Representation and Duality for Distributoids

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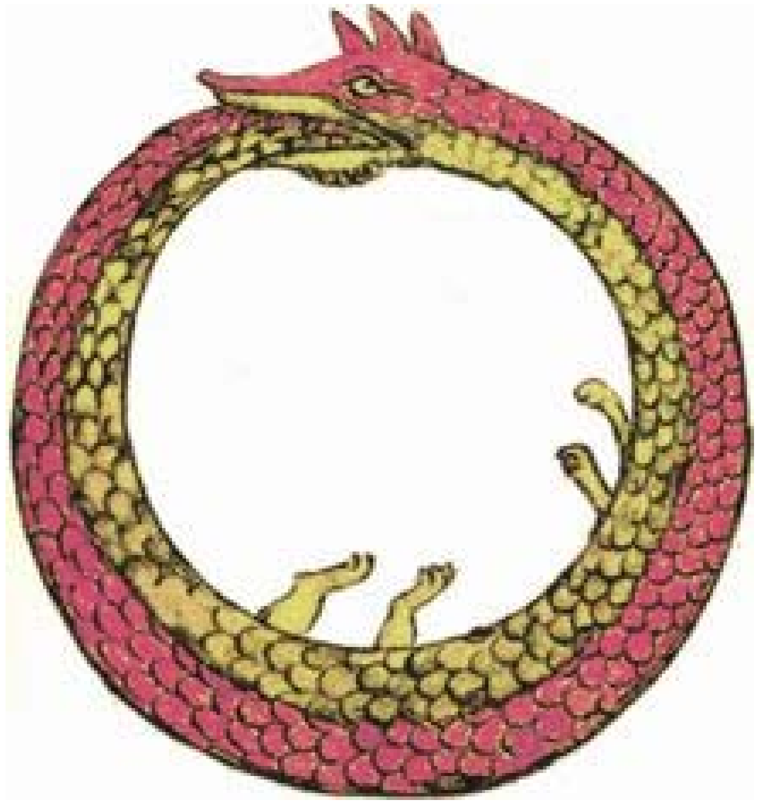
Duality



Ouroboros

As might be guessed from the workshop title, my talk will be about Duality in Algebra and Logic. The algebraic structures I will be talking about, distributoids and gaggles, are generalizations of the Boolean algebra of classical logic, and apply to many non-classical logics. And duality at the "object level" is built into their nature. They are closely related to Jonsson and Tarski's Boolean algebras with operators.

Distributoids have operations on a distributive lattices that distribute over join and meet, but in the process they are allowed to change each to its dual, joint to meet, and meet to join.

Gaggles tie together operations on the same distributoid into a family of operations using duality. I will spend most of this talk explaining distributoids and gaggles, but I will end the talk by talking about "meta level" duality for distributoids and gaggles, namely a topological duality.

Boolean Algebra with Operators (BAO)

We begin by talking about "Boolean algebras" with operators, which were my inspiration for distributoids. Jónsson and Tarski (1951-52) extended Stone's representation of Boolean algebras by considering Boolean algebras with certain kinds of operations on them which they called "operators":

- $(B, \wedge, \vee, -, \{o_i\}_{i \in I})$
- $(B, \wedge, \vee, -)$ is a Boolean algebra $[0 = a \wedge -a]$
- Each o_i is
- i) an *operation* on B that is ii) *normal* and iii) *additive*, i.e.

  i) For some n, $o_i: B^n \rightarrow B$

  ii) For each $m \leq n$, $f_i(x_1, \ldots, 0_m, \ldots, x_n) = 0$

  iii) $f_i(x_1, \ldots, (a \vee b)_m, \ldots, x_n) = f_i(x_1, \ldots, a_m, \ldots, x_n) \vee f_i(x_1, \ldots, b_m, \ldots, x_n)$

  It follows from (iii) that each operation f_i is monotonic:

$$a \leq b \text{ implies } f_i(x_1, \ldots, a_m, \ldots, x_n) \leq f_i(x_1, \ldots, b_m, \ldots, x_n)$$

Concrete BA

- Let U be a set, and consider a collection of subsets of U closed under intersection, union, and relative complement (to U). This is a Boolean algebra. M. Stone (1936) showed that every BA is isomorphic to such a “field of sets.” This is referred to as Stone’s Representation of BA’s.
- Jónsson and Tarski (1951 – actually their abstract was in 1949) extended this result to BAO’s by adding relations on U and defining “generalized image operators” as follows.

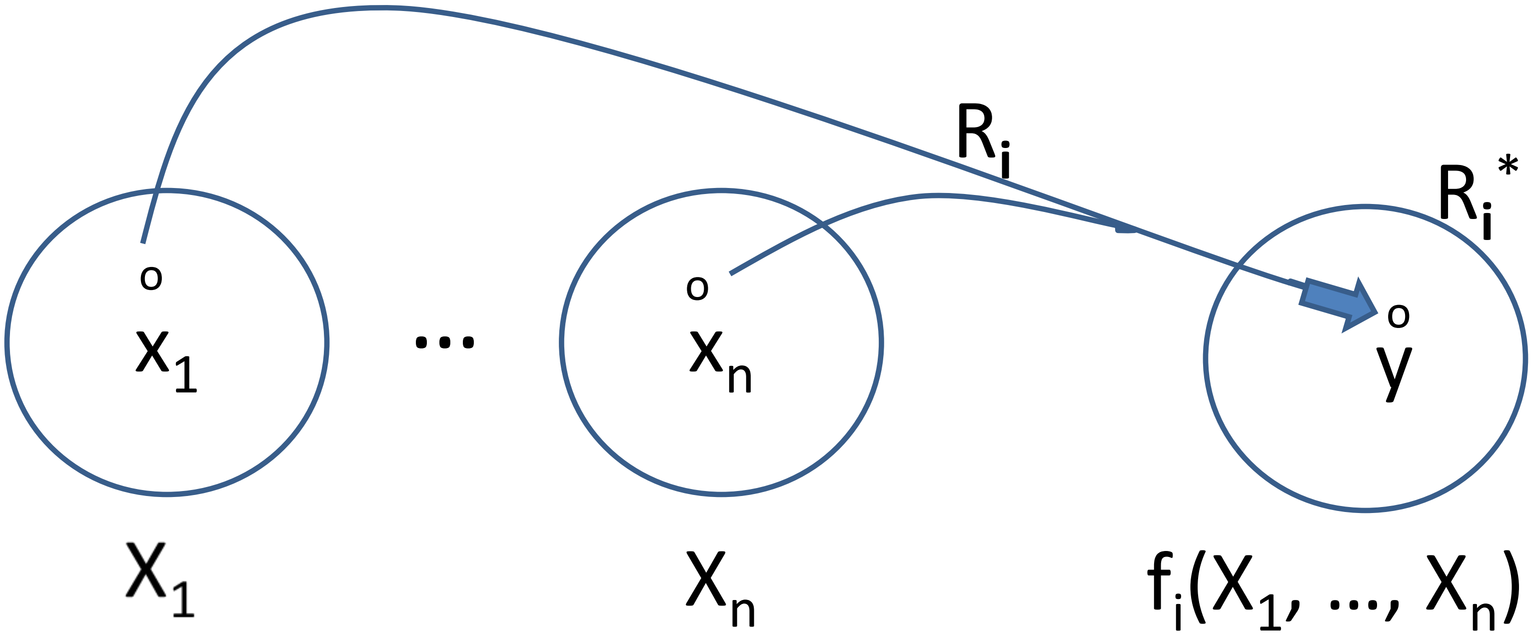
Generalized Image Operators

Let U be a non-empty set and let $\{R_i\}_{i \in I}$ be an indexed set of relations on U. Where R_i is of degree n + 1, and $X_1, \ldots, X_n \subseteq U$, define $f_i(X_1, \ldots, X_n) = R_{\mathbf{i}}^*(X_1, \ldots, X_n) = \{y: \exists\, x_1 \in X_1 \ldots \exists\, x_n \in X_n$ s.t. $R_{\mathbf{i}}\, x_1, \ldots, x_n\, y\}$.

This is a "generalized image operator," and includes the familiar special case for binary R:

$$f(X) = R^*X = \{y: \exists\, x \in X, R\, x\, y\}.$$

If we think of X as a set of "possible worlds," a "UCLA proposition" (the set of worlds in which it is true), then R can be thought of as (the converse of) Kripke's relative possibility relation, and o can be thought of as the possibility operator.



$$f_i(X_1, \ldots, X_n) = R_i^*(X_1, \ldots, X_n) =$$
$$\{y\text{: } \exists\, x_1 \in X_1, \ldots, \exists\, x_n \in X_n \text{ s.t. } R_i(x_1, \ldots, x_n, y)$$

Unary Logical Paradigms of (Co-) Distribution

These, especially the first two, are very familiar from modal logic.

Let (U, R) be a Kripke frame, i.e., U is a non-empty set of states and R is a binary relation on U ("accessibility"). For $A \subseteq U$ define:

Distributes over $\vee$

$\Diamond A = \{\chi : \exists\alpha\ (\chi R\alpha\ \&\ \alpha \in A)\}$ "Possible A"

$\Diamond(A \cup B) = \Diamond A \cup \Diamond B$ $\vee \mapsto \vee$

Distributes over $\wedge$

$\Box A = \{\chi : \forall\alpha\ (\text{not-}\chi R\alpha \text{ or } \alpha \in A)\}$ "Necessary A"

$\Box(A \cap B) = \Box A \cap \Box B$ $\wedge \mapsto \wedge$

Co-distribute over $\vee$

${}^{\perp}A = \{\chi : \forall\alpha\ (\text{not-}\chi R\alpha \text{ or } \alpha \notin A)\}$ "Impossible A"

${}^{\perp}(A \cup B) = {}^{\perp}A \cap {}^{\perp}B$ $\vee \mapsto \wedge$

Co-distributes over $\wedge$

$?A = \{\chi : \exists\alpha\ (\chi R\alpha\ \&\ \alpha \notin A)\}$ "Possible not A"

$?(A \cap B) = ?A \cup ?B$ $\wedge \mapsto \vee$

Unary Logical Paradigms of (Co-) Distribution

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Distribution Types

Co-distribute over $\vee$

${}^{\perp}A = \{\chi : \forall\alpha\, (\text{not-}\chi R\alpha \text{ or } \alpha \notin A)\}$ "Impossible A"

${}^{\perp}(A \cup B) = {}^{\perp}A \cap {}^{\perp}B$ $\vee \mapsto \wedge$

Co-distributes over $\wedge$

$?A = \{\chi : \exists\alpha\, (\chi R\alpha \;\&\; \alpha \notin A)\}$ "Possible not A"

$?(A \cap B) = ?A \cup ?B$ $\wedge \mapsto \vee$

Now consider binary operations. They can have 8 different distribution types, generalizing the Jónsson-Tarski requirement that the operations distribute over $\vee$.

$$
\begin{array}{llll}
\circ : & (\vee, \vee) \mapsto \vee & + : & (\wedge, \wedge) \mapsto \wedge \\
\rightarrow : & (\vee, \wedge) \mapsto \wedge & \rightharpoondown : & (\vee, \wedge) \mapsto \vee \\
\leftarrow : & (\wedge, \vee) \mapsto \wedge & \leftharpoondown : & (\wedge, \vee) \mapsto \vee \\
\dagger : & (\vee, \vee) \mapsto \wedge & \multimap : & (\wedge, \wedge) \mapsto \vee
\end{array}
$$

But what about "normality." Remember that Jónsson and Tarski required $o_i(x_1, \ldots, 0_m, \ldots, x_n) = 0$. ("One bad apple spoils the barrel.") But this rules out such natural logical operations as necessity, negation, and implication. Necessity preserves 1, negation inverts 0 to 1, and implication does both:

$$x_1 \rightarrow 1 = 1$$
$$0 \rightarrow x_2 = 1$$

Let's rewrite the distribution types, putting 0 in place of $\vee$ and 1 in place of $\wedge$:

$\circ$:	$(0,\ 0) \mapsto 0$	$+$:	$(1,\ 1) \mapsto 1$
$\rightarrow$:	$(0,\ 1) \mapsto 1$	$\rightharpoondown$:	$(0,\ 1) \mapsto 0$
$\leftarrow$:	$(1,\ 0) \mapsto 1$	$\leftharpoondown$:	$(1,\ 0) \mapsto 0$
$\dagger$:	$(0,\ 0) \mapsto 1$	ϕ :	$(1,\ 1) \mapsto 0$

Consider the distribution type of $\longrightarrow$. It can now be read: *if first argument is 0 evaluate as 1; if second argument is 1 evaluate as 1,* i.e.,

$$x_1 \rightarrow 1 = 1$$
$$0 \rightarrow x_2 = 1.$$

And similarly with the other distribution types.

Let's focus on one on just one of these binary operations, $\to$.

Suppose we are trying to algebraize an implication operation $\to$. It is natural to have a distributoid because $\to$ co-distributes over $\vee$ in the antecedent and distributes over $\wedge$ in the consequent:

$(x \vee y) \to z = (x \to z) \wedge (y \to z)$ *co-distributes*

$x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ *distributes*

So the distribution type is: $\vee, \wedge \mapsto \wedge$

If we take a ternary accessibility relation R and define

$A \to B = \{\chi: \forall\alpha, \beta: R\alpha\chi\beta \Rightarrow (\alpha \in A \Rightarrow \beta \in B)\}$, then it can be shown that

$(A \cup B) \to C = (A \to C) \cap (B \to C)$ *co-distributes*

$A \to (B \cap C) = (A \to B) \cap (A \to C)$ *distributes*

The definition of $\to$ is analogous to the satisfaction clause for relevant implication in the Routley-Meyer semantics for relevance logic:

$\chi \Vdash A \to B$ *iff* $\forall\alpha, \beta$: $R\alpha\chi\beta \Rightarrow (\alpha \in A \Rightarrow \beta \in B)$

Two incidental differences:

1. In the Routley-Meyer semantics A and B are sentences, not sets.
2. And χ is in the first, not the second position: $R\chi\alpha\beta$.

One person's first or second position is another person's third. It doesn't hurt to require the "abstraction variable," in this case χ, to always be in the last position.

This definition of $\rightarrow$ is analogous to the Jónsson-Tarski representation of an *n*-ary operator in terms of an $n + 1$-place relation. **BUT the representation of $\rightarrow$ is not a generalized image operator.** We need a schematic way to determine the definition of the representation using The distribution type of the operator we are representing, in this case:

$$\vee, \wedge \mapsto \wedge$$

Let us begin to untangle this by rewriting the definition:

$$\begin{aligned} A \rightarrow B &= \{\chi : \forall\alpha, \beta: \ R\alpha\beta\chi \Rightarrow (\alpha \in A \Rightarrow \beta \in B)\} \\ &= \{\chi : \forall\alpha, \beta: \ \text{not-}R\alpha\beta\chi \text{ or } \alpha \notin A \text{ or } \beta \in B)\} \end{aligned}$$

This is analogous to the Jónsson-Tarski representation of an *n*-ary operator in terms of an $n + 1$-place relation. **BUT the representation of $\longrightarrow$ is not a generalized image operator.** We need a schematic way to determine the definition of the representation using the distribution type of the operator we are representing, in this case:

$$\vee, \wedge \longmapsto \wedge$$

Let us begin to untangle this by rewriting the definition

$$
\begin{aligned}
A \rightarrow B &= \{\chi : \forall \alpha, \beta : \; R\alpha\beta\chi \Rightarrow (\alpha \in A \Rightarrow \beta \in B)\} \\
&= \{\chi : \forall \alpha, \beta : \; \text{not-}R\alpha\beta\chi \text{ or } \alpha \notin A \text{ or } \beta \in B)\}
\end{aligned}
$$

Realization Condition

What we need to do is figure out how to translate a distribution type into its corresponding realization condition.

Here is a first try. Don’t worry, it will continue to get more complicated. ☺

Given an n-ary operator f_i of distribution type t: $\tau_1, ..., \tau_n \mapsto \tau$,

1) if $\tau = \wedge$, the realization condition for t is of the form:
$\forall \alpha_1, \ldots, \alpha_n :_: R\, \alpha_1, \ldots, \alpha_n, \chi$ or $\pm(\alpha_1 \in A_1)$ or … or $\pm(\alpha_n \in A_n)$
where each component $\pm(\alpha_i \in A_i)$ is either $\alpha_i \in A_i$ or $\alpha_i \notin A_i$ depending on whether τ_i is $\wedge$ or $\vee$ respectively;

2) if $\tau = \vee$, the realization condition for t is of the form:
$\exists \alpha_1, \ldots, \alpha_n :_: R\, \alpha_1, \ldots, \alpha_n, \chi$ & $\pm(\alpha_1 \in A_1)$ & … & $\pm(\alpha_n \in A_n)$
where each component $\pm(\alpha_i \in A_i)$ is either $\alpha_i \in A_i$ or $\alpha_i \notin A_i$ depending on whether τ_i is $\vee$ or $\wedge$ respectively;

Note that 1) and 2) are appropriately dual.

Let’s see how these clauses work for our first two unary paradigms, Possibility $\Diamond$ and necessity $\Box$.

1. Possibility distributes over $\vee$ $\quad \vee \mapsto \vee \quad$ $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$

Since the output type is $\vee$ we get an existentially quantified conjunction $\exists\alpha(R\alpha\chi \ \& \pm (\alpha \in A)]$, and since the input type is $\vee$ we can delete the $\pm$ and obtain $\Diamond A = \{\chi: \exists\alpha\ (\chi R\alpha \ \&\ \alpha \in A)\}$.

Perfect!

:

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~~Perfect!~~

Oops! Close but no cigar!

Problem: $\Diamond A$ requires $R\chi\alpha$ but our realization clause has instead $R\alpha\chi$.

Fix: Put in R^{-1} in place of R in Realization Condition.

Now let us look at our 2nd unary paradigm.

2. Necessity distributes over $\Box$ $\wedge \mapsto \wedge$ $\Box(A \wedge B) = \Box A \wedge \Box B$

Since the output type is $\wedge$ we get a universally quantified disjunction $\forall\alpha(R\alpha\chi$ or $\pm(\alpha \in A))$, and since the input type is $\wedge$ we can delete the $\pm$ and obtain $\Box A = \{\chi: \forall\alpha\ (\chi R\alpha$ or $\alpha \in A)\}$.

Now let us look at our 2nd unary paradigm.

2. Necessity distributes over $\Box$ $\wedge \mapsto \wedge$ $\Box(A \wedge B) = \Box A \wedge \Box B$

Since the output type is $\wedge$ we get a universally quantified disjunction $\forall\alpha(R\alpha\chi$ or $\pm(\alpha \in A))$, and since the input type is $\wedge$ we e can delete the $\pm$ and obtain $\Box A = \{\chi: \forall\alpha\ (\alpha R\chi$ or $\alpha \in A)\}$.

Problem 1: $\alpha R\chi$ should be negated.

Fix 1: Replace R with –R.

Problem 2: $\Box A$ requires $-R\chi\alpha$ but our realization clause has $\alpha R\chi$.

Fix 1: Same as before -- Replace R with R^{-1}.

General fix . Fix: Put $(-R)^{-1}$ in place of R in Realization Condition.

But why did we use the relation R in our first statement of the Realization Condition, why not –R, or R^{-1}, or $(–R)^{-1}$? It turns out that any of these relations would work just as well. We have in effect 4 different menus, differing only in these 4 ways. It is like choosing food from different menus.

MENU A

$$\beta \in +_{\downarrow} A \qquad\qquad \beta \in \diamond_{\downarrow} A$$

$$(\wedge \mapsto \wedge)\ \forall\alpha(\alpha \in A \text{ or } \alpha R \beta) \qquad\qquad (\vee \mapsto \vee)\ \exists\alpha(\alpha \in A \mathbin{\&} \alpha R \beta)$$

$$\beta \in A^{\perp} (= \perp_{\downarrow} A) \qquad\qquad \beta \in ?_{\downarrow} A$$

$$(\vee \mapsto \wedge)\ \forall\alpha(\alpha \in \overline{A} \text{ or } \alpha R \beta) \qquad\qquad (\wedge \mapsto \vee)\ \exists\alpha(\alpha \in \overline{A} \mathbin{\&} \alpha R \beta)$$

MENU B: $\overline{R}$ in place of R

$$\beta \in \Box_{\downarrow} A$$

$$(\wedge \mapsto \wedge)\ \forall\alpha(\alpha \in A \text{ or } \alpha \overline{R} \beta) \qquad\qquad (\vee \mapsto \vee)\ \exists\alpha(\alpha \in A \mathbin{\&} \alpha \overline{R} \beta)$$

$$(\vee \mapsto \wedge)\ \forall\alpha(\alpha \in \overline{A} \text{ or } \alpha \overline{R} \beta) \qquad\qquad (\wedge \mapsto \vee)\ \exists\alpha(\alpha \in \overline{A} \mathbin{\&} \alpha \overline{R} \beta)$$

MENU C: R^{-1} in place of R

$$\beta \in +A \qquad\qquad \beta \in \diamond A$$

$$(\wedge \mapsto \wedge)\ \forall\alpha(\alpha \in A \text{ or } \beta R \alpha) \qquad\qquad (\vee \mapsto \vee)\ \exists\alpha(\alpha \in A \mathbin{\&} R \alpha)$$

$$\beta \in {}^{\perp}A (= {}_{\perp}A) \qquad\qquad \beta \in ?A$$

$$(\vee \mapsto \wedge)\ \forall\alpha(\alpha \in \overline{A} \text{ or } \beta R \alpha) \qquad\qquad (\wedge \mapsto \vee)\exists\alpha(\alpha \in \overline{A} \mathbin{\&} \beta R \alpha)$$

MENU D: $(\overline{R})^{-1}$ in place of R

$$\beta \in \Box A$$

$$(\wedge \mapsto \wedge)\ \forall\alpha(\alpha \in A \text{ or } \beta \overline{R} \alpha) \qquad\qquad (\vee \mapsto \vee)\ \exists\alpha(\alpha \in A \mathbin{\&} \beta \overline{R} \alpha)$$

$$(\vee \mapsto \wedge)\ \forall\alpha(\alpha \in \overline{A} \text{ or } \beta \overline{R} \alpha) \qquad\qquad (\wedge \mapsto \vee)\ \exists\alpha(\alpha \in \overline{A} \mathbin{\&} \beta \overline{R} \alpha)$$

Things get more complicated when we get to ternary relations and beyond. There is no such thing as *the inverse* of $R\alpha\beta\gamma$. But there are all the permutations of its terms. It turns out that we do not need all of these, but only the ones that exchange the last term with any other.

$$R^{-1}\alpha_1 \ldots \alpha_i \ldots \alpha_n\gamma = R\alpha_1 \ldots \gamma \ldots \alpha_n\alpha_i$$

Representation Theorem for Distributoids. Every distributoid Isomorphic to a distributoid on a ring of sets with each operation f being defined according to its distribution type using an accessibility relation.

The essence of the proof requires a "canonical model" consisting of all the prime filters on the distributoid and defining on them for each *n*-ary operation f the *canonical accessibility relation*

$$R_f = \{(P_1, \ldots, P_n, Q) \text{ as follows:}$$

- If the output type τ of f is $\vee$, then R_f is a universally quantified disjunction $\pm(x_1 \in P_1), \ldots$ r $\pm(x_n \in P_n)$, $f(x_1, \ldots, x_n) \in Q$, where each component $\pm(x_i \in P_i)$ is either $x_i \in P_i$ or $x_i \notin P_i$ depending on whether τ_i is $\wedge$ or $\vee$ respectively;
- If the output type τ of f is $\wedge$, then R_f is an existentially quantified conjunction of $\pm(x_1 \in P_1), \ldots$ r $\pm(x_n \in P_n)$, $f(x_1, \ldots, x_n) \in Q$, where each component $\pm(x_i \in P_i)$ is either $x_i \in P_i$ or $x_i \notin P_i$ depending on whether τ_i is $\vee$ or $\wedge$ respectively;

Consider a (distributive) lattice-ordered residuated groupoid $(S, \leq \circ, \leftarrow, \rightarrow)$:

$$a \leq c \leftarrow b \quad \textit{iff} \quad a \circ b \leq c \quad \textit{iff} \quad b \leq a \rightarrow c$$

It is part of the customary definition of “lattice-ordered groupoid” that $\circ$ distributes over join in each argument. It can be proven that:

$$(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z) \quad \textit{co-distributes}$$
$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z) \quad \textit{distributes}$$

Symmetrically for $\leftarrow$.

So distr. types. $\circ$: $(\vee, \vee) \mapsto \vee$ Head of “family”

$\rightarrow$: $(\vee, \wedge) \mapsto \wedge$

$\leftarrow$: $(\wedge, \vee) \mapsto \wedge$

$\rightarrow$ and $\leftarrow$ are “contrapositives” of $\circ$ in the sense that their distrib. types are obtainable from the distrib. type of $\circ$ by interchanging an input type with an output type while dualizing them.

Contrapositives allow us to group operators into natural "families" as we saw with the lattice-ordered groupoid.

Consider the dual binary case $(S, \leq, +, \leftharpoondown, \rightharpoondown)$,

where $+$ distributes over $\wedge$:

$$a \geq c \leftharpoondown b \text{ iff } a + b \geq c \text{ iff b} \geq a \rightharpoondown c$$

It can be proven that:

$$(y \vee z) \rightharpoondown x = (y \rightharpoondown x) \vee (z \rightharpoondown x) \quad \textit{distributes}$$

$$z \rightharpoondown (x \wedge y) = (z \rightharpoondown x) \vee (z \rightharpoondown y) \quad \textit{co-distributes}$$

Symmetrically for $\leftharpoondown$.

Distr. types

$+ : \quad (\wedge, \wedge) \mapsto \wedge$ **Head of "family"**

$\rightharpoondown : \quad (\vee, \wedge) \mapsto \vee$

$\leftharpoondown : \quad (\wedge, \vee) \mapsto \vee$

Are we missing any distribution types? We have these two families:

$\circ$: $(\vee, \vee) \mapsto \vee$ $\quad$ $+$: $(\wedge, \wedge) \mapsto \wedge$

$\rightarrow$: $(\vee, \wedge) \mapsto \wedge$ $\quad$ $\neg$: $(\wedge, \vee) \mapsto \vee$

$\leftarrow$: $(\wedge, \vee) \mapsto \wedge$ $\quad$ ⌐ : $(\vee, \wedge) \mapsto \vee$

But we are missing:

$\dagger$: $(\vee, \vee) \mapsto \wedge$ $\quad$ ϕ : $(\wedge, \wedge) \mapsto \vee$

These are each their own contrapositive.

A lattice-ordered residuated groupoid can be realized as a set U with a ternary relation $R \subseteq U^3$:

$A \circ B \;=_{df}\; \{\chi: \exists\alpha \in A,\ \beta \in B: \ R\alpha\beta\chi\}$

$A \rightarrow B \;=_{df}\; \{\chi: \forall\alpha,\ \beta: \ R\alpha\chi\beta \Rightarrow (\alpha \in A \Rightarrow \beta \in B)\}$

$B \leftarrow A \;=_{df}\; \{\chi: \forall\alpha, \beta: \ R\chi\alpha\beta \Rightarrow (\alpha \in A \Rightarrow \beta \in B)\}$

Fact: $A \subseteq C \leftarrow B$ *iff* $A \circ B \subseteq C$ *iff* $B \subseteq A \rightarrow C$

Note that when $\circ$ is commutative ($A \circ B = B \circ A$)

then $A \rightarrow B = B \leftarrow A$.

Dually (subtraction):

$$A + B =_{\mathrm{df}} \{\chi : \forall \alpha, \beta : S\alpha\beta\chi \Rightarrow \alpha \in A \text{ or } \beta \in B\}$$
$$A \rightharpoondown B =_{\mathrm{df}} \{\chi : \exists \alpha \in A, \beta \notin B : S\alpha\chi\beta\}$$
$$B \leftharpoondown A =_{\mathrm{df}} \{\chi : \exists \alpha \notin A, \beta \in B : S\chi\alpha\beta\}$$

Fact: $A \supseteq C \leftharpoondown B$ *iff* $A + B \supseteq C$ *iff* $B \supseteq A \rightharpoondown C$

The definition of $A \circ B$ reflects the definition of "intensional conjunction" (often called "fusion") in the Routley-Meyer semantics for relevance logic.

Since in that context it is taken to be commutative, the definition of $A \rightarrow B$ can be seen as reflecting the Routley-Meyer definition of relevant implication.

A distributive-lattice-ordered residuated groupoid serves as a good paradigm for my "Generalized Galois Logics." Their acronymn is "ggl." It is pronounced "gaggle."

“Gaggle,” not “giggle”

Definition of a Gaggle

If $\tau = \vee$, $S(f, a_1, \ldots, a_n, b)$ abbreviates $f(a_1, \ldots, a_n) \leq b$; and
if $\tau = \wedge$, it abbreviates $b \leq f(a_1, \ldots, a_n)$.

Two operations f and g satisfy the *Abstract Law of Residuation* (or *Galois Connection*) when f and g are contrapositives (w.r.t. some *ith*-place) and

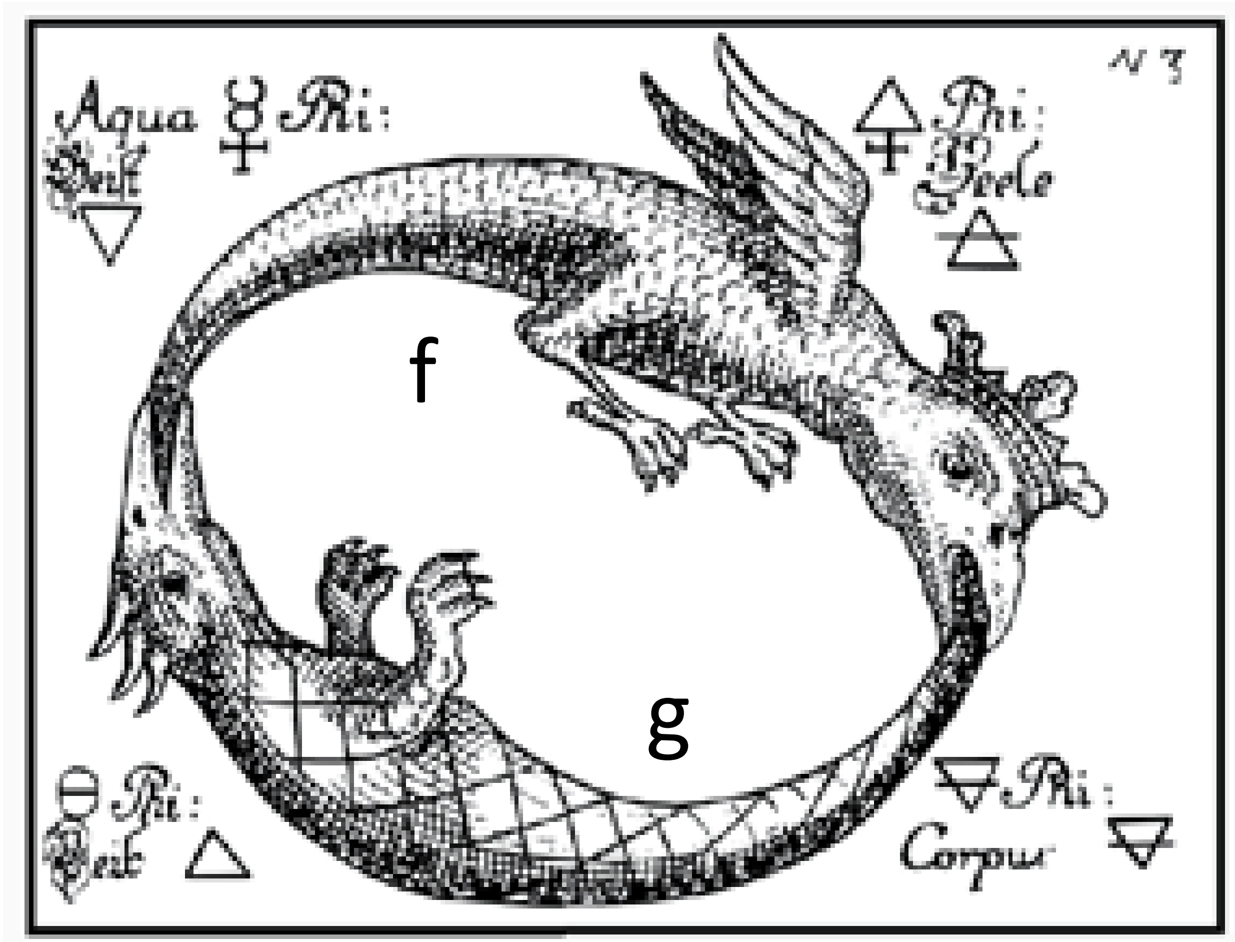
$$S(f, a_1, \ldots, a_i, \ldots, a_n, b) \textit{ iff } S(g, a_1, \ldots, b \ldots, a_n, a_i,)$$

Two operations f and g are *relatives* when they satisfy the Abstract Law of Residuation w.r.t. some *ith*-place.

The family of operations $\{f_i\}_{i \in I}$ is *founded* when there is an operator $h \in \{f_i\}_{i \in I}$ (the *head*) such that any other operation $g \in \{f_i\}_{i \in I}$ is a relative of f.

Now we can define a *Gaggle* as a distributoid $(D, \wedge, \vee, \{f_{i\in I}\})$ s.t. $\{f_{i\in I}\})$ is a founded family.

Gaggle



Double Ouroboros

Some examples of Gaggles

Distributive lattice-ordered forward possibility and backward necessity $(S, \leq, \Diamond, \Box\downarrow)$

Distributive lattice-ordered forward necessity and backwards possibility $(S, \leq, \Box, \Diamond\downarrow)$

Distrib. lattice-ordered residuated groupoid

$(S, \leq \pm, \leftarrow, \rightarrow)$

Distrib. lattice-ordered dual residuated groupoid

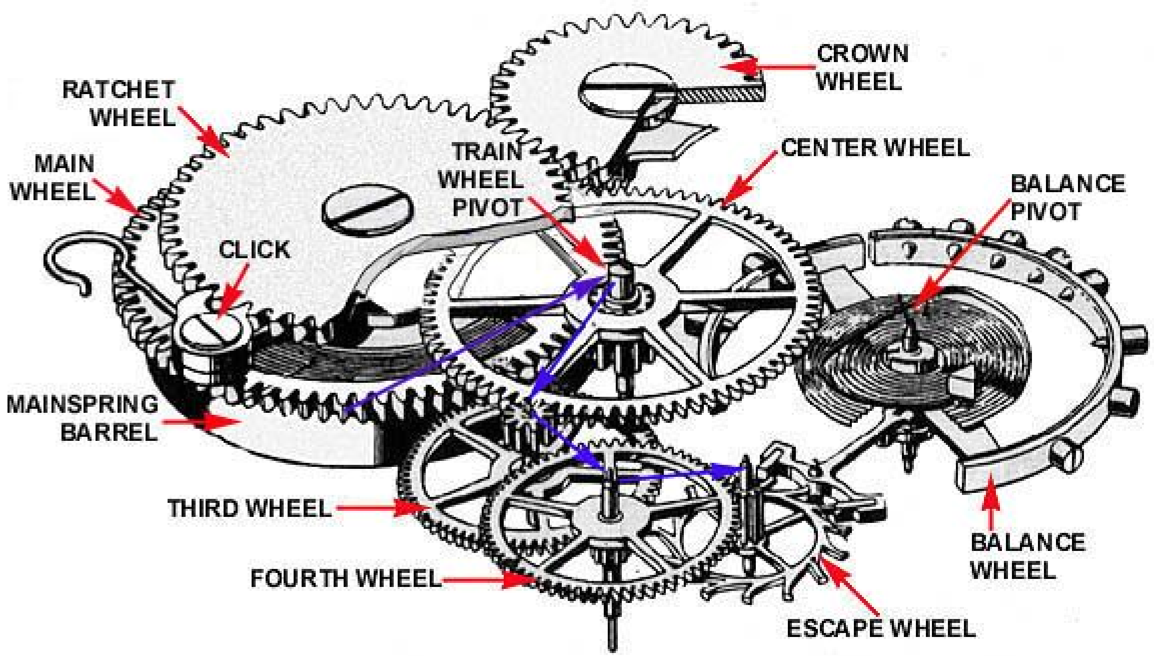
$(S, \leq, +, \leftharpoondown, \rightharpoondown)$

Not only can examples of lattice-ordered residuated groupoids be constructed from a ternary relation, but it turns out that (up to Isomorphism) all lattice-ordered residuated groupoids can be obtained this way. And this can be generalized to arbitrary gaggles, using $n + 1$ – placed relations to interpret n – ary operations.

Representation Theorem for Gaggles. Every gaggle is isomorphic to a gaggle on a ring of sets with operations being defined according to their distribution type using a single $n + 1$-placed accessibility relation.

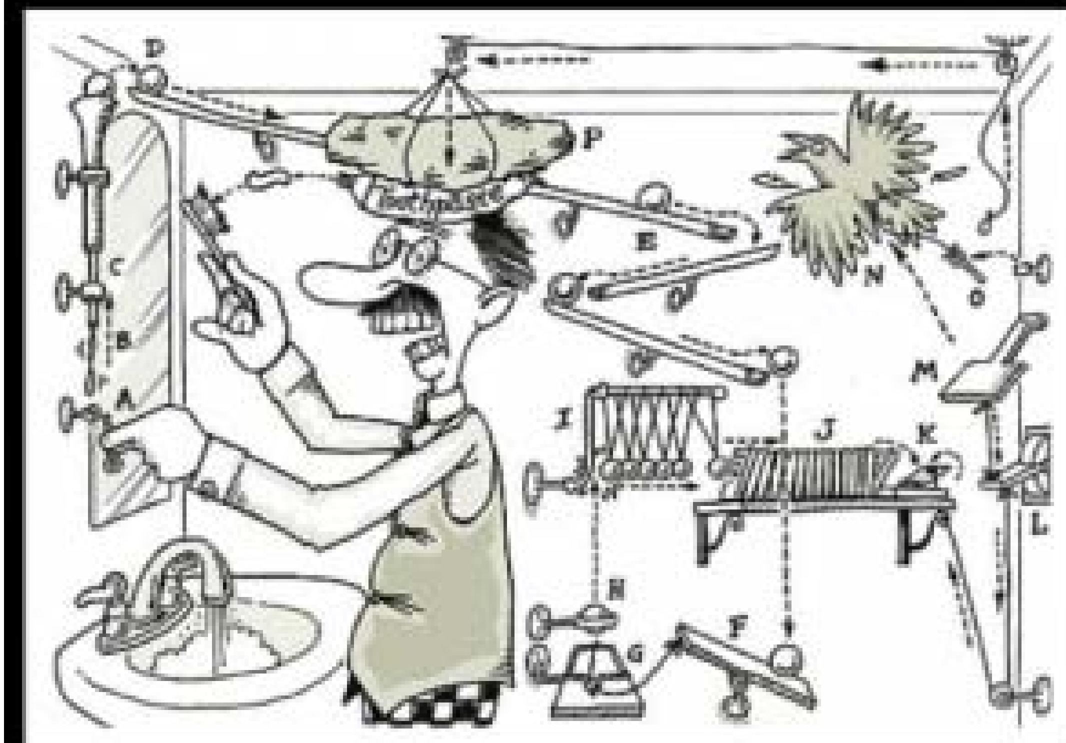
The proof is just as for distributoids except instead of multiple canonical accessibility relations we need only one, R_f, where f is the head of the family.

The definition of a *distributoid* (and a *gaggle*) is super delicate, with lots of interacting parts, much like the movement of a fine watch. At least I like to think of it that way.



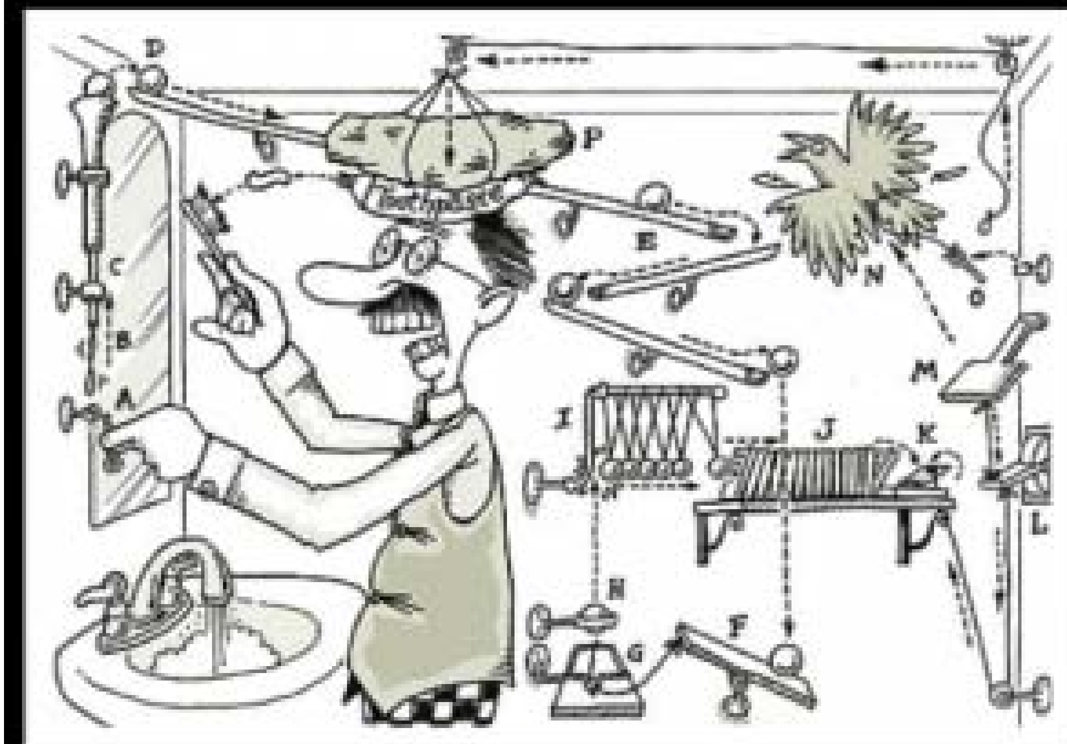
But maybe you think it is more like a Rube Goldberg machine?

Over-engineered toothpaste tube squeezer



But maybe you think it is more like a Rube Goldberg machine? ☺

Over-engineered toothpaste tube squeezer



It does take a little time to get comfortable with them.

Now we turn to another kind of duality, which I labeled “meta duality.” It has not to do with a duality within a structure, but rather a duality between structures, and in particular the duality between a structure and its representation.

WARNING

BUMPY ROAD AHEAD

Few definitions, assumes some topology.

Stone’s Topological Duality for Boolean algebras

DEFINITION (STONE SPACE) Let $\mathfrak{T} = \langle X; \mathfrak{O} \rangle$ be a topology. $\mathfrak{T}$ is *a Stone space* if compact and totally disconnected.

LEMMA (STONE SPACE OF A **BA**) Let $\mathfrak{A} = \langle$A; $\vee$, $-\rangle$ be a Boolean algebra. Also, let $\mathcal{M}$□ be the set of maximal filters of $\mathfrak{A}$. Let ha $=$ { F $\in$ $\mathcal{M}$□a $\in$ F } and B $=$ { ha : a $\in$ A }. Then $\mathfrak{T} = \langle \mathcal{U}; \tau$ (B)$\rangle$ is a topological space that is a Stone space. $\mathtt{Ts}(\mathfrak{A})$ denotes $\mathfrak{T}$ whenever $\mathfrak{A}$ is a **BA**.

LEMMA (ALGEBRA OF A STONE SPACE) Let $\mathfrak{T} = \langle$X; $\mathfrak{O}\rangle$ be a Stone space, that is, a compact totally disconnected topological space, and let OC$(\mathfrak{T}) =$ { O: O $\in \mathfrak{O}$ & $-$O $\in \mathfrak{O}$} . Then $\mathfrak{A} = \langle$OC$(\mathfrak{T})$; $\cap, \cup, -\rangle$ is a Boolean algebra. $\mathtt{Ab}(\mathfrak{T})$ denotes $\mathfrak{A}$ whenever $\mathfrak{T}$ is a Stone space.

THEOREM (TOPOLOGICAL REPRESENTATION OF **BA**S) If $\mathfrak{A}$ is a Boolean algebra then $\mathtt{Ab}(\mathtt{Ts}(\mathfrak{A})) \cong \mathfrak{A}$.

THEOREM (ALGEBRAIC REALIZATION OF STONE SPACES) If $\mathfrak{T}$ is a Stone space then $\mathtt{Ts}(\mathtt{Ab}(\mathfrak{T})) \rightleftharpoons\!\rightleftharpoons \mathfrak{T}$.

Duality of Boolean algebras and Stone Spaces

$$
\begin{tikzcd}
\mathfrak{A}_1 \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "1\text{-}1"] & \mathrm{T_S}\mathfrak{A}_1 \arrow[r] \arrow[dr, phantom, "1\text{-}1"] & \mathrm{A_B}\mathrm{T_S}\mathfrak{A}_1 \arrow[d, "\varphi^{-1\,-1}"'] \\
\mathfrak{A}_2 \arrow[r] & \mathrm{T_S}\mathfrak{A}_2 \arrow[r] \arrow[u, "\varphi^{-1}"] & \mathrm{A_B}\mathrm{T_S}\mathfrak{A}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{T_S}\mathrm{A_B}\mathfrak{T}_1 \arrow[d, "\psi^{-1\,-1}"'] \arrow[dr, phantom, "1\text{-}1"] & \mathrm{A_B}\mathfrak{T}_1 \arrow[l] \arrow[dr, phantom, "1\text{-}1"] & \mathfrak{T}_1 \arrow[l] \arrow[d, "\psi"'] \\
\mathrm{T_S}\mathrm{A_B}\mathfrak{T}_2 & \mathrm{A_B}\mathfrak{T}_2 \arrow[l] \arrow[u, "\psi^{-1}"] & \mathfrak{T}_2 \arrow[l]
\end{tikzcd}
$$

Priestley's Topological Duality for Distributive Lattices

DEFINITION (PRIESTLEY SPACE) If $\mathfrak{T} = \langle X; \mathfrak{O}, \leq \rangle$ is an ordered, compact, totally order disconnected topological space, then $\mathfrak{T}$ is a *Priestley space*.

LEMMA (PRIESTLEY SPACE OF A DISTRIBUTIVE LATTICE) Let $\mathfrak{A} = \langle A; \vee, \wedge, 0, 1 \rangle$ *be a* bounded distributive lattice. Let $\mathcal{P}$ be the set of prime filters of $\mathfrak{A}$.
Let ha $= \{F \in \mathcal{P}$: a $\in$ F $\}$ and S $= \{$ ha : a $\in$ A $\} \cup \{-$ha : a $\in$ A $\}$. The ordered topological space $\mathfrak{T} = \langle \mathcal{P}; \tau$ (S)$,\subseteq \rangle$ is compact and totally order disconnected. Thus, $\mathtt{T}$P$(\mathfrak{A})$ denotes $\mathfrak{T}$ — the Priestley space of $\mathfrak{A}$ — when $\mathfrak{A}$ is *a* bounded distributive lattice.

LEMMA (ALGEBRA OF A PRIESTLEY SPACE) Let $\mathfrak{T} = \langle X; \mathfrak{O}, \leq \rangle$ be a Priestley space and let OC$(\mathfrak{T})\uparrow = \{$ O: O $\in \wp$(X) & O $\in \mathfrak{O}$ & –O $\in \mathfrak{O}\}$. $\mathfrak{A} = \langle$OC$(\mathfrak{T})\uparrow; \cap, \cup, \emptyset$, X$\rangle$ is a distributive lattice. $\mathtt{A}$DL$(\mathfrak{T})$ denotes $\mathfrak{A}$ when $\mathfrak{T}$ is a Priestley space.

THEOREM (TOPOLOGICAL REPRESENTATION OF DISTRIBUTIVE LATTICES) If $\mathfrak{A} = \langle A; \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice, then $\mathtt{A}$DL$(\mathtt{T}$P$(\mathfrak{A})) \cong \mathfrak{A}$.

THEOREM (ALGEBRAIC REALIZATION OF PRIESTLEY SPACES) If $\mathfrak{T}$ is a Priestly space then $\mathtt{T}$P $(\mathtt{A}$DL $(\mathfrak{T})) \rightleftharpoons \mathfrak{T}$.

Givant, Goldblatt, Halmos, Hansoul: Topological Duality for BAOs

Givant, S. R.: *Duality theories for Boolean algebras with operators*, Springer, 2014.
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Note the alphabetical order. The history, differences, and Influences are complicated and could be a whole talk, or even seminar, on their own.

The material on duality of gaggles also borrows heavily from Katalin Bimbó's and my book *Generalized Galois Logics: Relational Semantics for Nonclassical Logical Caculi*, CSLI Lecture Notes, University of Chicago Press, 2008.

DEFINITION (GAGGLE SPACE) Let $\mathfrak{T} = \langle X; \mathfrak{O}, \leq, R_{n+1} \rangle$ be a topology that is a Priestley space with an additional relation R_{n+1} used to define *n*-ary operations in the family $\{f_i\}_{i \in I}$ with each operation f having a distribution type $t: \tau_1, \ldots, \tau_n \mapsto \tau_{n+1}$ is defined on clopen sets,

requiring $O_1, \ldots, O_n \in OC(\mathfrak{T}) \Rightarrow f(O_1, \ldots, O_n) \in OC(\mathfrak{T})$,

with f defined using the realization condition below.

1) if $\tau = \wedge$, the realization condition for f is of the form:
$\forall \alpha_1, \ldots, \alpha_n :: R\, \alpha_1, \ldots, \alpha_n, \chi$ or $\pm(\alpha_1 \in O_1)$ or … or $\pm(\alpha_n \in O_n)$ where each component $\pm(\alpha_i \in A_i)$ is either $\alpha_i \in A_i$ or $\alpha_i \notin A_i$ depending on whether τ_i is $\wedge$ or $\vee$ respectively;

2) 2) if $\tau = \vee$, the realization condition for f is of the form:
$\exists \alpha_1, \ldots, \alpha_n :: R\, \alpha_1, \ldots, \alpha_n, \chi$ & $\pm(\alpha_1 \in O_1)$ & … & $\pm(\alpha_n \in O_n)$ where each component $\pm(\alpha_i \in A_i)$ is either $\alpha_i \in A_i$ or $\alpha_i \notin A_i$ depending on whether τ_i is $\vee$ or $\wedge$ respectively.

Hindsight tells me that “Gaggle Space” is a bad name for two reasons:

1) We shall be showing that it is also a dual space for distributoids, not Just the more specialized gaggles.

2) Similar spaces are named after their inventors: e. g. Stone spaces and Priestley spaces. (It goes without saying that these inventors did not name them after themselves.)

So I shall here refer to “Gaggle Spaces” as *Bimbó spaces* to honor Katalin Bimbó.

LEMMA (BIMBÓ SPACE OF A GAGGLE) Let $\mathfrak{A}$ be a gaggle, and let h and S be as in a Priestley space, i.e., $ha = \{F \in \mathcal{P} : a \in F\}$ and $S = \{ ha : a \in A \} \cup \{-ha : a \in A$. Then T = ($\mathcal{P}$; τ(S), ⊆, R) , where R is the canonical accessibilty relation , is a Bimbó space. $T_B(\mathfrak{A})$ denotes the Bimbó space of a gaggle $\mathfrak{A}$.

LEMMA. (GAGGLE OF A BIMBÓ SPACE) If $\mathfrak{T}$ = (X; O, ≤, R) is a Bimbó space, then $\mathfrak{A}$ = (OC($\mathfrak{T}$) ; ∩, ∪, ∅, X, f) is a gaggle. Therefore, $A_{ggl}(\mathfrak{T})$ denotes $\mathfrak{A}$ when $\mathfrak{T}$ is a Bimbó space.

THEOREM (TOPOLOGICAL REPRESENTATION OF GAGGLES)
If $\mathfrak{A}$ is a gaggle then $A_{ggl}(T_B(\mathfrak{A})) \cong \mathfrak{A}$.

THEOREM. (ALGEBRAIC REALIZATION OF BIMBÓ SPACES)
If $\mathfrak{T}$ is a Bimbó space then $T_B(A_{ggl}(\mathfrak{T})) \rightleftharpoons \mathfrak{T}$.

Finally we turn, or return, to duality for distributoids. Somehow though Kata Bimbó and I proved duality results for gaggles in our 2008 book *Generalized Galois Logics*, we overlooked distributoids, perhaps they seemed to simple.

DEFINITION (DISTRIBUTOID SPACE) Let T = (hX; O, ·, $\{R_i\}_{i\,\in)}$ be a topology that is a Priestley space with additional relations $\{R_i\}_{i\,\in I}$ used to define operations $\{f_i\}_{i\,\in\, I}$ with each operation f_i having a distribution type t_{ii}: $\tau_1, \ldots, \tau_n \mapsto \tau_{n+1}$ is defined on clopen sets, requiring $O_1, \ldots, O_n \in OC(\mathfrak{T})\uparrow \Rightarrow f(O_1, \ldots, O_n) \in OC(\mathfrak{T})$, with f defined using the realization condition below.

1) if $\tau = \wedge$, the realization condition for f is of the form:
$\forall\, \alpha_1, \ldots, \alpha_n :_{:} R\, \alpha_1, \ldots, \alpha_n, \chi$ or $\pm(\alpha_1 \in O_1)$ or … or $\pm(\alpha_n \in O_n)$ where each component $\pm(\alpha_i \in A_i)$ is either $\alpha_i \in A_i$ or $\alpha_i \notin A_i$ depending on whether τ_i is $\wedge$ or $\vee$ respectively;

2) 2) if $\tau = \vee$, the realization condition for f is of the form:
$\exists\, \alpha_1, \ldots, \alpha_n :_{:} R\, \alpha_1, \ldots, \alpha_n, \chi\ \&\ \pm(\alpha_1 \in O_1)\ \&\ \ldots\ \&\ \pm(\alpha_n \in O_n)$
where each component $\pm(\alpha_i \in A_i)$ is either $\alpha_i \in A_i$ or $\alpha_i \notin A_i$ depending on whether τ_i is $\vee$ or $\wedge$ respectively.

LEMMA (DISTRIBUTOID SPACE OF A DISTRIBUTOID) Let $\mathfrak{A}$ be a distributoid, and let h and S be as in a Priestley space, i.e., $ha = \{F \in \mathcal{P} : a \in F\}$ and $S = \{ ha : a \in A \} \cup \{-ha : a \in A$. Then $T = (\mathcal{P}; \tau(S), \subseteq, \{R_i\}_{i\in I})$, where $\{R_i\}_{i\in I}$ is the set canonical accessibilty relations, is a distributoid space. $T_{ds}(\mathfrak{A})$ denotes the distributoid space of a distributoid $\mathfrak{A}$.

LEMMA. (DISTRIBUTOID OF A DISTRIBUTOID SPACE) If $\mathfrak{T} = (X; O, \leq, \{R_i\}_{i\in I})$ is an *ordered topological* space, then $\mathfrak{A} = (OC(\mathfrak{T}) ; \cap, \cup, \emptyset, X, \{f_i\}_{i\in I})$ is a distributoid. $A_d(\mathfrak{T})$ denotes $\mathfrak{A}$ when $\mathfrak{T}$ is a distributoid space.

THEOREM (TOPOLOGICAL REPRESENTATION OF DISTRIBUTOIDS)
If $\mathfrak{A}$ is a distributoid then $A_d(T_{DS}(\mathfrak{A})) \cong \mathfrak{A}$.

THEOREM. (ALGEBRAIC REALIZATION OF DISTRIBUTOID SPACES)
If $\mathfrak{T}$ is a distributoid space then $T_{DS}(A_d(\mathfrak{T})) \rightleftharpoons \mathfrak{T}$

Oh, I forgot to mention that there is a generalization of gaggles, *Symmetric Gaggles, which* may be viewed as an interacting combination of a gaggle and a dual gaggle, both on the same distributive lattice. We will leave that for another occasion.

Oh, also forgot to mention distributoids, gaggles, and symmetric Gaggles defined on posets and lattices. On posets is easy, on lattices hard. Chrysafis Hartonas has done some good recent work here. Just google!

Some References to “my” work

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Thank you!

And as the first speaker let me thank on behalf of all of us the organizers of this special workshop:

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