Functional duals for residuation algebras

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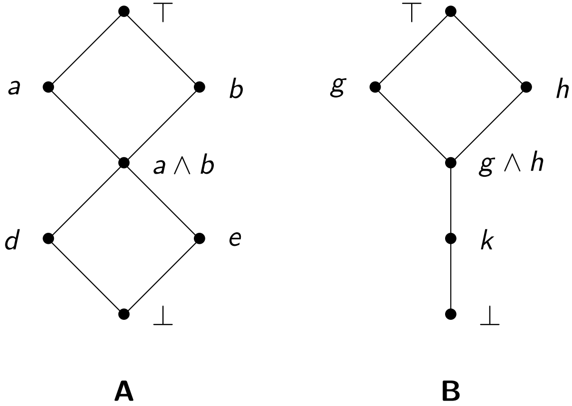
SYSMICS Workshop on Duality in Algebra and Logic
Chapmen University, Orange, CA

17 September 2018

Consider two Heyting algebras:



Introduction (cont.)

B has an interesting property that **A** does not: If F, G are prime filters of **B**, then so is their join $F \vee G$ in the lattice of filters of **B**.

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- **B** $\models (x \to y) \vee (y \to x) = 1$, or
- **B** is a subdirect product of linearly-ordered Heyting algebras, or
- **B** $\models x \to (y \vee z) = (x \to y) \vee (x \to z)$.

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Then either $g \wedge (g \to a) \in F \vee G$ or $g \wedge (g \to b) \in F \vee G$, so either $a \in F \vee G$ or $b \in F \vee G$.

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Consequently, the analysis above applies to, e.g., MV-algebras, MTL-algebras, etc.

...But it does not axiomatize semilinearity in this setting.

Introduction (cont.)

Recall that all residuated lattices satisfy

$$x(y \vee z) = xy \vee xz \tag{$\cdot\vee$}$$

$$(x \vee y)z = xz \vee yz \tag{$\vee\cdot$}$$

$$x\backslash(y \wedge z) = x\backslash y \wedge x\backslash z \tag{$\backslash\wedge$}$$

$$(x \wedge y)/z = x/z \wedge y/z \tag{$\wedge/$}$$

$$x/(y \vee z) = x/y \wedge x/z \tag{$/\vee$}$$

$$(x \vee y)\backslash z = x\backslash z \wedge y\backslash z \tag{$\vee\backslash$}$$

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...On the other hand, none of the following generally hold in residuated lattices:

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$$(x \vee y)/z = x/z \vee y/z \qquad (\vee/)$$

$$x\backslash(y \vee z) = x\backslash y \vee x\backslash z \qquad (\backslash\vee)$$

...But even together they don't guarantee semilinearity.

Introduction (cont.)

Motivating Question:

Under what circumstances is the dual of a residuated operation on lattice functional, and what role do identities like the above play? When is the dual relation not just functional but total?

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In order to address some of these unanswered questions, we explore functionality through the lens of canonical extensions.

Canonical extensions

Definition:

Let L be a lattice. Recall that the *canonical extension* of L is a completion $\sigma\colon L \to L^\delta$ of L such that

- Every element of L^δ is both a join of meets of elements of L and a meet of joins of elements of L, and
- Given any subsets $S, T \subseteq L$ with $\bigwedge S \leq \bigvee T$ in L^δ, there exist finite sets $S' \subseteq S$ and $T' \subseteq T$ such that $\bigwedge S' \leq \bigvee T'$.

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Theorem:

The canonical extension of a lattice exists and is unique up to an isomorphism fixing L.

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3. For all $a, b, c \in A$,

$$a \leq c/b \iff b \leq a\backslash c$$

Residuation algebras (cont.)

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But a residuation algebra may not have an operation $\cdot$ with

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However, such an operation is definable in every **complete** residuation algebra.

Canonical extensions of residuation algebras

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We denote these extensions by $\backslash^\pi$ and $/^\pi$.

$$x \backslash^\pi y := \bigvee \{x' \backslash y' : x', y' \in A \text{ and } x \leq x' \text{ and } y' \leq y\}$$

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The canonical extension of a residuation algebra with these operations is also a (complete) residuation algebra, so multiplication $\cdot$ may be defined.

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2. $\geq$ is the converse order inherited from A^{δ}
3. and R is a ternary relation on $J^{\infty}(A^{\delta})$ defined for $x, y, z \in J^{\infty}(A^{\delta})$ by

$$R(x, y, z) \iff x \leq y \cdot z$$

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- R to be *functional and defined everywhere* if $y \cdot z \in J^{\infty}(A^{\delta})$ for all $y, z \in J^{\infty}(A^{\delta})$. In this case, we say that A_{+}^{δ} is *total*.

Characterizing functionality

Theorem (Gehrke 2016, F. and Palmigiano 2018):

The following conditions are equivalent for any residuation algebra $\mathbf{A} = (A, \wedge, \vee, \backslash, / \bot, \top)$.

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1. The relational structure $\mathbf{A}_+^\delta$ is functional.
2. $\forall a, b, c \in A, \forall x \in J^\infty(A^\delta)$,
   $x \leq a \Rightarrow \exists a'[a' \in A \;\&\; x \leq a' \;\&\; a\backslash(b \vee c) \leq (a'\backslash b) \vee (a'\backslash c)]$.

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3. For all $x \in J^\infty(A^\delta)$, the map $x\backslash^\pi(-) : O(A^\delta) \to O(A^\delta)$ is $\vee$-preserving, where $O(A^\delta)$ denotes the join-closure of A in A^δ.

Benefits of working in the canonical extension

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Let L be a lattice and $k \in K(L^\delta)$ be finitely prime. Then $k \in J^\infty(L^\delta)$.

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Let L be a lattice and $k \in K(L^\delta)$ be finitely prime. Then $k \in J^\infty(L^\delta)$.

And a piece that relies on the identity.

Lemma 2:

Let $\mathbf{A}$ be a residuation algebra such that $\mathbf{A}$ satisfies $a\backslash(b \vee c) \leq (a\backslash b) \vee (a\backslash c)$. Then if $x, y \in J^\infty(A^\delta)$, either $x \cdot y = \bot$ or $x \cdot y$ is finitely prime.

Functionality by identities

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Let **A** be a residuation algebra satisfying $a\backslash(b \vee c) \leq (a\backslash b) \vee (a\backslash c)$. Then $\mathbf{A}_{+}^{\delta}$ is functional.

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Proof: Let $x, y \in J^\infty(A^\delta)$. Then $x, y \in K(A^\delta)$ by the general theory of canonical extensions, and also $x \cdot y \in K(A^\delta)$. By Lemma 2, if $x \cdot y \neq \bot$ then $x \cdot y$ is finitely prime. And by Lemma 1, this proves that $x \cdot y \in J^\infty(A^\delta)$.

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By compactness, there exist finite sets $A \subseteq \{a \in L : k \leq a\}$ and $B \subseteq \{b \in L : b \not\geq k\}$ with

Proving Lemma 1

Lemma 1:

Let L be a lattice and $k \in K(L^\delta)$ be finitely prime. Then $k \in J^\infty(L^\delta)$.

The proof involves the following claim.

Claim: Let L be a lattice. If $k \in K(L^\delta)$ is finitely prime and $o = \bigvee\{b \in L \mid b \not\geq k\}$, then $k \not\leq o$.

We prove the claim by contradiction.

Suppose $\bigwedge\{a \in L : k \leq a\} = k \leq o$.

By compactness, there exist finite sets $A \subseteq \{a \in L : k \leq a\}$ and $B \subseteq \{b \in L : b \not\geq k\}$ with

$$a' = \bigwedge A \leq \bigvee B = b'$$

Proving Lemma 1 (cont.)

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This is a contradiction, so this proves the claim.

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For each $s \in S$ pick $a_s \in L$ with $a_s \geq s$ and $a_s \not\geq k$.

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Then $a_s \leq o = \bigvee\{a \in L \mid a \not\geq k\}$ for each $s \in S$.

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But this contradicts the claim.

Totality

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Corollary:

Let **A** be a residuation algebra satisfying $a \backslash (b \vee c) \leq (a \backslash b) \vee (a \backslash c)$. If **A** has no zero-divisors, then $\mathbf{A}_{+}^{\delta}$ is total.

Functionality is not equational

Proposition:

There is no universal first-order condition in the language of residuation algebras that characterizes functionality. In particular, there is no equational condition that suffices.

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$$A \cdot B = \{a + b : a \in A, b \in B\}$$

$$A \backslash B = \{c : A \cdot \{c\} \subseteq B\},$$

$$A/B = \{c : \{c\} \cdot B \subseteq A\}.$$

Functionality is not equational (cont.)

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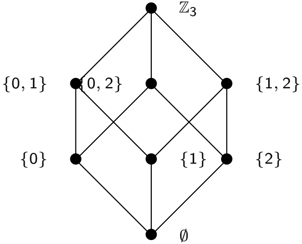

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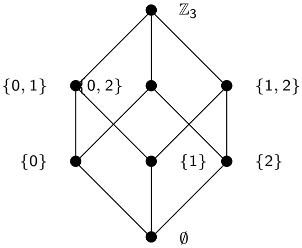

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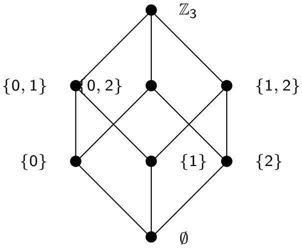

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And these are closed under $\bullet$

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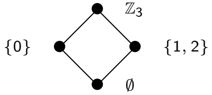

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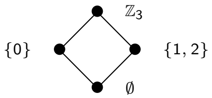

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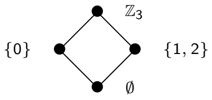

And $\uparrow\{1,2\}$ is prime in this subalgebra.

But $\uparrow\{1,2\} \bullet \uparrow\{1,2\} = \{\mathbb{Z}_3\}$, which is not prime.

Future and ongoing work

The modular account offered by the canonical extension approach implicates an important question: What about residuated lattices whose lattice reduct is non-distributive?

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Do residuation algebras whose duals are functional generate the variety of residuation algebras?

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