Is there a useful duality for residuated lattices?

Nick Galatos
University of Denver
ngalatos@du.edu

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Substructural logics

Algebraic semantics

A *residuated lattice*, or *residuated lattice-ordered monoid*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$a \cdot b \leq c \;\Leftrightarrow\; b \leq a \backslash c \;\Leftrightarrow\; a \leq c/b.$$

- Lattice-ordered groups: division is multiplication by inverse
- Heyting algebras: $x \cdot y = x \wedge y$
- MV-algebras: $x \cdot y = y \cdot x$, $x \vee y = (x \to y) \to y$.
- Relation algebras: multiplication is composition
- Ideals of rings: usual multiplication of ideals

RL: the variety of all residuated lattices
CRL: the variety of residuated lattices with coommutative multiplication
DRL: the variety of residuated lattices with distributive lattices

$$\frac{x \Rightarrow a \quad y{\circ}a{\circ}z \Rightarrow c}{y{\circ}x{\circ}z \Rightarrow c}\ (\mathsf{cut}) \qquad \frac{}{a \Rightarrow a}\ (\mathsf{Id})$$

$$\frac{y{\circ}a{\circ}z \Rightarrow c}{y{\circ}a \wedge b{\circ}z \Rightarrow c}\ (\wedge\mathsf{L}\ell) \quad \frac{y{\circ}b{\circ}z \Rightarrow c}{y{\circ}a \wedge b{\circ}z \Rightarrow c}\ (\wedge\mathsf{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge\mathsf{R})$$

$$\frac{y{\circ}a{\circ}z \Rightarrow c \quad y{\circ}b{\circ}z \Rightarrow c}{y{\circ}a \vee b{\circ}z \Rightarrow c}\ (\vee\mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee\mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee\mathsf{R}r)$$

$$\frac{x \Rightarrow a \quad y{\circ}b{\circ}z \Rightarrow c}{y{\circ}x \circ (a\backslash b){\circ}z \Rightarrow c}\ (\backslash\mathsf{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b}\ (\backslash\mathsf{R})$$

$$\frac{x \Rightarrow a \quad y{\circ}b{\circ}z \Rightarrow c}{y{\circ}(b/a) \circ x{\circ}z \Rightarrow c}\ (/\mathsf{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathsf{R})$$

$$\frac{y{\circ}a \circ b{\circ}z \Rightarrow c}{y{\circ}a \cdot b{\circ}z \Rightarrow c}\ (\cdot\mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot\mathsf{R})$$

$$\frac{y \circ z \Rightarrow a}{y{\circ}1{\circ}z \Rightarrow a}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathsf{R})$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$.

(C)	$[x \to (y \to z)] \to [y \to (x \to z)]$	$(xy = yx)$
(K)	$y \to (x \to y)$	$(x \leq 1)$
(W)	$[x \to (x \to y)] \to (x \to y)$	$(x \leq x^2)$

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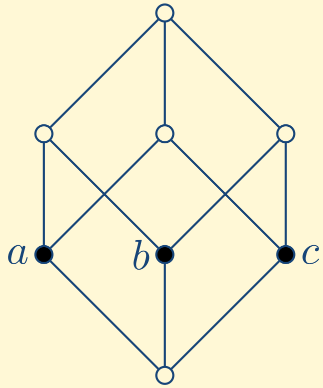
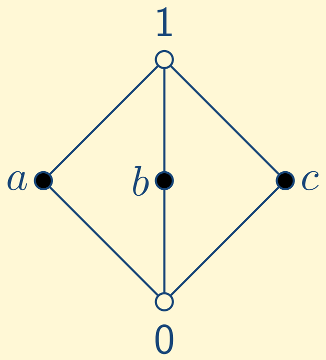
Examples of substructural logics include

- classical: (C)+(K)+(W)+ $\neg\neg\phi = \phi$ (DN)
- intuitionistic (Brouwer, Heyting): (C)+(K)+(W)
- many-valued (Łukasiewicz): (C)+(K)+ $(\phi \to \psi) \to \psi = \phi \vee \psi$
- MTL (Esteva, Godo): (C)+(K)+ $(\phi \to \psi) \vee (\psi \to \phi)$
- basic (Hajek): MTL+ $\phi(\phi \to \psi) = \phi \wedge \psi$
- relevance (Anderson, Belnap): (C)+(W)+ Distrib. (+ DN)
- (MA)linear logic (Girard): (C)

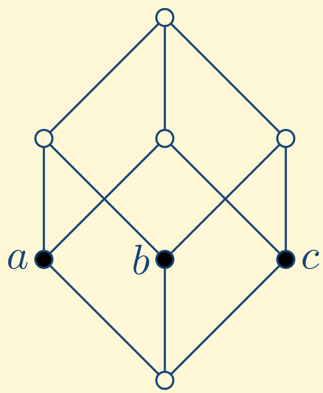
Lattice representation

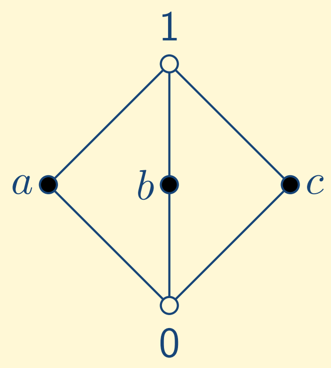
Lattices





For general (non-distributive) lattices, the poset of join irreducibles is not enough to recover the lattice.





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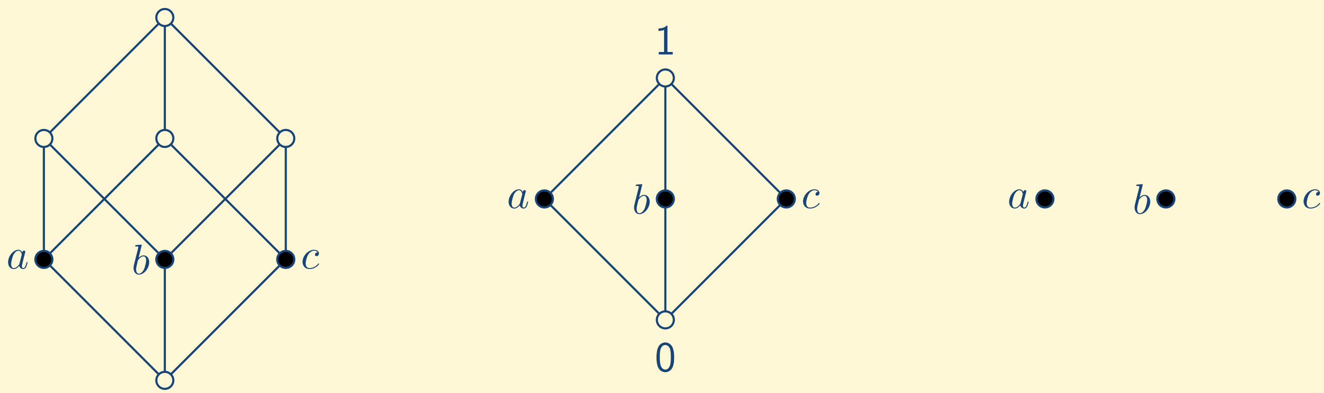
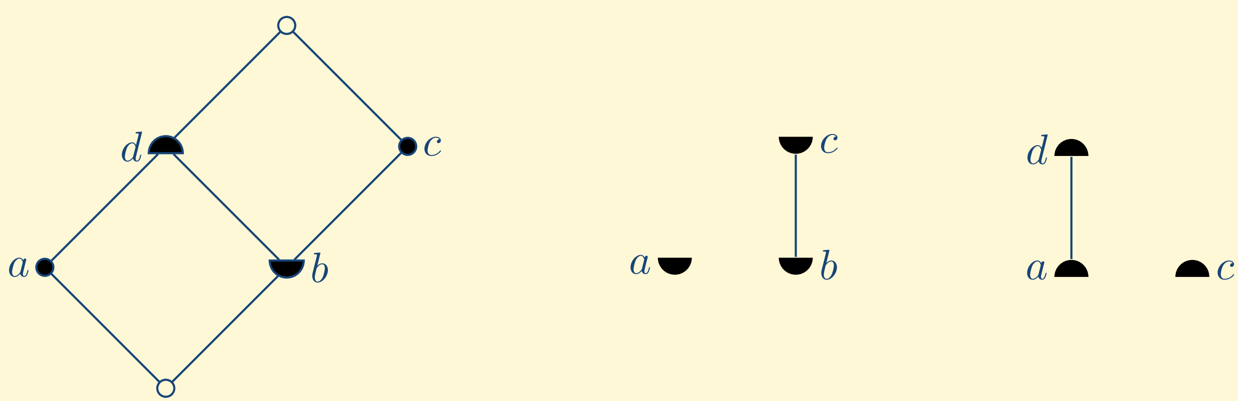
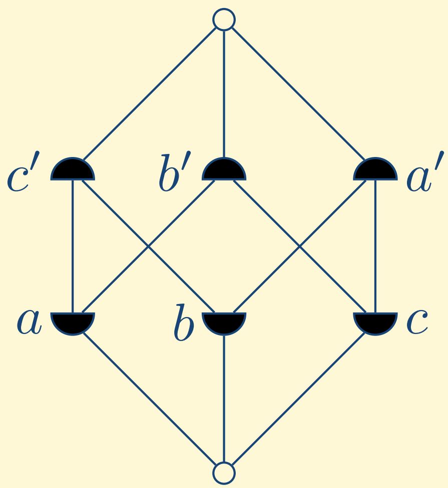
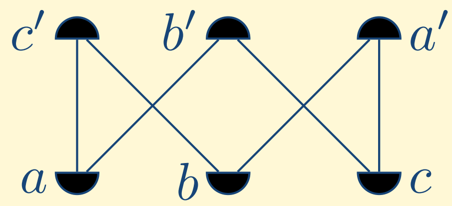


For general (non-distributive) lattices, the poset of join irreducibles is not enough to recover the lattice. We also need the *meet irreducibles*; we denote their poset by $M(\mathbf{L})$. For every distributive lattice $M(\mathbf{L})$ is isomorphic to $J(\mathbf{L})$. Note $\uparrow a \cup \downarrow c = \uparrow b \cup \downarrow a = \uparrow c \cup \downarrow d = L$.
Splitting pairs: $(a, c), (b, a), (c, d)$.

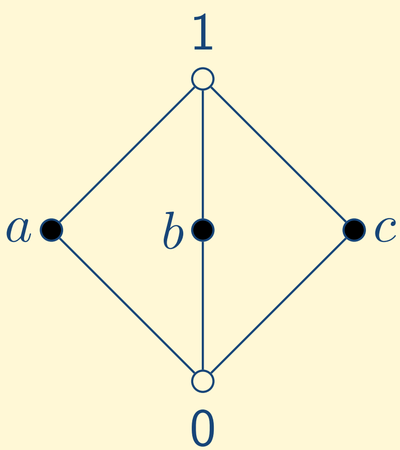


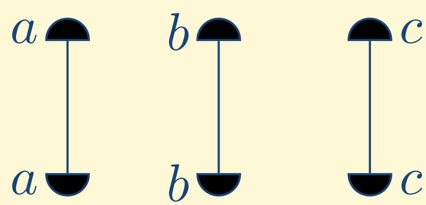
Contexts





$\sqsubseteq$	a'	b'	c'
a		$\times$	$\times$
b	$\times$		$\times$
c	$\times$	$\times$	





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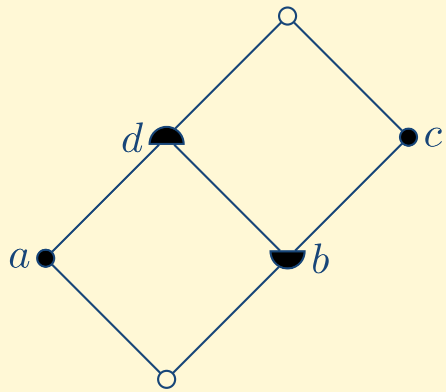
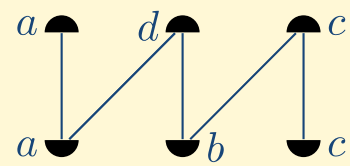
Dedekind-Birkhoff





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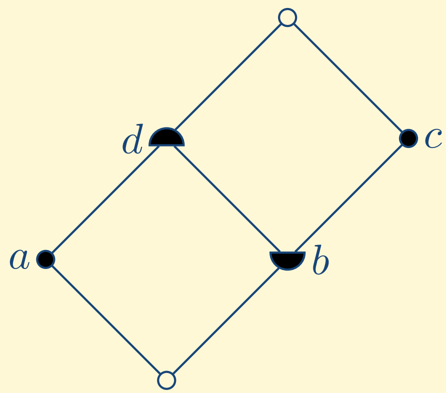
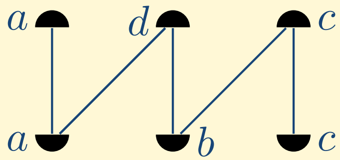
b		$\times$	$\times$
c			$\times$





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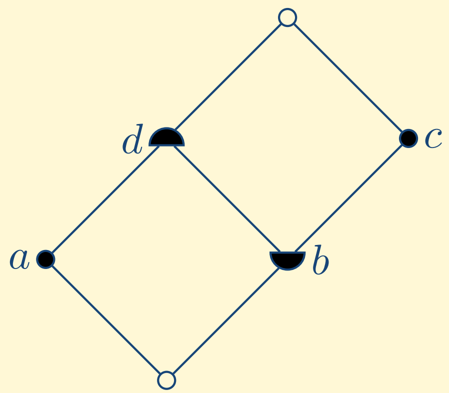
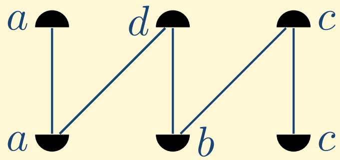
$\{a\}^{\lhd} = \{a\}$, $\{d\}^{\lhd} = \{a, b\}$, $\{c\}^{\lhd} = \{b, c\}$.





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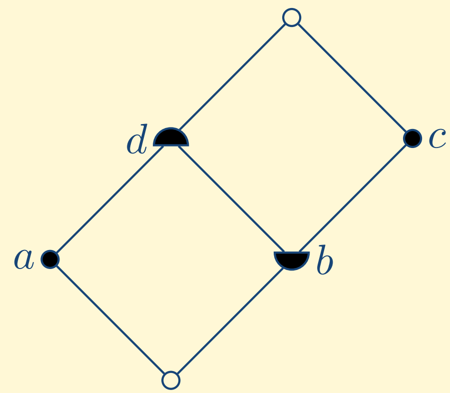

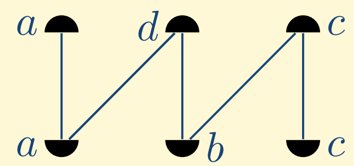

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These correspond to the meet generators of the original lattice and the lattice is obtained by intersections of these sets. In general we obtain the Dedekind-McNeille completion of the original lattice.

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$$X^{\rhd} = \{b \in W' : x \sqsubseteq b, \text{ for all } x \in X\}$$
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If $\mathbf{W}$ satisfies the condition (COM), then $\mathbf{W}^+$ is a chain.

$$\frac{x \sqsubseteq z \qquad y \sqsubseteq w}{x \sqsubseteq w \;\; OR \;\; y \sqsubseteq z}\ \text{(COM)}$$

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$$\frac{x \sqsubseteq a \quad a \sqsubseteq z}{x \sqsubseteq z}\ (\mathsf{CUT}) \qquad \frac{}{a \sqsubseteq a}\ (\mathsf{Id})$$

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Corollary. The map $q : \mathbf{S} \to \mathbf{W}^+$, $q(a) = \{a\}^{\lhd}$ is a homomorphism: $q(a \wedge_{\mathbf{B}} b) = q(a) \wedge_{\mathbf{W}^+} q(b)$ and $q(a \vee_{\mathbf{B}} b) = q(a) \vee_{\mathbf{W}^+} q(b)$.

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Corollary. The map $q : \mathbf{S} \to \mathbf{W}^+$, $q(a) = \{a\}^{\triangleleft}$ is a homomorphism: $q(a \wedge_{\mathbf{B}} b) = q(a) \wedge_{\mathbf{W}^+} q(b)$ and $q(a \vee_{\mathbf{B}} b) = q(a) \vee_{\mathbf{W}^+} q(b)$. If $\sqsubseteq$ is antisymmetric on S, then q is injective.

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Application (DM-completion/embedding): Given a lattice $\mathbf{L}$, $\mathbf{W_L} = (L, \leq, L)$ is a lattice frame and the pair $(\mathbf{W_L}, \mathbf{L})$ is a Genzen lattice frame. $\mathbf{W_L^+}$ is the Dedekind-MacNeille completion of $\mathbf{L}$ and $q : \mathbf{L} \to \mathbf{W_L^+}$ is an embedding.

Cut elimination for lattices

$$\frac{}{a \leq a} \qquad \frac{a \leq b \quad b \leq a}{a = b} \qquad \frac{a \leq b \quad b \leq c}{a \leq c}$$

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Theorem. (Cut elimination) $\mathbf{Lat}$ and $\mathbf{Lat}^{\mathbf{cf}}$ ($\mathbf{Lat}$ without cut) prove the same sequents.

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Theorem. (Cut elimination) $\mathbf{Lat}$ and $\mathbf{Lat^{cf}}$ ($\mathbf{Lat}$ without cut) prove the same sequents. We consider the lattice frame $\mathbf{W}$, where $W = Fm$, $W' = Fm$ and $a \sqsubseteq b$ iff $a \le b$ is provable in $\mathbf{Lat^{cf}}$.

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Cut elimination for lattices

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Lemma. For all $a, b \in S$, then $a \wedge_{\mathbf{B}} b \in q(a) \wedge_{\mathbf{W}^+} q(b) \subseteq q(a \wedge_{\mathbf{B}} b)$ and $a \vee_{\mathbf{B}} b \in q(a) \vee_{\mathbf{W}^+} q(b) \subseteq q(a \vee_{\mathbf{B}} b)$. $(\mathbf{W}, \mathbf{Fm})$ is cf-Gentzen.

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Corollary. The homomorphism $h : \mathbf{Fm} \to \mathbf{W}^+$ extending the variable assignment $p \mapsto q(p)$ satisfies $a \in h(a) \subseteq q(a)$.

So, if $\mathbf{W}^+ \models a \leq b$, then $a \in h(a) \subseteq h(b) \subseteq q(b) = \{b\}^{\lhd}$, so $a \sqsubseteq b$.

Residuated frames

A *residuated frame* is a structure $\mathbf{W} = (W, \circ, \varepsilon, \sqsubseteq, W')$ where

- $(W, \sqsubseteq, W')$ is a lattice frame
- $(W, \circ, \varepsilon)$ is a monoid
- there exist $\backslash\!\!\backslash$ and $/\!\!/$ such that for all $x, y \in W$ and $z \in W'$

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Corollary. If $\mathbf{W}$ is a residuated frame then the *Galois/dual algebra* $\mathbf{W}^+ = (\gamma[\mathcal{P}(W)], \cap, \cup_\gamma, \circ_\gamma, \gamma(1), \backslash, /)$ is a residuated lattice, where
$X \circ Y = \{x \circ y : x \in X, y \in Y\}$,
$X \backslash Y = \{z : X \circ \{z\} \subseteq Y\}$
$Y/X = \{z : \{z\} \circ X \subseteq Y\}$.

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Simple equations

Consider the equation ε:

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Theorem: If $\mathbf{W}$ satisfies R(ε) iff $\mathbf{W}^+$ satisfies ε.

Lemma. Every equation over $\{\vee, \cdot, 1\}$ is equivalent to a conjunction of *simple* equations: $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are $\{\cdot, 1\}$-terms and t_0 is linear.

$$\frac{x \sqsubseteq a \quad a \sqsubseteq z}{x \sqsubseteq z}\ (\mathsf{CUT}) \qquad \frac{}{a \sqsubseteq a}\ (\mathsf{Id})$$

$$\frac{a \sqsubseteq z \quad b \sqsubseteq z}{a \vee b \sqsubseteq z}\ (\vee\mathsf{L}) \qquad \frac{x \sqsubseteq a}{x \sqsubseteq a \vee b}\ (\vee\mathsf{R}\ell) \qquad \frac{x \sqsubseteq b}{x \sqsubseteq a \vee b}\ (\vee\mathsf{R}r)$$

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Gentzen frames

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If we have a common subset S of W and W' that supports a (partial) algebra $\mathbf{S} = (S, \wedge, \vee, \cdot, \backslash, /, 1)$, and for $a, b, c \in S$, $x, y \in W$, $z \in W'$,

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Gentzen frames

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If we have a common subset S of W and W' that supports a (partial) algebra $\mathbf{S} = (S, \wedge, \vee, \cdot, \backslash, /, 1)$, and for $a, b, c \in S$, $x, y \in W$, $z \in W'$, then we call $(\mathbf{W}, \mathbf{S})$ a Gentzen frame and we call $\mathbf{W}$ an $\mathbf{S}$-frame. Again, $q : \mathbf{S} \to \mathbf{W}^+$ is a homomorphism (in the full signature).

DM-completions

To a residuated lattice $\mathbf{A}$, we associate the Gentzen frame $(\mathbf{W_A}, \mathbf{A})$, where $\mathbf{W_A} = (A, \cdot, 1, \leq, A)$.

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Theorem. The map $x \mapsto x^{\lhd}$ is an embedding of $\mathbf{A}$ into $\mathbf{W}^+_{\mathbf{A}}$.

Corollary. The variety of residuated lattices is closed under DM-completions.

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This is an $\mathbf{A}$-frame, where the maps from A are $a \mapsto a$ and $a \mapsto (\varepsilon, a, \varepsilon)$.

Theorem. The map $x \mapsto x^{\triangleleft}$ is an embedding of $\mathbf{A}$ into $\mathbf{W}_\mathbf{A}^+$. If $\mathbf{A}$ has a multiplicative unit then the embeding preserves it. The embedding preserves exising joins $\bigvee X$ for which $y(\bigvee X)z = \bigvee(yx_iz)$ for all $y, z \in A$. The embedding preserves all existing residuals.

Pre-frames

Given a *frame* $\mathbf{W} = (W, \circ, \varepsilon, \sqsubseteq, W')$ which might not be residuated, we can construct a residuated frame $\widetilde{\mathbf{W}} = (W, \circ, \varepsilon, \widetilde{\sqsubseteq}, \widetilde{W'})$ out of it.

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So we define: $w \widetilde{\sqsubseteq} (x, z, y)$ iff $x \circ w \circ y \sqsubseteq z$.

We now check if the new frame is residuated:

$$\begin{aligned} w_1 \circ w_2 \widetilde{\sqsubseteq} (x, z, y) \quad & \text{iff } x \circ w_1 \circ w_2 \circ y \sqsubseteq z \\ & \text{iff } w_1 \widetilde{\sqsubseteq} (x, z, w_2 \circ y) \end{aligned}$$

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\begin{array}{lll}
w_1 \circ w_2 \widetilde{\sqsubseteq} (x, z, y) & \text{iff } x \circ w_1 \circ w_2 \circ y \sqsubseteq z & \\
& \text{iff } w_1 \widetilde{\sqsubseteq} (x, z, w_2 \circ y) & = (x, z, y) \mathbin{/\!/} w_2 \\
& \text{iff } w_2 \widetilde{\sqsubseteq} (x \circ w_1, z) & = w_1 \mathbin{\backslash\!\backslash} (x, z, y)
\end{array}
$$

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Often we will write $\sqsubseteq$ for the extension $\widetilde{\sqsubseteq}$.

Embedding of subreducts using preframes

To a partially-odrered semigroup $\mathbf{A} = (A, \leq, \cdot)$, we associate the Gentzen pre-frame $(\mathbf{W_A}, \mathbf{A})$, where $\mathbf{W_A} = (A_\varepsilon, \cdot, \sqsubseteq, A)$, $A_\varepsilon = A \cup \{\varepsilon\}$ for $\varepsilon \notin A$, where $a \circ b = ab$ for $a, b \in A$ and $\varepsilon \circ a = a \circ \varepsilon = a$. Also,

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Theorem. The map $x \mapsto x^{\lhd}$ is an embedding of $\mathbf{A}$ into $\mathbf{W}_{\mathbf{A}}^{+}$. If $\mathbf{A}$ has a multiplicative unit then the embeding preserves it. The embedding preserves exising joins $\bigvee X$ for which $y(\bigvee X)z = \bigvee(yx_iz)$ for all $y, z \in A$. The embedding preserves all existing residuals.

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It is easy to see that $(\mathbf{W_{FL}}, \mathbf{Fm})$ is a Gentzen frame. For example, consider

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FL

$$\frac{x \Rightarrow a \quad y{\circ}a{\circ}z \Rightarrow c}{y{\circ}x{\circ}z \Rightarrow c}\ (\mathsf{cut}) \qquad \frac{}{a \Rightarrow a}\ (\mathsf{Id})$$

$$\frac{y{\circ}a{\circ}z \Rightarrow c}{y{\circ}a \wedge b{\circ}z \Rightarrow c}\ (\wedge \mathsf{L}\ell) \quad \frac{y{\circ}b{\circ}z \Rightarrow c}{y{\circ}a \wedge b{\circ}z \Rightarrow c}\ (\wedge \mathsf{L}r) \quad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge \mathsf{R})$$

$$\frac{y{\circ}a{\circ}z \Rightarrow c \quad y{\circ}b{\circ}z \Rightarrow c}{y{\circ}a \vee b{\circ}z \Rightarrow c}\ (\vee \mathsf{L}) \quad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}\ell) \quad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee \mathsf{R}r)$$

$$\frac{x \Rightarrow a \quad y{\circ}b{\circ}z \Rightarrow c}{y{\circ}x \circ (a\backslash b){\circ}z \Rightarrow c}\ (\backslash \mathsf{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b}\ (\backslash \mathsf{R})$$

$$\frac{x \Rightarrow a \quad y{\circ}b{\circ}z \Rightarrow c}{y{\circ}(b/a) \circ x{\circ}z \Rightarrow c}\ (/\mathsf{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathsf{R})$$

$$\frac{y{\circ}a \circ b{\circ}z \Rightarrow c}{y{\circ}a \cdot b{\circ}z \Rightarrow c}\ (\cdot \mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot \mathsf{R})$$

$$\frac{y \circ z \Rightarrow a}{y{\circ}1{\circ}z \Rightarrow a}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathsf{R})$$

where $a, b, c \in Fm$, $x, y, z \in Fm^*$.

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Corollary. The system $\mathbf{FL}$ has the finite model property. The same holds for *reducing* simple extensions. The corresponding varieties of residuated lattices are generated by their finite members.

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

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Corollary. These varieties are generated as quasivarieties by their finite members. The corresponding logics have the *strong finite model property*.

Combining frames

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We will give applications of this construction in proving:

- Amagamation (and related properties)
- Interpolation
- Densification

A class $\mathcal{K}$ of similar algebras has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f_B : \mathbf{A} \to \mathbf{B}$ and $f_C : \mathbf{A} \to \mathbf{C}$, there is a $\mathbf{D} \in \mathcal{K}$ and embeddings $f'_B : \mathbf{B} \to \mathbf{D}$ and $f'_C : \mathbf{C} \to \mathbf{D}$ such that $f'_B \circ f_B = f'_C \circ f_C$. [Single embedding $f' : \mathbf{B} \cup \mathbf{C} \to \mathbf{D}$.]

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We consider the frames $\mathbf{W}_B = (B, \cdot, 1, \leq, B)$ and $\mathbf{W}_C = (C, \cdot, 1, \leq, C)$, and as before we construct the residuated frame $\mathbf{W} = \mathbf{W}_B \star \mathbf{W}_C$. For that we need

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A class $\mathcal{K}$ of similar algebras has the *amalgamation property* (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f_B : \mathbf{A} \to \mathbf{B}$ and $f_C : \mathbf{A} \to \mathbf{C}$, there is a $\mathbf{D} \in \mathcal{K}$ and embeddings $f'_B : \mathbf{B} \to \mathbf{D}$ and $f'_C : \mathbf{C} \to \mathbf{D}$ such that $f'_B \circ f_B = f'_C \circ f_C$. [Single embedding $f' : \mathbf{B} \cup \mathbf{C} \to \mathbf{D}$.]

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Modifications of the AP are known as follows.

- **Transferable injections**: f_B is assumed to be injective and f'_B is required to be injective.
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The AP proof works in the same way!

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It suffices to construct a residuated frame $\mathbf{W}$ from this data such that $(\mathbf{W}, \mathbf{B} \cup \{p\})$ is a Gentzen frame and $\mathbf{W}^+$ is a chain.

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Theorem. $\mathbf{FL_e}$ has the Craig interpolation property, i.e. if $\vdash_{\mathbf{FL_e}} \phi \to \psi$, then there is a χ such that

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Corollary. If $\vdash_{\mathbf{FL_e}} x \Rightarrow d$, then $x \sqsubseteq d$. It follows that $\mathbf{FL_e}$ has the IP.

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For $\mathbf{A} \in \mathcal{K}$, there is a $\mathbf{D} \in \mathcal{K}$ and an epimorphism $f : \mathbf{D} \to \mathbf{A}$ such that if $1 \leq_{\mathbf{D}} a \vee b$, then $1 \leq_{\mathbf{A}} f(a)$ or $1 \leq_{\mathbf{A}} f(b)$.

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This property holds for all subvarieties of CRL axiomatized with equations over $\{\vee, \cdot, 1\}$.

Given a 3-counter machine the commutative monoid word $q_i r_1^{n_1} r_2^{n_2} r_3^{n_3}$ represents the configuration where the machine is at state q_i and the contents of the three registers are respectively n_1, n_2, n_3.

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(An intermediate machine allows us to convert to powers of a carefully chosen integer K, so that the simple equation will not affect the computation of the machine.)

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Now $h : \mathbf{Fm} \to \mathbf{W}^+$ is the homomorphism extending $p \mapsto q(\{p\} \cup \{x : (x \Rightarrow p) \in S\})$.

Hilbert system for FL

$$a\backslash a \qquad \frac{a \quad a\backslash b}{b} \qquad (a\backslash b)\backslash[(c\backslash a)\backslash(c\backslash b)] \qquad \frac{a\backslash b}{(b\backslash c)\backslash(a\backslash c)} \qquad \frac{a}{(a\backslash b)\backslash b}$$

$$a\backslash[(b/a)\backslash b] \qquad [((a\backslash b)/c)]\backslash[a\backslash(b/c)] \qquad \frac{b\backslash a}{a/b}$$

$$(a \wedge b)\backslash a \qquad (a \wedge b)\backslash b \qquad \frac{a \quad b}{a \wedge b} \qquad [(a\backslash b) \wedge (a\backslash c)]\backslash[a\backslash(b \wedge c)]$$

$$a\backslash(a \vee b) \qquad b\backslash(a \vee b) \qquad \frac{a\backslash c \quad b\backslash c}{(a \vee b)\backslash c}$$

$$b\backslash(a\backslash ab) \qquad [b\backslash(a\backslash c)]\backslash(ab\backslash c) \qquad 1 \qquad 1\backslash(a\backslash a) \qquad a\backslash(1\backslash a)$$

Strong separation

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Since h is a $\mathcal{L}$-homomorphism we have $h(1) = \gamma(\varepsilon)$. Moreover, $(\mathbf{W}, \mathbf{S}_{\mathcal{K}})$ is a Gentzen frame, so for every subformula d of $B \cup \{c\}$, $h(d) = \{d\}^{\lhd}$.

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Since h is a $\mathcal{L}$-homomorphism we have $h(1) = \gamma(\varepsilon)$. Moreover, $(\mathbf{W}, \mathbf{S}_{\mathcal{K}})$ is a Gentzen frame, so for every subformula d of $B \cup \{c\}$, $h(d) = \{d\}^{\lhd}$. Consequently, $h(1) \subseteq_{\mathbf{W}^+} h(d)$ iff $\gamma(\varepsilon) \subseteq_{\mathbf{W}^+} \{d\}^{\lhd}$ iff $\varepsilon \in \{d\}^{\lhd}$ iff $\varepsilon \; N \; d$. This is equivalent to $B \vdash_{\mathcal{K}\mathbf{HL}} d$, so we have that $B \vdash_{\mathcal{K}\mathbf{HL}} b$, for all $b \in B$ implies $B \vdash_{\mathcal{K}\mathbf{HL}} c$. Thus, we obtain $B \vdash_{\mathcal{K}\mathbf{HL}} c$.

Variants of frames

Distributive frames

A *distributive residuated frame* is a structure
$\mathbf{W} = (W, \circ, 1, \owedge, \sqsubseteq, W')$

- $(W, \sqsubseteq, W')$ is a lattice frame
- $(W, \circ, 1)$ is a monoid
- $(W, \owedge)$ is a commutative, idempotent semigroup
- both $\circ$ and $\owedge$ are residuated and the following condition holds:

$$\frac{x \sqsubseteq z}{x \owedge y \sqsubseteq z}\ (\owedge i)$$

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Applications include:

- Simple equations are: All equations over $\{\wedge, \vee, \cdot, 1\}$.
- A new *distributive* completion
- Cut-elimination (DFL and BI-logic)
- FMP, FEP

An *involutive frame* is a structure $\mathbf{W} = (W, \circ, \varepsilon, {}^{\sim}, {}^{-}, \sqsubseteq)$, where

- $(W, \sqsubseteq, W)$ is a lattice frame
- $(W, \circ, \varepsilon)$ is a monoid
- $x^{\sim -} = x = x^{- \sim}$
- $(x \circ y)^{\sim\sim} = (x^{\sim\sim} \circ y^{\sim\sim})$
- $\circ$ is residuated with $x \bbslash z = (z^{-} \circ x)^{\sim}$ and $z \sslash y = (y \circ z^{\sim})^{-}$

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To the conditions for a Gentzen frame we add:

$$\frac{x \sqsubseteq a}{\sim a \sqsubseteq x^{\sim}}\ (\sim\mathrm{L}) \qquad \frac{x \sqsubseteq a^{\sim}}{x \sqsubseteq \sim a}\ (\sim\mathrm{R}) \qquad \frac{x \sqsubseteq a}{-a \sqsubseteq x^{-}}\ (-\mathrm{L}) \qquad \frac{x \sqsubseteq a^{-}}{x \sqsubseteq -a}\ (-\mathrm{R})$$

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Applications include:

- A new *involutive* completion
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Existence of ComIDM

An ℓ-bimonoid is a structure $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, +, 0)$, with a lattice and two commutative monoid reducts, such that multiplciation distributes over joins, addition over meets and $x(y+z) \leq xy + z$. Given such an algebra, we will construct an involutive $\mathbf{A}$-frame $\mathbf{F_A}$.

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Theorem. If $\mathbf{A}$ is an ℓ-bimonoid, then $\mathbf{F_A}$ is faithful involutive $\mathbf{A}$-frame. So, $\mathbf{A}$ embeds into the InCRL $\mathbf{F_A^+}$.

A $BiFL_e$-*algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \rightarrow, 1, +, -, 0)$, where $(A, \wedge, \vee, \cdot, \rightarrow, 1)$ and $(A, \vee, \wedge, +, -, 0)$ are commutative residuated lattices.

BiFL frames

A *$BiFL_e$-algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 1, +, -, 0)$, where $(A, \wedge, \vee, \cdot, \to, 1)$ and $(A, \vee, \wedge, +, -, 0)$ are commutative residuated lattices.

An FL_e^+-algebra is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 1, +, 0)$, where $(A, \wedge, \vee, \cdot, \to, 1)$ is a commutative residuated lattice and $x + (y \wedge z) = (x + y) \wedge (x + z)$.

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A *(commutative) biresiduated frame* is a structure $\mathbf{W} = (W, \circ, \varepsilon, N, W', \oplus, \epsilon)$, where

- (W, N, W') is a lattice frame
- $(W, \circ, \varepsilon)$ and $(W', \oplus, \epsilon)$ are commutative monoids.
- $x \circ y \sqsubseteq z$ iff $y \sqsubseteq x \setminus\!\!\setminus z$, and $z \sqsubseteq x \oplus y$ iff $z \mathbin{/\!\!/} y \sqsubseteq x$.

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Using biresiduated frames we can prove that every FL_e^+-algebra can be embedded in a $BiFL_e$-algebra.

Hyper-frames

A *hyperresiduated frame* is a structure $\mathbf{H} = (W, W', \vdash, \circ, \varepsilon)$, where

- $(W, \circ, \varepsilon)$ is a monoid and W' is a set.
- $\vdash$ is an upward closed subset of H, the free semilattice over $W \times W'$.
- For all $x, y \in W$ and $z \in W'$ there exist elements $x \mathbin{\backslash\!\!\backslash} z, z \mathbin{/\!\!/} y \in W'$ such that for any $h \in H$,

$$\vdash (x \circ y, z) \mid h \iff \vdash (y, x \mathbin{\backslash\!\!\backslash} z) \mid h \iff \vdash (x, z \mathbin{/\!\!/} y) \mid h.$$

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- $\vdash$ is an upward closed subset of H, the free semilattice over $W \times W'$.
- For all $x, y \in W$ and $z \in W'$ there exist elements $x \mathbin{\backslash\!\!\backslash} z, z \mathbin{/\!\!/} y \in W'$ such that for any $h \in H$,

$$\vdash (x \circ y, z) \mid h \;\Leftrightarrow \vdash (y, x \mathbin{\backslash\!\!\backslash} z) \mid h \;\Leftrightarrow \vdash (x, z \mathbin{/\!\!/} y) \mid h.$$

The *dual algebra* $\mathbf{H}^+$ is the dual algebra of the residuated frame $r(\mathbf{H}) = (W \times H, W' \times H, \sqsubseteq, \bullet, (\varepsilon; \emptyset))$,

$$\begin{array}{rcl} (x; h_1) \bullet (y; h_2) & = & (x \circ y; h_1 \mid h_2) \\ (x; h_1) \mathbin{\backslash\!\!\backslash} (z; h_2) & = & (x \mathbin{\backslash\!\!\backslash} z; h_1 \mid h_2) \\ (z; h_2) \mathbin{/\!\!/} (x; h_1) & = & (z \mathbin{/\!\!/} x; h_1 \mid h_2) \\ (x; h_1) \sqsubseteq (z; h_2) & \Leftrightarrow & \vdash (x, z) \mid h_1 \mid h_2. \end{array}$$

Example. If $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ is an residuated lattice, then $\mathbf{H_A} = (A, A, \vdash, \cdot, 1)$ is a hyperresiduated frame, where:

$$\vdash (x_1, y_1) | \ldots | (x_n, y_n) \iff 1 \leq \gamma_1(x_1 \backslash y_1) \vee \cdots \vee \gamma_n(x_n \backslash y_n).$$

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Example. Let W be the free monoid over the set Fm of all formulas and n-negated formulas $n \in \mathbb{Z}$. We can define a hyperresiduated frame from a hypersequent version of $\mathbf{FL}$ in the natural way.

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