The Geometry of Free Algebras in Chang Variety: a Bridge from Semisimplicity to Infinitesimals

Antonio Di Nola

Department of Mathematics
University of Salerno

Talk based on a joint work with G. Lenzi and G. Vitale

Duality in Algebra and Logic

Treatment of Infinitesimals: an exotic or negligible topic in MV-algebras theory!

Treatment of Infinitesimals: an exotic or negligible topic in MV-algebras theory!

Is it true?

Treatment of Infinitesimals: an exotic or negligible topic in MV-algebras theory!

Is it true?

For sure: It is **problematic**

Treatment of Infinitesimals: an exotic or negligible topic in MV-algebras theory!

Is it true?

For sure: It is **problematic**

But: **fascinating** and very **promising**.

A dualism

(1) Archimedean Algebras: A tractable class

—**vs**—

(2) Non Archimedean Algebras: A wild class

A dualism

(1) Archimedean Algebras: A tractable class

—**vs**—

(2) Non Archimedean Algebras: A wild class

How to tame (2)?

In MV-algebraic setting:

$$\textbf{Archimedean} = \textbf{Semisimple}$$

(•) Boolean Algebras are all semisimple

Duality in Algebra and Logic

(•) Boolean Algebras are all semisimple

(•) $[0, 1]$ is archimedean (more, it is simple)

Duality in Algebra and Logic

- (•) Boolean Algebras are all semisimple
- (•) $[0,1]$ is archimedean (more, it is simple)
- (•) Archimedean algebras have a good state theory

- (•) Boolean Algebras are all semisimple
- (•) $[0, 1]$ is archimedean (more, it is simple)
- (•) Archimedean algebras have a good state theory
- (•) Complete and σ-complete MV-algebras are archimedean

Duality in Algebra and Logic

- (•) Boolean Algebras are all semisimple
- (•) $[0, 1]$ is archimedean (more, it is simple)
- (•) Archimedean algebras have a good state theory
- (•) Complete and σ-complete MV-algebras are archimedean
- (•) Semisimple MV-algebras are dual with closed sets of a Tychonoff space (Marra-Spada duality)

Duality in Algebra and Logic

- (•) Boolean Algebras are all semisimple
- (•) $[0, 1]$ is archimedean (more, it is simple)
- (•) Archimedean algebras have a good state theory
- (•) Complete and σ-complete MV-algebras are archimedean
- (•) Semisimple MV-algebras are dual with closed sets of a Tychonoff space (Marra-Spada duality)
- (•) Finitely presented MV-algebras are archimedean and dual with rational polyhedra of $[0, 1]^n$

Duality in Algebra and Logic

- (•) Boolean Algebras are all semisimple
- (•) $[0,1]$ is archimedean (more, it is simple)
- (•) Archimedean algebras have a good state theory
- (•) Complete and σ-complete MV-algebras are archimedean
- (•) Semisimple MV-algebras are dual with closed sets of a Tychonoff space (Marra-Spada duality)
- (•) Finitely presented MV-algebras are archimedean and dual with rational polyhedra of $[0,1]^n$
- (•) Free MV-algebras are archimedean

Duality in Algebra and Logic

- (•) Boolean Algebras are all semisimple
- (•) $[0,1]$ is archimedean (more, it is simple)
- (•) Archimedean algebras have a good state theory
- (•) Complete and σ-complete MV-algebras are archimedean
- (•) Semisimple MV-algebras are dual with closed sets of a Tychonoff space (Marra-Spada duality)
- (•) Finitely presented MV-algebras are archimedean and dual with rational polyhedra of $[0,1]^n$
- (•) Free MV-algebras are archimedean
- (•) Propositional Łukasiewicz Logic is complete with respect to $[0,1]$ (Standard competeness)

(•) Why non-archimedean MV-algebras?

(•) Why non-archimedean MV-algebras?

Indeed

$$\textbf{there} \quad \textbf{are} \quad \textbf{interesting} \quad \textbf{non-archimedean} \quad \textbf{MV-algebras}$$

(•) Why non-archimedean MV-algebras?

Indeed

$$\textbf{there} \quad \textbf{are} \quad \textbf{interesting} \quad \textbf{non} - \textbf{archimedean} \quad \textbf{MV} - \textbf{algebras}$$

• We will try to justify some deep interest in non-archimedean MV-algebras.

The smallest non-archimedean MV-algebra:

$$Chang's \quad Algebra \quad C$$

• C is a kind of **virus** algebra: a lot of **peculiar phenomena** arise from the existence of non-archimedean MV-algebras

• C is essentially made by infinitesimals

$$\mathbf{0} \bullet \cdots \cdot \quad \cdot \quad \cdot \qquad\qquad\qquad \cdot \quad \cdot \quad \cdot \quad \cdots \bullet \mathbf{1}$$

The class of MV-algebras which generalizes the algebra C is given by

$$\textbf{Perfect} \quad MV - algebras,$$

those MV-algebras which are generated by their **radical** (i.e. algebras generated by their infinitesimal elements)

Duality in Algebra and Logic

The class of MV-algebras which generalizes the algebra C is given by

$$\textbf{Perfect} \quad MV - algebras,$$

those MV-algebras which are generated by their **radical** (i.e. algebras generated by their infinitesimal elements)

• The category of Perfect MV-algebras and the category of Abelian ℓ-groups are equivalent by a functor Δ:

from Abelian ℓ-groups $\xrightarrow{\Delta}$ to Perfect MV-algebras.

The class of MV-algebras which generalizes the algebra C is given by

$$\textbf{Perfect} \quad MV - algebras,$$

those MV-algebras which are generated by their **radical** (i.e. algebras generated by their infinitesimal elements)

• The category of Perfect MV-algebras and the category of Abelian ℓ-groups are equivalent by a functor Δ:

from Abelian ℓ-groups $\xrightarrow{\Delta}$ to Perfect MV-algebras.

• This equivalence allows a remarkable exchange of results between Perfect MV-algebras and the time-honored theory of lattice ordered abelian groups

Some facts about infinitesimals in MV-algebras theory:

Some facts about infinitesimals in MV-algebras theory:

(•) Every MV-algebra can be represented by an MV-algebra of functions valued in ${}^*[0, 1]$

Some facts about infinitesimals in MV-algebras theory:

- (•) Every MV-algebra can be represented by an MV-algebra of functions valued in $^*[0,1]$
- (•) The first order Łukasiewicz Logic is non standard-complete: there are formulas which are **true** and **not provable**

Some facts about infinitesimals in MV-algebras theory:

- (•) Every MV-algebra can be represented by an MV-algebra of functions valued in $^*[0,1]$
- (•) The first order Łukasiewicz Logic is non standard-complete: there are formulas which are **true** and **not provable**
- (•) Such formulas are **co-infinitesimals** in the Lindenbaum algebra, *FOL*, of First Order Łukasiewicz Logic
- (•) It is worth to study the structure and the properties of the **Perfect skeleton** of *FOL*, still unknown.

Duality in Algebra and Logic

• An intuitive interpretation of infinitesimals and co-infinitesimals in MV-algebras:

(•) **Infintesimals** as **almost false**

(•) **Co − infintesimals** as **almost true**

The logic of infinitesimal as approximation of truth

• take a formula α evaluated in $[0,1]$ $v(\alpha) = 1/3$ or $v(\alpha) = 1/2$

then α can be reasonably considered a formula which is quasi true (or also quasi false)

• but if $v(\alpha)$ is evaluated in an infinitesimal, then α has to be considered quasi false and not quasi true

• because, in this case, the behaviour of α is very much similar to the absolute falsity: $v(\alpha) = 0$

Indeed:

• when α is evaluated in $[0, 1]$ and $v(\alpha) \neq 0$, then there is $n \mid nv(\alpha) = 1$

• this cannot be when $v(\alpha)$ is infinitesimal.

• Evaluating a formula α on a perfect algebra A, we can interpret $v(\alpha)$:

(1) as measuring **how much** α is close to be true, if $v(\alpha) \in Co - Rad(A)$

(2) as measuring **how much** α is close to be false, if $v(\alpha) \in Rad(A)$

• This can be seen as a **quantitative information** about the truth value of α

We would like to justify the term **quantitative** thinking of abelian ℓ-groups (in duality with Perfect MV-algebras) as algebras of magnitudes (Mundici: lattice ordered abelian groups (l-groups) describe magnitude-valued functions defined on compact spaces)

• Reamarkable fact is that:There is a specific logic for the concept of quasi true, which is a conservative extension of Łukasiewicz logic.

The extension L_P of ŁL is given by adding the further axiom:

$$(x \oplus x) \odot (x \oplus x) \leftrightarrow (x \odot x) \oplus (x \odot x)$$

It happens that L_P is complete with respect to Chang'algebra C.

The equational class of MV-algebras obtained by adding the condition:

$$(2x)^2 = 2(x^2)$$

will be denoted by $V(C)$ (the variety generated by C).

Why the variety $V(C)$ is interesting? :

(-) For any MV-algebra, all the above defined skeletons, invariant, are members of $V(C)$;

(-) Every MV-algebra A has the greatest subalgebra belonging to $V(C)$ (the $V(C)$ **skeleton** of A: $Sk - V(C)(A)$, still an invariant of A;

(-) The maps sending A to its above skeletons, respectively, are functors;

(-) The logic L_P associated to $V(C)$ has nice properties that we will see later on;

(-) Every $V(C) - algebra$ is **generated** by the **union** of its **Boolean** skeleton and its **Perfect** skeleton.

(-) $Max(A)$ is homeomorphic to $Max(B(A))$, hence a Stone space

(-) $B(A)$ is a retract with respect to the $Rad(A)$

Duality in Algebra and Logic

- The categories of **Boolean Algebras** and **Perfect MV − algebras** contain very different objects,
- in some extent the categories are *opposites* categories
- Boolean algebras are semisimple (i.e. have $Radical = \{0\}$)
- Perfect MV-algebras are just generated by their Radical

The opposite poles of V(C)

What in between?

$$Boolean\ Algebras\ \Rightarrow ???\ \Leftarrow\ Perfect\ MV-algebras$$

The axiom $2x^2 = (2x)^2$ reconciles the extreme poles.

L_P is:

(-) L_P is complete with respect to all **PerfectMV-chains**

(-) L_P is complete with respect to $\Gamma(Zx_{lex}R,(1,0))$

(-) L_P is structurally complete

(-) the tautology problem is coNP-complete

- To speak of $V(C)$ algebras, as a slogan, we can say that:

$V(C)$ algebras are made by **clouds** of **infinitesimals** around **idempotents**

- better: $V(C)$ algebras **fuse** together boolean elements with infinitesimals

- that is: **qualitative** information **fused** with **quantitative** information

T_{LEX} Transformation
A machinery to deal with infinitesimals

$$\mathbf{T_{LEX} - transform \quad and \quad Free \quad Algebras}$$

(•) A **functorial** way to move from:

Finitely Generated Free MV-algebras $\rightarrow$ to Finitely Generated Free $V(C)$-algebras

(•) We try to build a geometry of free $V(C)$-algebras by paralleling the way used for Free MV-algebras

(•) this means to pass from a geometry based on an Archimedean algebra to a non archimedean one.

(•) Note that: $[0,1]$ generates the variety MV

and

(•) $T_{LEX}([0,1]) = \Gamma(ZX_{lex}R, (1,0))$ generates the variety $V(C)$.

(•) McNaughton Theorem (for MV-algebras)

Free MV-algebras are MV-algebras of functions from $[0,1]^n$ to $[0,1]$ which are continuous, piecewise linear with integer coefficients.

• we can parallel the above theorem.

(1) Given a formula α, by McNaughton theorem we get a McNaughton function f_α

(1) Given a formula α, by McNaughton theorem we get a McNaughton function f_α

(2) Then we have the constituents $\{f_\alpha^1, ..., f_\alpha^k\}$, of f_α, a family of linear functions and a simplicial complex of $[0,1]^n$, $\{\sigma_1, ..., \sigma_k\}$

(1) Given a formula α, by McNaughton theorem we get a McNaughton function f_α

(2) Then we have the constituents $\{f_\alpha^1, \ldots, f_\alpha^k\}$, of f_α, a family of linear functions and a simplicial complex of $[0,1]^n$, $\{\sigma_1, \ldots, \sigma_k\}$

(3) By DNL-Lettieri Normal Form, we get $\phi_1, \ldots, \phi_k$ **MV-polynomials** with corresponding McNaughton functions $f_{\phi_1}, \ldots, f_{\phi_k}$ whose restrictions to $\{\sigma_1, \ldots, \sigma_k\}$, respectively, coincide with $\{f_\alpha^1, \ldots, f_\alpha^k\}$

- For a vertex v of $[0,1]^n$

- For a vertex v of $[0,1]^n$

- We can consider all the simplexes having v as a vertex (v-simplicial complex)

• For a vertex v of $[0,1]^n$

• We can consider all the simplexes having v as a vertex (v-simplicial complex)

• (Def.) An abstract simplicial complex F_v is called abstract simplicial **fan** iff F_v is a v-simplicial complex.

• Note that v-simplicial complex and abstact fans are combinatorially equivalent

• We have seen that for each McNaughton function f_α we can construct a simplicial complex Λ_{f_α}

• We have seen that for each McNaughton function f_α we can construct a simplicial complex Λ_{f_α}

• Then there is a canonical way to associate to each v-complex an abstract simplicial fan

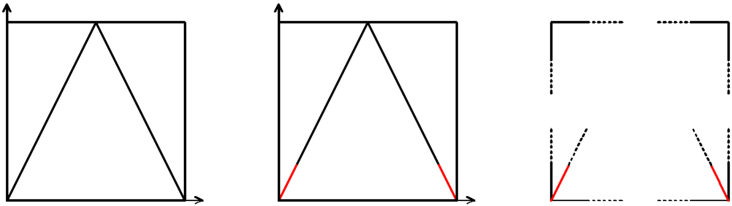
• We have seen that for each McNaughton function f_α we can construct a simplicial complex Λ_{f_α}

• Then there is a canonical way to associate to each v-complex an abstract simplicial fan

• We can evaluate the **polynomials** $\phi_1, ..., \phi_k$ by functions in $\Delta(R)^{\Delta(R)^n}$ and to get a map:

$$T_{Lex} : Free_{MV}(n) \longrightarrow \Delta(R)^{\Delta(R)^n}$$

Example



Theorem:

$T_{Lex}(Free_{MV}(n))$ is the free n-generated algebra in $V(C)$

Theorem:

$T_{Lex}(Free_{MV}(n))$ is the free n-generated algebra in $V(C)$

- T_{Lex} of ideals are ideals, i.e. $T_{Lex}(I) := \{T_{Lex}(f) \mid f \in I\}$

Theorem:

$T_{Lex}(Free_{MV}(n))$ is the free n-generated algebra in $V(C)$

• T_{Lex} of ideals are ideals, i.e. $T_{Lex}(I) := \{T_{Lex}(f) \mid f \in I\}$

• T_{Lex} is a functor from the category of free finitely generated MV-algebras to the category of free finitely generated $V(C)$-algebras.

As consequences of the above theorem, and by applications of T_{Lex} transform, we have:

McNaughton-type Theorem for Free-V(C)-algebras

(•) Free $V(C)$-algebras are algebras of functions f from $(T_{LEX}([0,1]))^n$ to $T_{LEX}([0,1])$ such that:

- f is continuous and there are a finite number of distinct linear functions with integer coefficients $\lambda_1, ..., \lambda_n$ such that for each $(x_1, ..., x_n) \in (T_{LEX}([0,1]))^n$ there exists $j \in \{1, ..., n\}$ such that:

$$f(x_1, ..., x_n) = \lambda_j(x_1, ..., x_n).$$

$$V(C) - Semisimple \quad Algebras$$

(•) a V(C)-algebra A, is called $V(C)$-**simple** iff A is subalgebra of $\Delta(\mathbb{R})$.

(•) a V(C)-algebra A, is called $V(C)$-**simple** iff A is subalgebra of $\Delta(\mathbb{R})$.

(•) The category $\mathbf{V(C)_{ss}}$ of $V(C)$-**semisimple** algebras:

($\bullet$) a V(C)-algebra A, is called $V(C)$-**simple** iff A is subalgebra of $\Delta(\mathbb{R})$.

($\bullet$) The category $\mathbf{V(C)_{ss}}$ of $V(C)$-**semisimple** algebras:

($-$) Objects of $\mathbf{V(C)_{ss}}$ are $V(C)$-algebras isomorphic with a subdirect product of copies of $\Delta(\mathbb{Z})$,
($-$) Morphisms of $\mathbf{V(C)_{ss}}$ are definable maps.

($\bullet$) a V(C)-algebra A, is called $V(C)$-**simple** iff A is subalgebra of $\Delta(\mathbb{R})$.

($\bullet$) The category $\mathbf{V(C)_{ss}}$ of $V(C)$-**semisimple** algebras:

($-$) Objects of $\mathbf{V(C)_{ss}}$ are $V(C)$-algebras isomorphic with a subdirect product of copies of $\Delta(\mathbb{Z})$,
($-$) Morphisms of $\mathbf{V(C)_{ss}}$ are definable maps.

($\bullet$) Note that Free $V(C)$-algebras are $V(C)$-semisimple

(•) Let **MS** be the Marra-Spada functor between **semisimple MV-algebras** and closed sets in $[0,1]^\alpha$ with α cardinal equiped with definable maps as mophisms.

(•) Let **MS** be the Marra-Spada functor between **semisimple MV-algebras** and closed sets in $[0,1]^\alpha$ with α cardinal equiped with definable maps as mophisms.

(•) In a natural way we can replicate the above result in Chang variety, considering the category $\mathbf{V(C)_{ss}}$.

($\bullet$) Let **MS** be the Marra-Spada functor between **semisimple MV-algebras** and closed sets in $[0,1]^\alpha$ with α cardinal equiped with definable maps as mophisms.

($\bullet$) In a natural way we can replicate the above result in Chang variety, considering the category $\mathbf{V(C)_{ss}}$.

($\bullet$) Indeed, $\mathbf{T_{LEX}}|_{\mathbf{MV_{ss}}}$ is a functor from $\mathbf{MV_{ss}}$ to $\mathbf{V(C)_{ss}}$.

(•) $\mathbb{Z}$-**closed sets** in $\Delta(\mathbb{Z})^\alpha$

Def: A subset S of $\Delta(\mathbb{Z})^\alpha$ iff $Z(I(S))$ is a closed set in the Zarisky topology.
We have that:

Semisimple MV-algebras

Semisimple MV-algebras

$$\Big\updownarrow^{\textit{Marra}-\textit{Spada}}$$

Closed subsets of $[0,1]^n$

$$\textbf{Semisimple MV-algebras}$$
$$\updownarrow^{Marra-Spada}$$
$$\textbf{Closed subsets of } [0,1]^n$$
$$\downarrow^{\mathcal{T}_{\mathcal{LEX}}}$$
$$\mathbb{Z}\textbf{-closed sets} \text{ in } \Delta(\mathbb{Z})^{\alpha}$$

Semisimple MV-algebras

$$\updownarrow^{\textit{Marra}-\textit{Spada}}$$

Closed subsets of $[0,1]^n$

$$\downarrow^{\mathcal{T}_{\mathcal{LEX}}}$$

$\mathbb{Z}$**-closed sets** in $\Delta(\mathbb{Z})^{\alpha}$

Semisimple MV-algebras

$$\begin{array}{c} \textbf{Semisimple MV-algebras} \\ \big\updownarrow{\scriptstyle Marra-Spada} \\ \textbf{Closed subsets of } [0,1]^n \\ \big\downarrow{\scriptstyle \mathcal{T}_{\mathcal{LEX}}} \\ \mathbb{Z}\textbf{-closed sets in } \Delta(\mathbb{Z})^\alpha \end{array}$$

$$\begin{array}{c} \textbf{Semisimple MV-algebras} \\ \big\downarrow{\scriptstyle T_{LEX}} \\ \textbf{V(C)-semisimple algebras} \end{array}$$

$$\begin{array}{c} \textbf{Semisimple MV-algebras} \\ \updownarrow{\scriptstyle Marra-Spada} \\ \textbf{Closed subsets of } [0,1]^n \\ \downarrow{\scriptstyle \mathcal{T_{LEX}}} \\ \mathbb{Z}\textbf{-closed sets in } \Delta(\mathbb{Z})^\alpha \end{array} \qquad \begin{array}{c} \textbf{Semisimple MV-algebras} \\ \downarrow{\scriptstyle T_{LEX}} \\ \textbf{V(C)-semisimple algebras} \\ \updownarrow{\scriptstyle V(C)-Marra-Spada} \\ \mathbb{Z}\textbf{-closed sets in } \Delta(\mathbb{Z})^\alpha \end{array}$$

Finitely presented MV-algebras in $V(C)$

(•) **Def**: A fan P is a rational fan iff every cone belonging to P is defined by linear inequalities with rational coefficients.

Theorem

For any $A \in V(C)$ the following conditions are equivalent:

1. *A is finitely presented;*
2. *for some rational fan P in $\Delta(\mathbb{R})^n$ $A \cong \mathbf{T_{LEX}}(M_n)|_P$;*
3. *$A \cong LIND^1_\theta$ for some θ satisfable formula.*

- Note that:

passing from the variety MV to $V(C)$, in the case of finitely presented algebras , the role of **Rational polyhedra**, is converted into the role of **Rational fans**, via T_{LEX} transform.

$\mathbf{T}_{LEX}$ Transformation

MV-algebras	$\xrightarrow{T_{Lex}}$	$V(C)$-algebras
Free	$\xrightarrow{T_{Lex}}$	Free
Semisimple	$\xrightarrow{T_{Lex}}$	$V(C)$-semisimple
Closed subsets of $[0,1]^\alpha$	$\xrightarrow{T_{Lex}}$	$\mathbb{Z}$-**closed sets** in $\Delta(\mathbb{Z})^\alpha$
Finitely Presented	$\xrightarrow{T_{Lex}}$	Finitely Presented
Rational Polyedra in $[0,1]^n$	$\xrightarrow{T_{Lex}}$	Rational Fans in $\Delta(\mathbb{Z})^n$

On the structure of $V(C)$ algebras

Every MV-algebra A in the variety $V(C)$ is generated by:

- its Boolean skeleton $B(A)$

and its

- Perfect skeleton $Perf(A)$.

Usually it is **not enough** to fix a Boolean algebra B and a perfect MV-algebra *Perf* to univocally determine an MV-algebra A such that $B(A) \cong B$ and $Perf(A) \cong P$.

($\bullet$) Additional information is needed

Normal Form in $V(C)$-algebras

1. For every MV algebra $A \in V(C)$ there are a **unique** $b \in B(A)$ and a **unique** $\epsilon \in Rad(A)$ such that

$$x = (b \wedge \epsilon^*) \vee \epsilon$$

.

Finally, we let

$$N(x) = (b_x, \epsilon_x)$$

and we call $N(x)$ **the normal form of** x.

In order to describe the effect of MV-algebraic operations on the boolean and infinitesimal components we consider the polynomial: $b_1, b_2 \in B(A)$ and $h_1, h_2 \in Perf(A)$

$$T(b_1, b_2, h_1, h_2) =$$

$$(b_1 \wedge b_2^*) \wedge (h_1 \ominus h_2) \oplus (b_1^* \wedge b_2) \wedge (h_2 \ominus h_1) \oplus (b_1^* \wedge b_2^*) \wedge (h_1 \oplus h_2).$$

($\bullet$) The polynomial T results to be an invariant for $V(C)$-algebras.

A representation theorem Let A be a $V(C)$ algebra and $D_A = B(A) \times Rad(A)$. Then on D_A we define the following operations:

$$(b, \varepsilon) \mp (b', \varepsilon') = (b \vee b', T_A(b, b', \varepsilon, \varepsilon'))$$

$$\neg(b, \varepsilon) = (b^*, \varepsilon).$$

Theorem

$D_A = (D_A, \mp, (0,0), \neg)$ is an MV-algebra and D_A is isomorphic to A.

By the above theorem

• we can see that starting with a $V(C)$ algebra A,

• we can get a triple of pieces of information that, up to isomorphism, completely codifies the MV-algebraic structure of A.

• So the triple is a complete invariant for $V(C)$ algebras, given by $B(A), Rad(A), T_A$.

Conclusions. We have seen that:

(-) In the wild class of non-archimedean MV-algebras, V(C)-algebras have a quite tractable behaviour.

Conclusions. We have seen that:

(-) In the wild class of non-archimedean MV-algebras, V(C)-algebras have a quite tractable behaviour.

(-) It seems clear that their behaviour parallels that of archimedean ones, at least in the relationships existing among logic, free algebras and their geometry.

Conclusions. We have seen that:

(-) In the wild class of non-archimedean MV-algebras, V(C)-algebras have a quite tractable behaviour.

(-) It seems clear that their behaviour parallels that of archimedean ones, at least in the relationships existing among logic, free algebras and their geometry.

(-) $\mathbf{T_{LEX}}$ functor provides useful and powerful mechanism to generate, manipulate and manage the infinitesimals, i.e. the algebraic and analytical representatives of perturbations of *clear* truth values.

(-) $\mathbf{T_{LEX}}$ functor provides useful and powerful mechanism to generate, manipulate and manage the infinitesimals, i.e. the algebraic and analytical representatives of perturbations of *clear* truth values.

(-) The presented results open the door to an analogous treatment for all varieties of MV-algebras generated by a single non-archimedean chain, showing how, in some regards, to pass from results on archimedean MV-algebras to non-archimedean ones.

Some perspectives:

1. Having a convincing representation of free $V(C)$-algebras, via McNaughton-like functions
2. Recalling the interpretation of elements of free MV-algebras as **generalized events**, as suggested by D. Mundici
3. then we can look at any element of free $V(C)$-algebras as special case of generalized event, and of course, an element of free Boolean algebras as classical event

Duality in Algebra and Logic

1. Hence we can think of $V(C)$-events as obtained by a **fusion** of classical events with **almost true** events or **almost false** event
2. An analysis of the structure of $V(C)$-algebras can show how such a fusion can be performed

Duality in Algebra and Logic

1. Hence we can think of $V(C)$-events as obtained by a **fusion** of classical events with **almost true** events or **almost false** event
2. An analysis of the structure of $V(C)$-algebras can show how such a fusion can be performed
3. This open the door to questions, for example, about setting, in the framework of $V(C)$-algebras, of notions as, $V(C) - \sigma$-complete algebra, $V(C)$-random variable, $V(C)$-tribe, and so on, via T_{Lex} transformation of the classical, or MV-algebraic corresponding notions.