beta}{C\alpha \vdash C\beta}$$

and satisfies sPCP.

Denote $\bigwedge_{i\in I} \Diamond_i p$, by $\Diamond p$. Finite approximations of Cp:

$$C^0 p = \Diamond p, \quad C^{n+1} p = \Diamond(p \wedge C^n p)$$

adopt the fixed point axiom above and add an infinitary rule

$$\{C^n p \mid n \in N\} \vdash Cp$$

We need to ensure monotonicity and PCP again, plus, the following:

$$\frac{\Gamma \vdash_\omega \beta}{\circ\Gamma \vdash_\omega \circ\beta}$$

for any definable box-type operator (meet-preserving) $\circ$ (combinations of $\neg\Diamond_i\neg$).

By the above theory, the resulting logic allows for a canonical model construction and is thus strongly complete.