Lindenbaum-style proof of completeness for infinitary logics

Part II

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Digression 1: Classes of infinitary logics

A logic L has the

- CIPEP (completely $\cap$-prime extension property) if
  completely $\cap$-prime theories form a basis of $\mathrm{Th}(\mathrm{L})$
- IPEP ($\cap$-prime ext. property) if $\cap$-prime theories form a basis of $\mathrm{Th}(\mathrm{L})$

Theorem

Given any algebraizable logic L *and theory* T, *we have:*

1. $\mathbf{LindT}_T \in \mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$ *iff* T *is completely* $\cap$*-prime.*
2. $\mathbf{LindT}_T \in \mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ *iff* F *is* $\cap$*-prime.*

Digression 1: Classes of infinitary logics

A logic L has the

- CIPEP (completely $\cap$-prime extension property) if completely $\cap$-prime theories form a basis of $\mathrm{Th}(\mathrm{L})$
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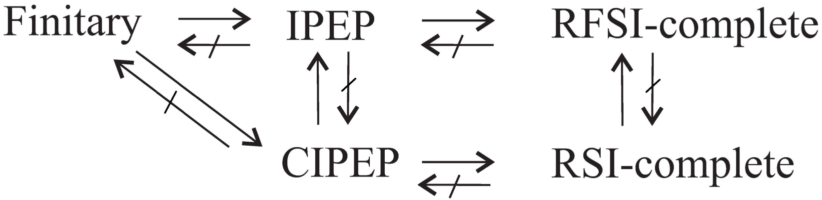
A logic L is

- RSI-complete if $\mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$
- RFSI-complete if $\mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}}$



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Three kinds of disjunction

A connective $\vee$ (primitive of defined) is called strong disjunction in $\vdash$ if:

$$\varphi \vdash \varphi \vee \psi \qquad \psi \vdash \varphi \vee \psi \tag{PD}$$

$$\frac{\Gamma, \Phi \vdash \chi \qquad \Gamma, \Psi \vdash \chi}{\Gamma, \{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\} \vdash \chi} \tag{sPCP}$$

Three kinds of disjunction

A connective $\vee$ (primitive of defined) is called disjunction in $\vdash$ if:

$$\varphi \vdash \varphi \vee \psi \qquad \psi \vdash \varphi \vee \psi \tag{PD}$$

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi} \tag{PCP}$$

In a finitary logic each disjunction is strong but not vice-versa

If $\vee$ is a disjunction, then T is prime iff $\varphi \vee \psi \in T$ implies $\varphi \in T$ or $\psi \in T$.

Three kinds of disjunction

A connective $\vee$ (primitive of defined) is called weak disjunction in $\vdash$ if:

$$\varphi \vdash \varphi \vee \psi \qquad \psi \vdash \varphi \vee \psi \tag{PD}$$

$$\frac{\varphi \vdash \chi \qquad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi} \tag{wPCP}$$

There is finitary logic with a weak disjunction but no disjunction

Note that a weak disjunction suffices for a meaningful definition of $\Vdash_{\mathrm{L}}$:

$$\Gamma \Vdash_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{there is a finite non-empty } \Delta' \subseteq \Delta \text{ and } \Gamma \vdash_{\mathrm{L}} \bigvee \Delta'.$$

Three kinds of disjunction

A connective $\vee$ is weak disjunction in $\vdash$ iff:

$$\mathrm{Th_L}(\varphi) \cap \mathrm{Th_L}(\psi) = \mathrm{Th_L}(\varphi \vee \psi)$$

Thus the intersection of two principal theories is principal

Some characterizations

Let L be a logic with axiomatization $\mathcal{AS}$. Then $\vee$ is a strong disjunction iff

$$\varphi \vdash_{\mathrm{L}} \varphi \vee \psi \qquad \varphi \vee \psi \vdash_{\mathrm{L}} \psi \vee \varphi \qquad \varphi \vee \varphi \vdash_{\mathrm{L}} \varphi$$

$$\{\gamma \vee \chi \mid \gamma \in \Gamma\} \vdash_{\mathrm{L}} \varphi \vee \chi \qquad \text{for each } \Gamma \rhd \varphi \text{ from } \mathcal{AS}$$

Digression 2: Łukasiwicz logic and its relatives

$[0,1]_Ł$: the standard MV-algebra with domain $[0,1]$ and operations

$$x \to y = \min\{1, 1 - x + y\}$$
$$x \,\&\, y = \max\{0, x + y - 1\}$$
$$x \vee y = \max\{x, y\}$$
$$\neg x = 1 - x$$

Ł: the logic axiomatized by modus ponens and 4 Łukasiewicz axioms

Fact: the equivalence $\Gamma \vdash_Ł \varphi$ iff $\Gamma \models_{[0,1]_Ł} \varphi$ holds for finite Γs only

BTLSMVA: the extension of Ł by the rule

$$\{\neg\varphi \to \varphi \& \overset{n}{\ldots} \& \varphi \mid n \geq 1\} \rhd \varphi$$

Fact?: $\Gamma \vdash_{BTLSMVA} \varphi$ iff $\Gamma \models_{[0,1]_Ł} \varphi$ holds for all Γs.

Proving completeness of BTLSMVA

1) We know that it is countably axiomatizable

2) And $\vee$ is its strong disjunction

We can easily show that it is a strong disjunction in Ł:

$$\varphi \vdash_{\text{Ł}} \varphi \vee \psi \qquad \varphi \vee \psi \vdash_{\text{Ł}} \psi \vee \varphi \qquad \varphi \vee \varphi \vdash_{\text{Ł}} \varphi \qquad \varphi \vee \chi, (\varphi \to \psi) \vee \chi \vdash_{\text{Ł}} \psi \vee \chi$$

Thus we can show that:

$$\frac{\neg\varphi \to \varphi^n \vdash_{\text{Ł}} \neg(\varphi \vee \chi) \to (\varphi \vee \chi)^n \qquad \chi \vdash_{\text{Ł}} \neg(\varphi \vee \chi) \to (\varphi \vee \chi)^n}{(\neg\varphi \to \varphi^n) \vee \chi \vdash_{\text{Ł}} \neg(\varphi \vee \chi) \to (\varphi \vee \chi)^n}$$

Then $\quad \{(\neg\varphi \to \varphi^n) \vee \chi \mid n \geq 0\} \vdash_{\text{Ł}_\infty} \varphi \vee \chi$

Proving completeness of $\mathrm{BTLSMVA}$

1) We know that it is countably axiomatizable

2) And $\vee$ is its strong disjunction

3) Thus if $\Gamma \nvdash_{\mathrm{BTLSMVA}} \varphi$, there is a prime theory $T \supseteq \Gamma$ st. $\varphi \notin T$

Take Lindenbaum-Tarski algebra of T: we know it is relatively finitely subdirectly irreducible $\mathrm{BTLSMVA}$-algebra

Thus it is a simple MV-chain and so it us embeddable into $[0, 1]_{Ł}$

Back to work: some more characterizations

Let us consider a logic L with a weak disjunction $\vee$. TFAE

1. $\vee$ is a strong disjunction
2. for each rule $\Gamma \rhd \varphi$ of some axiomatic system of L we have:

$$\{\gamma \vee \chi \mid \gamma \in \Gamma\} \vdash \varphi \vee \chi$$

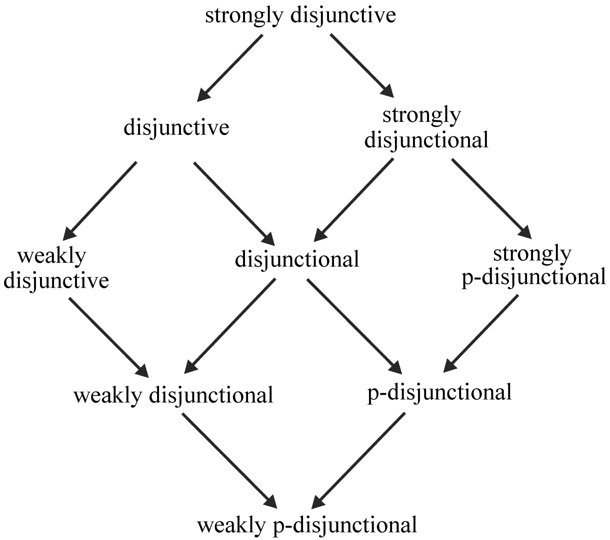
3. $\Vdash_{\mathrm{L}}$ enjoys the Strong-Cut for finite Δs, i.e.,

$$\frac{\{\Gamma \Vdash_{\mathrm{L}} \Delta \cup \{\varphi\} \mid \varphi \in \Phi\} \qquad \Gamma \cup \Phi \Vdash_{\mathrm{L}} \Delta}{\Gamma \Vdash_{\mathrm{L}} \Delta}.$$

4. the lattice of all theories is a frame, i.e.,

$$T \cap \bigvee_{S \in \mathcal{S}} S = \bigvee_{S \in \mathcal{S}} (T \cap S).$$

Digression 3: logics and disjunctions



An alternative summary of Part I

Let L be countably axiomatizable logics with a weak disjunction $\vee$. TFAE

1. $\Vdash_{\mathrm{L}}$ has the Pair Extension Property for finite Δs.
2. $\Vdash_{\mathrm{L}}$ enjoys the Strong-Cut for finite Δs.
3. $\vee$ is a strong disjunction.

Let L be countably axiomatizable logics with a weak disjunction $\vee$. TFAE

1. $\Vdash_{\mathrm{L}}$ has the Pair Extension Property.
2. $\Vdash_{\mathrm{L}}$ enjoys the Strong-Cut.
3. L is finitary.

Pair extension implies Strong-Cut

We want to show that

$$\frac{\{\Gamma \Vdash_{\mathrm{L}} \Delta \cup \{\varphi\} \mid \varphi \in \Phi\} \qquad \Gamma \cup \Phi \Vdash_{\mathrm{L}} \Delta}{\Gamma \Vdash_{\mathrm{L}} \Delta}.$$

Assume that $\Gamma \nVdash_{\mathrm{L}} \Delta$ and $\langle \Gamma', \Delta' \rangle$ is the full pair extending $\langle \Gamma, \Delta \rangle$

If $\Phi \subseteq \Gamma'$, then $\Gamma \cup \Phi \nVdash_{\mathrm{L}} \Delta$, a contradiction.

Let $\varphi \in \Phi \setminus \Gamma'$, then $\Gamma \nVdash_{\mathrm{L}} \Delta \cup \{\varphi\}$, a contradiction.

Strong Cut implies finitarity

Consider $\{\gamma_1, \gamma_2, \dots\} \rhd \varphi$ is proper infinitary rule

$\Delta = \{p_1, p_2, \dots\}$: infinite set of variables not occurring in $\{\varphi, \gamma_1, \gamma_2, \dots\}$

Claim: there is n such that

$$\{\gamma_i \vee p_i \mid i \geq 1\} \vdash_{\mathrm{L}} \varphi \vee p_1 \vee \cdots \vee p_n$$

To prove the claim we simply use Strong-Cut to obtain:

$$\frac{\{\{\gamma_i \vee p_i \mid i \geq 1\} \Vdash_{\mathrm{L}} \Delta \cup \{\gamma_i\} \mid i \geq 1\} \qquad \{\gamma_1, \gamma_2, \dots\} \Vdash_{\mathrm{L}} \{\varphi\}}{\{\gamma_i \vee p_i \mid i \geq 1\} \Vdash_{\mathrm{L}} \Delta \cup \{\varphi\}}$$

Strong Cut implies finitarity

Consider $\{\gamma_1, \gamma_2, \dots\} \rhd \varphi$ is proper infinitary rule

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$$\{\gamma_i \vee p_i \mid i \geq 1\} \vdash_{\mathrm{L}} \varphi \vee p_1 \vee \cdots \vee p_n$$

Now we consider a substitution σ:

$$\sigma(p) = \begin{cases} p & \text{if } p \notin \Delta \\ \varphi & \text{if } p = p_i \text{ for } i \leq n \\ \gamma_n & \text{if } p = p_i \text{ for } i > n \end{cases}$$

$$\{\gamma_1 \vee \varphi, \dots, \gamma_n \vee \varphi\} \cup \{\gamma_i \vee \gamma_n \mid i > n\} \vdash_{\mathrm{L}} \varphi \vee \varphi \vee \cdots \vee \varphi$$

$$\{\gamma_1, \dots, \gamma_n\} \vdash_{\mathrm{L}} \varphi$$

Lets us generalize our setting now

First we drop the structurality assumption

Then we try to live without disjunction ...

For each logic $\vdash_{\mathrm{L}}$ with we define a relation $\Vdash_{\mathrm{L}}$:

$$\Gamma \Vdash_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{there is a finite } \Delta' \subseteq \Delta \text{ and } \bigcap_{\psi \in \Delta'} \mathrm{Th}_{\mathrm{L}}(\psi) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$$

To make it work we need to assume two things: ...

1) Intersection of two finitely generated theories is finitely generated

2) L is framal, i.e. the lattice of its theories is a frame:

$$T \cap \bigvee_{S \in \mathcal{S}} S = \bigvee_{S \in \mathcal{S}} (T \cap S).$$

Properties of symmetrizations of framal logics

Assume that $\vdash_{\mathrm{L}}$ is framal, then:

1. If $\langle \Gamma, \Delta \rangle$ is a full pair, then Γ is prime theory and
if Γ is a prime theory, then $\langle \Gamma, Fm \setminus \Gamma \rangle$ is full pair

Proof: Clearly Γ is a theory; assume it is reducible, then for some $\varphi, \psi \notin \Gamma$:

$$\begin{aligned}\Gamma &= \mathrm{Th}_{\mathrm{L}}(\Gamma \cup \{\varphi\}) \cap \mathrm{Th}_{\mathrm{L}}(\Gamma \cup \{\psi\}) \\ &= (\Gamma \vee \mathrm{Th}_{\mathrm{L}}(\varphi)) \cap (\Gamma \vee \mathrm{Th}_{\mathrm{L}}(\psi)) \\ &= \Gamma \vee (\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi))\end{aligned}$$

As $\varphi, \psi \in \Delta$ we have contradiction with $\Gamma \nVdash_{\mathrm{L}} \Delta$

Properties of symmetrizations of framal logics

Assume that $\vdash_{\mathrm{L}}$ is framal, then:

1. If $\langle \Gamma, \Delta \rangle$ is a full pair, then Γ is prime theory and
   if Γ is a prime theory, then $\langle \Gamma, Fm \setminus \Gamma \rangle$ is full pair

2. $\Vdash_{\mathrm{L}}$ enjoys the Strong-Cut for finite Δs:

$$\frac{\{\Gamma \Vdash_{\mathrm{L}} \Delta \cup \{\varphi\} \mid \varphi \in \Phi\} \qquad \Gamma \cup \Phi \Vdash_{\mathrm{L}} \Delta}{\Gamma \Vdash_{\mathrm{L}} \Delta}.$$

Let us set $D = \bigcap_{\delta \in \Delta} \mathrm{Th}_{\mathrm{L}}(\delta)$ then $D \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma) \vee \mathrm{Th}_{\mathrm{L}}(\Phi)$ and for each $\varphi \in \Phi$:

$$D \cap \mathrm{Th}_{\mathrm{L}}(\varphi) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$$

$$D \cap \mathrm{Th}_{\mathrm{L}}(\Phi) = D \cap \bigvee_{\varphi \in \Phi} \mathrm{Th}_{\mathrm{L}}(\varphi) = \bigvee_{\varphi \in \Phi} D \cap \mathrm{Th}_{\mathrm{L}}(\varphi) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$$

Thus $D \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$, i.e., $\Gamma \Vdash_{\mathrm{L}} \Delta$

A more general result

Theorem (Lindenbaum Lemma for certain infinitary structural logics)

Let $\vdash$ be a countably axiomatizable structural logic with a strong disjunction.

If $\Gamma \nvdash \varphi$, then there is a prime theory $T \supseteq \Gamma$ such that $\varphi \notin T$.

A more general result

Theorem (Lindenbaum Lemma for certain infinitary logics)

Let $\vdash$ be a countably axiomatizable logic which is framal and the intersection of two finitely generated theories if finitely generated.

If $\Gamma \nvdash \varphi$, then there is a prime theory $T \supseteq \Gamma$ such that $\varphi \notin T$.

The proof is almost the same . . .

Enumerate all rules $\Lambda_i \rhd \varphi_i$

Define increasing sequence of pairs $\langle \Gamma_i, \Delta_i \rangle$ starting with $\langle \Gamma_0, \Delta_0 \rangle = \langle \Gamma, \Delta \rangle$

Induction step, we distinguish two cases:

- If $\langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle$ is a pair, then $\langle \Gamma_{i+1}, \Delta_{i+1} \rangle = \langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle$.
- If $\langle \Gamma_i \cup \{\varphi_i\}, \Delta_i \rangle$ is not a pair, then there has to be $\chi_i \in \Lambda_i$ such that $\langle \Gamma_i, \Delta_i \cup \{\chi_i\} \rangle$ is a pair so we set $\langle \Gamma_{i+1}, \Delta_{i+1} \rangle = \langle \Gamma_i, \Delta_i \cup \{\chi_i\} \rangle$.

  Why there is such χ_i?

$$\dfrac{\dfrac{\{\Gamma_i \Vdash \Delta_i \cup \{\varphi_i\} \cup \{\chi_i\} \mid \chi_i \in \Lambda_i\} \qquad \Gamma_i \cup \Lambda_i \Vdash \Delta_i \cup \{\varphi_i\}}{\Gamma_i \Vdash \Delta_i \cup \{\varphi_i\}} \qquad \Gamma_i \cup \{\varphi_i\} \Vdash \Delta_i}{\Gamma_i \Vdash \Delta_i}$$

Assume that we have a ‘dummy’ rule $\psi \rhd \psi$, thus each ψ is in some Γ_i or Δ_i

Proof (cont.) define $\Gamma' = \bigcup \Gamma_i$ and $\Delta' = \bigcup \Delta_i$

Claim: for each ψ we have: if $\Gamma' \vdash \psi$ than $\psi \in \Gamma_j$ for some j.

Proof of the Claim: let us fix a proof of ψ from Γ'; we prove it for each formula labeling some of its nodes

If the node is a leaf the claim is trivial

Consider node obtained using rule $\Lambda_i \rhd \varphi_i$

If we proceed by the first case in our induction step we have $\varphi_i \in \Gamma_{i+1}$

Assume we proceed by the second case: then $\chi_i \in \Lambda_i \cap \Delta_{i+1}$

As $\Gamma' \vdash \chi_i$ (it labels a node preceding φ_i), then by IP: $\Gamma_j \vdash \chi_i$ for some j

Thus $\Gamma_{\max\{i+1,j\}} \Vdash_{\mathrm{L}} \Delta_{\max\{i+1,j\}}$, a contradiction.

Proof (cont.) define $\Gamma' = \bigcup \Gamma_i$ and $\Delta' = \bigcup \Delta_i$

Claim: for each ψ we have: if $\Gamma' \vdash \psi$ than $\psi \in \Gamma_j$ for some j.

The conclusion of the proof: we prove that $\langle \Gamma', \Delta' \rangle$ is a pair.

If not then $\bigcap_{\varphi \in \Delta''} \mathrm{Th}_{\mathrm{L}}(\varphi) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma')$ for some finite $\Delta'' \subseteq \Delta'$.

We know that $\bigcap_{\varphi \in \Delta''} \mathrm{Th}_{\mathrm{L}}(\varphi) = \mathrm{Th}_{\mathrm{L}}(D)$ for some finite D

Thus by the Claim there is j such that: $\Gamma_j \vdash \delta$ for each $\delta \in D$

Then $\bigcap_{\varphi \in \Delta''} \mathrm{Th}_{\mathrm{L}}(\varphi) = \mathrm{Th}_{\mathrm{L}}(D) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma_j)$

As $\Delta'' \subseteq \Delta_i$ for some i we have a contradiction:

$$\Gamma_{\max\{i,j\}} \Vdash_{\mathrm{L}} \Delta_{\max\{i,j\}}.$$

Digression 4: some incoherent thoughts

- on countable axiomatizability
- on relation to the proof of existence of Henkin extension
- on relation to Rasiowa–Sikorski Lemma

Closure operators on lattices

Let $\boldsymbol{U}$ be a algebraic lattice, C is closure operator on $\boldsymbol{U}$ if

- $x \leq y$ implies $C(x) \leq c(y)$
- $x \leq C(x)$
- $C(x) = C(C(x))$

We say that C is algebraic if

$K(\boldsymbol{U})$ = compact elements of $\boldsymbol{U}$

$$C(x) = \bigvee_{y \leq x,\ y \in K(\boldsymbol{U})} C(y)$$

The image of C is a complete meet-subsemilattice $\boldsymbol{C}$ of $\boldsymbol{U}$, where

$$x \vee^{\boldsymbol{C}} y = c(x \vee^{\boldsymbol{C}} y)$$

Lindenbaum lemma in this setting

Theorem (Abstracter Lindenbaum lemma)

Let C be an algebraic closure operator on algebraic lattice $\boldsymbol{U}$.
Then each element of $\boldsymbol{C}$ is a meet of meet-irreducible elements of $\boldsymbol{C}$.

Theorem (Abstracter 'our' Lindenbaum lemma)

Let C be a closure operator on algebraic lattice $\boldsymbol{U}$ such that

- *C is countably axiomatizable*
- *$\boldsymbol{C}$ is a frame*
- *$C[K(\boldsymbol{U})]$ is a subuniverse of $\boldsymbol{C}$*

Then each element of $\boldsymbol{C}$ is a meet of meet-irreducible elements of $\boldsymbol{C}$.

Axiomatizable ???

As $\boldsymbol{U}$ is algebraic we always have:

$$C(x) = \bigvee_{y \leq C(x),\ y \in K(\boldsymbol{U})} y$$

Axiomatic system $\mathcal{A}$: a collection of pairs $x \triangleright y$ where $y \in K(\boldsymbol{U})$

Proof of y from x: a well-founded tree labeled by elements of $K(\boldsymbol{U})$ st

- its root is labeled by y and leaves by elements $z \leq x$ and
- if a node is labeled by z and D is the set of labels of its preceding nodes, then $\bigvee D \triangleright z \in \mathcal{A}$

We define:

$$C_{\mathcal{A}}(x) = \bigvee_{x \vdash_{\mathcal{A}} y,\ y \in K(\boldsymbol{U})} y$$

Then $C_{\mathcal{A}}$ is the least co C on $\boldsymbol{U}$ s.t. for each $x \triangleright y \in \mathcal{A}$ we have $y \leq C(x)$.

Note: $C = C_{\{x \triangleright y \mid y \in K(\boldsymbol{U}),\ y \leq C(x)\}}$.

M-Logic

Logic: a relation $\vdash$ between sets of formulas and formulas st:

- $\{\varphi\} \vdash \varphi$. (Reflexivity)
- If $\Gamma \vdash \varphi$, then $\Gamma \cup \Delta \vdash \varphi$ (Monotonicity)
- If $\Delta \vdash \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$ (Cut)

Some logics could satisfy additional property:

- If $\Gamma \vdash \varphi$, then $\Gamma' \vdash \varphi$ for some *finite* $\Gamma' \subseteq \Gamma$ (Finitarity)

M-Logic

M-Logic: a relation $\Vdash$ between sets of formulas and sets of formulas st:

- $\{\varphi\} \Vdash \{\varphi\}$ (Reflexivity)
- If $\Gamma \Vdash \Delta$, then $\Gamma \cup \Sigma \Vdash \Delta \cup \Sigma'$ (Monotonicity)
- If $\Gamma, \Sigma \Vdash \Delta, Fm \setminus \Sigma$ for each $\Sigma \subseteq Fm$, then $\Gamma \Vdash \Delta$ (PEP)

Some m-logics could satisfy additional properties:

- If $\Gamma \Vdash \Delta$, then $\Gamma' \Vdash \Delta$ for some *finite* $\Gamma' \subseteq \Gamma$ (Left-Finitarity)
- If $\Gamma \Vdash \Delta$, then $\Gamma \Vdash \Delta'$ for some *finite* $\Delta' \subseteq \Delta$ (Right-Finitarity)

Variants of Cut rule

Any m-logic $\Vdash$ has the Strong-Cut:

$$\frac{\{\Gamma \Vdash \Delta \cup \{\varphi\} \mid \varphi \in \Phi\} \qquad \Gamma \cup \Phi \Vdash \Delta}{\Gamma \Vdash \Delta}.$$

But not vice-versa!

In presence of both finitarity conditions, the PEP can be equivalently replaced simply by:

$$\frac{\Gamma \Vdash \Delta \cup \{\varphi\} \qquad \Gamma \cup \{\varphi\} \Vdash \Delta}{\Gamma \Vdash \Delta}.$$

Possible symmetrizations of a logic $\vdash_{\mathrm{L}}$

$\Gamma \Vdash^{1}_{\mathrm{L}} \Delta$ iff there is $\delta \subseteq \Delta$ and $\Gamma \vdash_{\mathrm{L}} \delta$

$\Gamma \Vdash^{1}_{\mathrm{L}} \Delta$ iff there is $\delta' \subseteq \Delta$ and $\mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$

$\Gamma \Vdash^{\mathrm{fin}}_{\mathrm{L}} \Delta$ iff there is a finite $\Delta' \subseteq \Delta$ and $\Gamma \vdash_{\mathrm{L}} \bigvee \Delta'$

$\Gamma \Vdash^{\mathrm{fin}}_{\mathrm{L}} \Delta$ iff there is a finite $\Delta' \subseteq \Delta$ and $\bigcap_{\delta \in \Delta'} \mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$

$\Gamma \Vdash^{\omega}_{\mathrm{L}} \Delta$ iff $\bigcap_{\delta \in \Delta} \mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$

$\Gamma \Vdash^{\mathrm{sw}}_{\mathrm{L}} \Delta$ iff for each evaluation s.t. $e[\Gamma] \subseteq \{1\}$ there is $\delta \in \Delta$ s.t. $e(\delta) = 1$

$\Gamma \Vdash^{s}_{\mathrm{L}} \Delta$ iff for each evaluation s.t. $e[\Gamma] \subseteq \{1\}$ we have $\sup_{\delta \in \Delta} e(\delta) = 1$

In structural setting: $\Vdash^{\mathrm{fin}}_{\mathrm{L}}$ is (finitary) m-logic iff L is finitary

$\Vdash^{\mathrm{sw}}_{\mathrm{L}}$ is always m-logic

Possible symmetrizations of a logic $\vdash_{\mathrm{L}}$

$\Gamma \Vdash^1_{\mathrm{L}} \Delta$ iff there is $\delta \subseteq \Delta$ and $\Gamma \vdash_{\mathrm{L}} \delta$

$\Gamma \Vdash^1_{\mathrm{L}} \Delta$ iff there is $\delta' \subseteq \Delta$ and $\mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$

$\Gamma \Vdash^{\mathrm{fin}}_{\mathrm{L}} \Delta$ iff there is a finite $\Delta' \subseteq \Delta$ and $\Gamma \vdash_{\mathrm{L}} \bigvee \Delta'$

$\Gamma \Vdash^{\mathrm{fin}}_{\mathrm{L}} \Delta$ iff there is a finite $\Delta' \subseteq \Delta$ and $\bigcap_{\delta \in \Delta'} \mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$

$\Gamma \Vdash^{\omega}_{\mathrm{L}} \Delta$ iff $\bigcap_{\delta \in \Delta} \mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$

$\Gamma \Vdash^{\mathrm{sw}}_{\mathrm{L}} \Delta$ iff for each evaluation s.t. $e[\Gamma] \subseteq \{1\}$ there is $\delta \in \Delta$ s.t. $e(\delta) = 1$

$\Gamma \Vdash^{s}_{\mathrm{L}} \Delta$ iff for each evaluation s.t. $e[\Gamma] \subseteq \{1\}$ we have $\sup_{\delta \in \Delta} e(\delta) = 1$

Relationships (some inclusions require certain framework):

$$\Vdash^1_{\mathrm{L}} \subsetneq \Vdash^{\mathrm{fin}}_{\mathrm{L}} \subsetneq \Vdash^{\mathrm{sw}}_{\mathrm{L}} \subsetneq \Vdash^{\omega}_{\mathrm{L}} \qquad \Vdash^{\mathrm{sw}}_{\mathrm{L}} \subsetneq \Vdash^{s}_{\mathrm{L}}$$

Possible symmetrizations of a logic $\vdash_{\mathrm{L}}$

$$\Gamma \Vdash^{1}_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{there is } \delta \subseteq \Delta \text{ and } \Gamma \vdash_{\mathrm{L}} \delta$$

$$\Gamma \Vdash^{1}_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{there is } \delta' \subseteq \Delta \text{ and } \mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$$

$$\Gamma \Vdash^{\mathrm{fin}}_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{there is a finite } \Delta' \subseteq \Delta \text{ and } \Gamma \vdash_{\mathrm{L}} \bigvee \Delta'$$

$$\Gamma \Vdash^{\mathrm{fin}}_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{there is a finite } \Delta' \subseteq \Delta \text{ and } \bigcap_{\delta \in \Delta'} \mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$$

$$\Gamma \Vdash^{\omega}_{\mathrm{L}} \Delta \quad \text{iff} \quad \bigcap_{\delta \in \Delta} \mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$$

$$\Gamma \Vdash^{\mathrm{sw}}_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{for each evaluation s.t. } e[\Gamma] \subseteq \{1\} \text{ there is } \delta \in \Delta \text{ s.t. } e(\delta) = 1$$

$$\Gamma \Vdash^{s}_{\mathrm{L}} \Delta \quad \text{iff} \quad \text{for each evaluation s.t. } e[\Gamma] \subseteq \{1\} \text{ we have } \sup_{\delta \in \Delta} e(\delta) = 1$$

Note that $\Vdash^{\mathrm{sw}}_{\mathrm{CL}}$ can be given syntactically

$$\Gamma \Vdash^{\mathrm{sw}}_{\mathrm{CL}} \Delta \quad \text{iff} \quad \Gamma, \{\neg\delta \mid \delta \in \Delta\} \vdash_{\mathrm{CL}} \bot$$

Possible symmetrizations of a logic $\vdash_{\mathrm{L}}$

$\Gamma \Vdash^1_{\mathrm{L}} \Delta$ iff there is $\delta \subseteq \Delta$ and $\Gamma \vdash_{\mathrm{L}} \delta$

$\Gamma \Vdash^1_{\mathrm{L}} \Delta$ iff there is $\delta' \subseteq \Delta$ and $\mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$

$\Gamma \Vdash^{\mathrm{fin}}_{\mathrm{L}} \Delta$ iff there is a finite $\Delta' \subseteq \Delta$ and $\Gamma \vdash_{\mathrm{L}} \bigvee \Delta'$

$\Gamma \Vdash^{\mathrm{fin}}_{\mathrm{L}} \Delta$ iff there is a finite $\Delta' \subseteq \Delta$ and $\bigcap_{\delta \in \Delta'} \mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$

$\Gamma \Vdash^{\omega}_{\mathrm{L}} \Delta$ iff $\bigcap_{\delta \in \Delta} \mathrm{Th}_{\mathrm{L}}(\delta) \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$

$\Gamma \Vdash^{\mathrm{sw}}_{\mathrm{L}} \Delta$ iff for each evaluation s.t. $e[\Gamma] \subseteq \{1\}$ there is $\delta \in \Delta$ s.t. $e(\delta) = 1$

$\Gamma \Vdash^{s}_{\mathrm{L}} \Delta$ iff for each evaluation s.t. $e[\Gamma] \subseteq \{1\}$ we have $\sup_{\delta \in \Delta} e(\delta) = 1$

And we can do it even in $\mathrm{BTLSMVA}$

$\Gamma \Vdash^{\mathrm{sw}}_{B\ldots} \Delta$ iff for each function $n \colon \Delta \to \omega$ we have $\Gamma, \{\neg(\delta^{n(\delta)}) \mid \delta \in \Delta\} \vdash_{\mathrm{B}\ldots} \bot$

Some references

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