Focusing via display

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Overview

1. The need of structural reasoning
2. The multi-type approach comes in handy
3. Which logic for linguistic analysis?
4. Focusing via display

The need of structural reasoning

Parsing as deduction

[Lambek 58]: string of words, [Lambek 61]: bracketed strings (phrases),
[Ajdukiewicz 35, Bar-Hillel 53]: AB-grammar

- Parts of speech (noun, verb...) $\rightsquigarrow$ logical formulas - types.
- Grammaticality judgement $\rightsquigarrow$ logical deduction - computation.

np	$\otimes$	$(np\backslash s)$	$\otimes$	$(((np\backslash s)\backslash(np\backslash s))/np)$	$\otimes$	(np/n)	$\otimes$	n	$\vdash$	s
time		flies		like		an		arrow		

- transitive verb 'love': $(np\backslash s)/np$
  - kids · (love · games)
- conjunction words 'and/but': *chameleon* word $(X\backslash X)/X$
  - $X = s$: (kids like sweets)$_s$ but (parents prefer liquor)$_s$
  - $X = np\backslash s$: kids (like sweets)$_{np\backslash s}$ but (hate vegetables)$_{np\backslash s}$
- relative pronoun 'that': $(n\backslash n)/(s/np)$, i.e. it looks for a noun n to its left and an *incomplete* sentence to its right (s/np: it misses a np, the *gap* at the right)

L: Global Associativity $\rightsquigarrow$ peripheral gaps $\surd$

$$\dfrac{\dfrac{\text{games}}{n} \quad \dfrac{\dfrac{\text{that}}{(n\backslash n)/(s/np)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\text{kids}}{np} \quad \dfrac{\dfrac{\text{like}}{(np\backslash s)/np} \quad [\textvisiblespace \vdash np]^1}{\text{like}\cdot \textvisiblespace \vdash np\backslash s}[/E]}{\text{kids}\cdot(\text{like}\cdot\textvisiblespace)\vdash s}[\backslash E]}{(\text{kids}\cdot\text{like})\cdot\textvisiblespace\vdash s}[A]}{\text{kids}\cdot\text{like}\vdash s/np}[/I]^1}{\text{that}\cdot(\text{kids}\cdot\text{like})\vdash n\backslash n}[/E]}{\text{games}\cdot(\text{that}\cdot(\text{kids}\cdot\text{like}))\vdash n}[\backslash E]$$

$\lambda x_3.((\textsc{games}\ x_3) \wedge ((\textsc{like}\ x_3)\ \textsc{kids}))$

The multi-type approach comes in handy

The language of Lambek calculus

$$
\begin{aligned}
A &::= p \mid A \otimes A \mid A \;/\; A \mid A \setminus A \\
X &::= A \mid X \mathbin{\hat{\otimes}} X \mid X \mathbin{\check{/}} X \mid X \mathbin{\check{\setminus}} X
\end{aligned}
.
$$

Lambek calculus NL (L = NL + Associativity)

- Identity and Cut rules (preorder)

$$\text{Id}\ \frac{}{A \vdash A} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \text{Cut}$$

- Display rules (residuation, adjunction)

$$\text{rp}\ \frac{X \vdash Z \check{/} Y}{\dfrac{X \hat{\otimes} Y \vdash Z}{Y \vdash X \check{\backslash} Z}}\ \text{rp}$$

- Logical rules (arity and tonicity)

$$\otimes_L\ \frac{A \hat{\otimes} B \vdash Y}{A \otimes B \vdash Y} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \hat{\otimes} Y \vdash A \otimes B}\ \otimes_R$$

$$\backslash_L\ \frac{X \vdash A \qquad B \vdash Y}{A \backslash B \vdash X \check{\backslash} Y} \qquad \frac{X \vdash A \check{\backslash} B}{X \vdash A \backslash B}\ \backslash_R$$

$$/_L\ \frac{X \vdash A \qquad B \vdash Y}{B / A \vdash Y \check{/} X} \qquad \frac{X \vdash B \check{/} A}{X \vdash B / A}\ /_R$$

Proper display calculi

[Wansing 98]: proper, [Belnap 82, 89]: display logic, [Mints 72, Dunn 73, 75]: structural connectives

Definition

A **proper DC** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation);
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters (Properness!);
5. **reduction strategy** exists when cut formulas are principal.

Theorem (Canonical!)

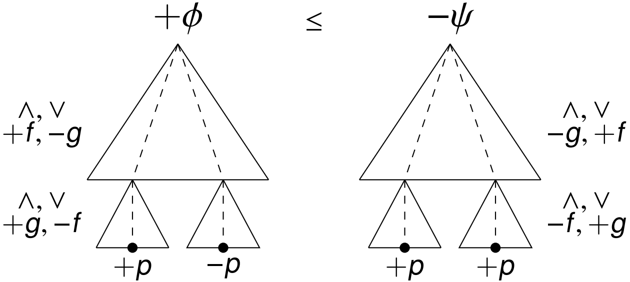
Cut elim. and subformula property hold for any **proper DC**.

Which logics are properly displayable?

[Ciabattoni et al. 15, Greco et al. 16]

Complete characterization:

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**: $\rightsquigarrow$ unified correspondence



Fact: cut-elim., subfm. prop., sound-&-completeness, conservativity **guaranteed** by metatheorem + ALBA-technology.

For many... but not for all.

[www.appliedlogictudelft.nl]

- The characterization theorem sets **hard boundaries** to the scope of proper display calculi.
- Interesting logics are **left out**:
  - DEL, PDL, Logic of Resources and Capabilities
  - Linear logic
  - (Lattice logic)
  - (First order logic)
  - Inquisitive logic
  - Semi De Morgan logic
  - Bi-lattice logic
  - Rough algebras

Can we **extend the scope** of proper display calculi?

Yes: proper display calculi $\rightsquigarrow$ proper **multi-type** calculi

Multi-type proper display calculi

[Greco et al. 14, ...]

Definition

A **proper DC** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation)
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters within each type (Properness!);
5. **reduction strategy** exists when cut formulas are principal.
6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

Theorem (Canonical!)

Cut elim. and subformula property hold for any **proper m.DC**.

Which logic for linguistic analysis?

L: Global Associativity $\rightsquigarrow$ overgeneration ✗

$$\text{kids} : np \qquad \text{like} : (np\backslash s)/np \qquad \text{call\_of\_duty} : np \qquad \text{but} : (s\backslash s)/s \qquad \text{parents} : np \qquad \text{hate} : (np\backslash s)/np \qquad [\,\llcorner \vdash np\,]^1$$

$$\text{hate} \cdot \llcorner \vdash np\backslash s \quad [/E]$$

$$\text{like} \cdot \text{call\_of\_duty} \vdash np\backslash s \quad [/E]$$

$$\text{parents} \cdot (\text{hate} \cdot \llcorner) \vdash s \quad [\backslash E]$$

$$\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty}) \vdash s \quad [\backslash E]$$

$$\text{but} \cdot (\text{parents} \cdot (\text{hate} \cdot \llcorner)) \vdash s\backslash s \quad [/E]$$

$$(\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty})) \cdot (\text{but} \cdot (\text{parents} \cdot (\text{hate} \cdot \llcorner))) \vdash s \quad [\backslash E]$$

$$(\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty})) \cdot (\text{but} \cdot ((\text{parents} \cdot \text{hate}) \cdot \llcorner)) \vdash s \quad [A]$$

$$(\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty})) \cdot ((\text{but} \cdot (\text{parents} \cdot \text{hate})) \cdot \llcorner) \vdash s \quad [A]$$

$$((\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty})) \cdot (\text{but} \cdot (\text{parents} \cdot \text{hate}))) \cdot \llcorner \vdash s \quad [A]$$

$$(\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty})) \cdot (\text{but} \cdot (\text{parents} \cdot \text{hate})) \vdash s/np \quad [/I]^1$$

$$\text{games} : n \qquad \text{that} : (n\backslash n)/(s/np)$$

$$\text{that} \cdot ((\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty})) \cdot (\text{but} \cdot (\text{parents} \cdot \text{hate}))) \vdash n\backslash n \quad [/E]$$

$$\text{games} \cdot (\text{that} \cdot ((\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty})) \cdot (\text{but} \cdot (\text{parents} \cdot \text{hate})))) \vdash n \quad [\backslash E]$$

$$\text{games} \cdot (\text{that} \cdot (((\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty})) \cdot \text{but}) \cdot (\text{parents} \cdot \text{hate}))) \vdash n \quad [A]$$

$$\text{games} \cdot (\text{that} \cdot ((((\text{kids} \cdot (\text{like} \cdot \text{call\_of\_duty})) \cdot \text{but}) \cdot \text{parents}) \cdot \text{hate})) \vdash n \quad [A]$$

$$\lambda y_4.((\textsc{games}\ y_4) \wedge ((\textsc{but}\ ((\textsc{hate}\ y_4)\ \textsc{parents}))\ ((\textsc{like}\ \textsc{call\_of\_duty})\ \textsc{kids})))$$

NL+SC (licence): Controlled Associativity ⇝ peripheral gaps √

[Moortgat 96]

$$\dfrac{\text{like}}{(np\backslash s)/np} \qquad \dfrac{[\,\llcorner\!\lrcorner \vdash \Box np]^2}{\langle \llcorner\!\lrcorner \rangle \vdash np}\ [\Box E]$$

$$\dfrac{(np\backslash s)/np \qquad \langle \llcorner\!\lrcorner \rangle \vdash np}{\text{like} \cdot \langle \llcorner\!\lrcorner \rangle \vdash np\backslash s}\ [/E]$$

$$\dfrac{\dfrac{\text{kids}}{np} \qquad \text{like} \cdot \langle \llcorner\!\lrcorner \rangle \vdash np\backslash s}{\text{kids} \cdot (\text{like} \cdot \langle \llcorner\!\lrcorner \rangle) \vdash s}\ [\backslash E]$$

$$\dfrac{\text{kids} \cdot (\text{like} \cdot \langle \llcorner\!\lrcorner \rangle) \vdash s}{(\text{kids} \cdot \text{like}) \cdot \langle \llcorner\!\lrcorner \rangle \vdash s}\ [MA]$$

$$\dfrac{[\,\llcorner\!\lrcorner \vdash \Diamond\Box np]^1 \qquad (\text{kids} \cdot \text{like}) \cdot \langle \llcorner\!\lrcorner \rangle \vdash s}{(\text{kids} \cdot \text{like}) \cdot \llcorner\!\lrcorner \vdash s}\ [\Diamond E]^2$$

$$\dfrac{(\text{kids} \cdot \text{like}) \cdot \llcorner\!\lrcorner \vdash s}{\text{kids} \cdot \text{like} \vdash s/\Diamond\Box np}\ [/I]^1$$

$$\dfrac{\dfrac{\text{that}}{(n\backslash n)/(s/\Diamond\Box np)} \qquad \text{kids} \cdot \text{like} \vdash s/\Diamond\Box np}{\text{that} \cdot (\text{kids} \cdot \text{like}) \vdash n\backslash n}\ [/E]$$

$$\dfrac{\dfrac{\text{games}}{n} \qquad \text{that} \cdot (\text{kids} \cdot \text{like}) \vdash n\backslash n}{\text{games} \cdot (\text{that} \cdot (\text{kids} \cdot \text{like})) \vdash n}\ [\backslash E]$$

$$\lambda y_3.((\text{GAMES}\ y_3) \wedge ((\text{LIKE}\ y_3)\ \text{KIDS}))$$

NL+SC (licence): Controlled Associativity and Exchange $\rightsquigarrow$ internal gaps √ (Global Assoc. $\rightsquigarrow$ undergeneration ×)

$$\dfrac{\text{like}}{(np\backslash s)/np} \qquad \dfrac{[\text{␣} \vdash \Box np]^2}{\langle \text{␣} \rangle \vdash np}\,[\Box E]$$

$$\dfrac{(np\backslash s)/np \qquad \langle \text{␣} \rangle \vdash np}{\text{like} \cdot \langle \text{␣} \rangle \vdash np\backslash s}\,[/E]$$

$$\dfrac{[\text{␣} \vdash np]^3 \qquad \text{like} \cdot \langle \text{␣} \rangle \vdash np\backslash s}{\text{␣} \cdot (\text{like} \cdot \langle \text{␣} \rangle) \vdash s}\,[\backslash E]$$

$$\dfrac{\text{␣} \cdot (\text{like} \cdot \langle \text{␣} \rangle) \vdash s}{\text{like} \cdot \langle \text{␣} \rangle \vdash np\backslash s}\,[\backslash I]^3 \qquad \dfrac{\text{a\_lot}}{(np\backslash s)\backslash(np\backslash s)}$$

$$\dfrac{\text{like} \cdot \langle \text{␣} \rangle \vdash np\backslash s \qquad (np\backslash s)\backslash(np\backslash s)}{(\text{like} \cdot \langle \text{␣} \rangle) \cdot \text{a\_lot} \vdash np\backslash s}\,[\backslash E]$$

$$\dfrac{\dfrac{\text{kids}}{np} \qquad (\text{like} \cdot \langle \text{␣} \rangle) \cdot \text{a\_lot} \vdash np\backslash s}{\text{kids} \cdot ((\text{like} \cdot \langle \text{␣} \rangle) \cdot \text{a\_lot}) \vdash s}\,[\backslash E]$$

$$\dfrac{\text{kids} \cdot ((\text{like} \cdot \langle \text{␣} \rangle) \cdot \text{a\_lot}) \vdash s}{\text{kids} \cdot ((\text{like} \cdot \text{a\_lot}) \cdot \langle \text{␣} \rangle) \vdash s}\,[MC]$$

$$\dfrac{\text{kids} \cdot ((\text{like} \cdot \text{a\_lot}) \cdot \langle \text{␣} \rangle) \vdash s}{(\text{kids} \cdot (\text{like} \cdot \text{a\_lot})) \cdot \langle \text{␣} \rangle \vdash s}\,[MA]$$

$$\dfrac{[\text{␣} \vdash \Diamond\Box np]^1 \qquad (\text{kids} \cdot (\text{like} \cdot \text{a\_lot})) \cdot \langle \text{␣} \rangle \vdash s}{(\text{kids} \cdot (\text{like} \cdot \text{a\_lot})) \cdot \text{␣} \vdash s}\,[\Diamond E]^2$$

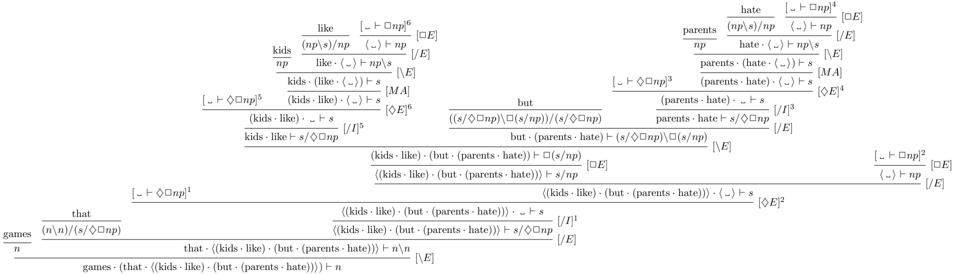
$$\dfrac{(\text{kids} \cdot (\text{like} \cdot \text{a\_lot})) \cdot \text{␣} \vdash s}{\text{kids} \cdot (\text{like} \cdot \text{a\_lot}) \vdash s/\Diamond\Box np}\,[/I]^1$$

$$\dfrac{\dfrac{\text{that}}{(n\backslash n)/(s/\Diamond\Box np)} \qquad \text{kids} \cdot (\text{like} \cdot \text{a\_lot}) \vdash s/\Diamond\Box np}{\text{that} \cdot (\text{kids} \cdot (\text{like} \cdot \text{a\_lot})) \vdash n\backslash n}\,[/E]$$

$$\dfrac{\dfrac{\text{games}}{n} \qquad \text{that} \cdot (\text{kids} \cdot (\text{like} \cdot \text{a\_lot})) \vdash n\backslash n}{\text{games} \cdot (\text{that} \cdot (\text{kids} \cdot (\text{like} \cdot \text{a\_lot}))) \vdash n}\,[\backslash E]$$

$$\lambda y_4.((\textsc{games}\ y_4) \wedge ((\textsc{a\_lot}\ (\textsc{like}\ y_4))\ \textsc{kids}))$$

NL+SC (licence & block): Controlled Associativity ⇝ peripheral & internal gaps √



$\lambda y_6.((\textsc{games}\ y_6) \wedge (((\textsc{but}\ \lambda x_3.((\textsc{hate}\ x_3)\ \textsc{parents}))\ \lambda x_4.((\textsc{like}\ x_4)\ \textsc{kids}))\ y_6))$

Lambek-Grishin calculus

$$A ::= p \mid A \otimes A \mid A \obslash A \mid A \oslash A \mid A \oplus A \mid A \;/\; A \mid A \;\backslash\; A$$

$$X \begin{cases} \mathcal{I} ::= A \mid \mathcal{I} \,\hat{\otimes}\, \mathcal{I} \mid \mathcal{O} \,\hat{\obslash}\, \mathcal{I} \mid \mathcal{I} \,\hat{\oslash}\, \mathcal{O} \\ \mathcal{O} ::= A \mid \mathcal{O} \,\check{\oplus}\, \mathcal{O} \mid \mathcal{I} \,\check{/}\, \mathcal{O} \mid \mathcal{O} \,\check{\backslash}\, \mathcal{I} \end{cases}$$

Full NL = NL + Display and Logical rules for the additional (dual) connectives.

- Grishin rules (interactions)

$$\text{G1}\ \frac{X \hat{\otimes} Y \vdash Z \check{\oplus} W}{Z \hat{\obslash} X \vdash W \check{/} Y} \qquad \frac{X \hat{\otimes} Y \vdash Z \check{\oplus} W}{Z \hat{\obslash} Y \vdash X \check{\backslash} W}\ \text{G2}$$

$$\text{G3}\ \frac{X \hat{\otimes} Y \vdash Z \check{\oplus} W}{Y \hat{\oslash} W \vdash X \check{\backslash} Z} \qquad \frac{X \hat{\otimes} Y \vdash Z \check{\oplus} W}{X \hat{\oslash} W \vdash Z \check{/} Y}\ \text{G4}$$

LG: Dual operators & Interaction rules $\rightsquigarrow$ beyond context-free grammars √ (indirect argument)

[Moot 07]

Fact: **NL** and **L** recognize only context-free languages.

Fact: To capture the dependencies in natural languages, one needs expressivity beyond context-free but below context-sensitive (e.g. crossing dependencies: $\{a^n b^m c^n d^m \mid n, m > 0\}$).

Many Rewriting systems / Formal Grammars handle such patterns: e.g. Tree Adjoining Grammars are broadly used.

Tree Adjoining Grammars can be modeled using **Grishin interaction principles**, and mildly context-sensitive patterns can be obtained within **LG**.

(For a direct argument: see “in situ questions” in Japanese.)

Focused Lambek-Grishin calculus fLG

[Moortgat 09]: fLG, [Andreoli 2001]: focused proof are complete for LL

fLG is a refinement of **LG** where:

1. Display, Structural and **invertible** Logical rules are retained;
2. Identity and **non invertible** Logical rules are replaced by their focused version (Axiom/Coaxiom, Tonicity);
3. four new rules are added (Focusing/Defocusing):

- Axiom / Coaxiom where p is negative and q is positive

$$\text{CoAx}\ \frac{}{\boxed{p} \vdash p} \qquad \frac{}{q \vdash \boxed{q}}\ \text{Ax}$$

- Focusing / Defocusing where A is negative and B is positive

$$\leftharpoondown\ \frac{\boxed{A} \vdash Y}{A \vdash Y} \qquad \frac{X \vdash A}{X \vdash \boxed{A}}\ \rightharpoondown$$

$$\leftharpoonup\ \frac{B \vdash Y}{\boxed{B} \vdash Y} \qquad \frac{X \vdash \boxed{B}}{X \vdash B}\ \rightharpoonup$$

- Tonicity rules

$$\oplus_L \frac{\boxed{A} \vdash X \qquad \boxed{B} \vdash Y}{\boxed{A \oplus B} \vdash X \cdot \oplus \cdot Y} \qquad \frac{X \vdash \boxed{A} \qquad Y \vdash \boxed{B}}{X \cdot \otimes \cdot Y \vdash \boxed{A \otimes B}} \otimes_R$$

$$/_L \frac{X \vdash \boxed{A} \qquad \boxed{B} \vdash Y}{\boxed{B / A} \vdash Y \cdot / \cdot X} \qquad \frac{X \vdash \boxed{A} \qquad \boxed{B} \vdash Y}{Y \cdot \oslash \cdot X \vdash \boxed{B \oslash A}} \oslash_R$$

$$\backslash_L \frac{X \vdash \boxed{A} \qquad \boxed{B} \vdash Y}{\boxed{A \backslash B} \vdash X \cdot \backslash \cdot Y} \qquad \frac{X \vdash \boxed{A} \qquad \boxed{B} \vdash Y}{X \cdot \oslash \cdot Y \vdash \boxed{A \oslash B}} \oslash_R$$

A simple proof search strategy

Backward chaining focused proof search:

1. apply logical invertible rules as much as possible (you may use structural rules);
2. pick a formula and put it in focus;
3. decompose the focused formula by means of non-invertible logical rules as much as possible;
4. go to 1.

Properties:

- each derivable sequent has at most one formula in focus
- three phases
  - positive: sequent with a positive formula in focus
  - negative: sequent with a negative formula in focus
  - neutral: sequents with no formula in focus
- neutral phases always alternate the move from a focused phase x to another y and $x \neq y$

fLG: Focusing & Bias $\rightsquigarrow$ Reading ambiguity ($\forall\exists$) $\checkmark$

$$\dfrac{\dfrac{y_0}{np \vdash \boxed{np}} \qquad \dfrac{\alpha_0}{\boxed{s} \vdash s}}{\boxed{np\backslash s} \vdash np \cdot\backslash\cdot s}\,(\backslash L) \qquad \dfrac{z_0}{np \vdash \boxed{np}}$$

$$\dfrac{\boxed{np\backslash s} \vdash np \cdot\backslash\cdot s \qquad np \vdash \boxed{np}}{\boxed{(np\backslash s)/np} \vdash (np \cdot\backslash\cdot s)\cdot/\cdot np}\,(/L)$$

$$\dfrac{\boxed{(np\backslash s)/np} \vdash (np \cdot\backslash\cdot s)\cdot/\cdot np}{(np\backslash s)/np \vdash (np \cdot\backslash\cdot s)\cdot/\cdot np}\,\leftharpoondown$$

$$\dfrac{(np\backslash s)/np \vdash (np \cdot\backslash\cdot s)\cdot/\cdot np}{\boxed{np} \vdash ((np\backslash s)/np)\cdot\backslash\cdot(np \cdot\backslash\cdot s)}\,\leftharpoondown \qquad \dfrac{\text{teacher}}{n \vdash \boxed{n}}$$

$$\dfrac{\boxed{np} \vdash ((np\backslash s)/np)\cdot\backslash\cdot(np \cdot\backslash\cdot s) \qquad n \vdash \boxed{n}}{\boxed{np/n} \vdash (((np\backslash s)/np)\cdot\backslash\cdot(np \cdot\backslash\cdot s))\cdot/\cdot n}\,(/L)$$

$$\dfrac{\boxed{np/n} \vdash (((np\backslash s)/np)\cdot\backslash\cdot(np \cdot\backslash\cdot s))\cdot/\cdot n}{np/n \vdash (((np\backslash s)/np)\cdot\backslash\cdot(np \cdot\backslash\cdot s))\cdot/\cdot n}\,\leftharpoondown$$

$$\dfrac{np/n \vdash (((np\backslash s)/np)\cdot\backslash\cdot(np \cdot\backslash\cdot s))\cdot/\cdot n}{\boxed{np} \vdash s\cdot/\cdot(((np\backslash s)/np)\cdot\otimes\cdot((np/n)\cdot\otimes\cdot n))}\,\leftharpoondown \qquad \dfrac{\text{student}}{n \vdash \boxed{n}}$$

$$\dfrac{\boxed{np} \vdash s\cdot/\cdot(((np\backslash s)/np)\cdot\otimes\cdot((np/n)\cdot\otimes\cdot n)) \qquad n \vdash \boxed{n}}{\boxed{np/n} \vdash (s\cdot/\cdot(((np\backslash s)/np)\cdot\otimes\cdot((np/n)\cdot\otimes\cdot n)))\cdot/\cdot n}\,(/L)$$

$$\dfrac{\boxed{np/n} \vdash (s\cdot/\cdot(((np\backslash s)/np)\cdot\otimes\cdot((np/n)\cdot\otimes\cdot n)))\cdot/\cdot n}{np/n \vdash (s\cdot/\cdot(((np\backslash s)/np)\cdot\otimes\cdot((np/n)\cdot\otimes\cdot n)))\cdot/\cdot n}\,\leftharpoondown$$

$$\dfrac{np/n \vdash (s\cdot/\cdot(((np\backslash s)/np)\cdot\otimes\cdot((np/n)\cdot\otimes\cdot n)))\cdot/\cdot n}{((np/n)\cdot\otimes\cdot n)\cdot\otimes\cdot(((np\backslash s)/np)\cdot\otimes\cdot((np/n)\cdot\otimes\cdot n)) \vdash \boxed{s}}\,\rightharpoondown$$

$$\mu\alpha_0.\langle\ \text{every} \upharpoonright (\widetilde{\mu} y_0.\langle\ \text{some} \upharpoonright (\widetilde{\mu} z_0.\langle\ \text{likes} \upharpoonright ((y_0 \backslash \alpha_0) / z_0)\ \rangle / \text{teacher})\ \rangle / \text{student})\ \rangle$$

$$\lambda\alpha_0.(\lceil\text{every}\rceil\ \langle\lambda y_0.(\lceil\text{some}\rceil\ \langle\lambda z_0.(\lceil\text{likes}\rceil\ \langle\langle y_0,\alpha_0\rangle, z_0\rangle), \lceil\text{teacher}\rceil\rangle), \lceil\text{student}\rceil\rangle)$$

$$\lambda\alpha_0.(\forall\ \lambda z_1.((\Rightarrow\ (\text{STUDENT}\ z_1))\ (\exists\ \lambda y_2.((\wedge\ (\text{TEACHER}\ y_2))\ (\alpha_0\ ((\text{LIKES}\ y_2)\ z_1))))))$$

fLG: Focusing & Bias ⇝ Reading ambiguity (∃∀) √

$$\dfrac{\dfrac{\dfrac{z_0}{np \vdash \boxed{np}} \quad \dfrac{\alpha_0}{\boxed{s} \vdash s}}{\boxed{np\backslash s} \vdash np \cdot\backslash\cdot s}(\backslash L) \quad \dfrac{y_0}{np \vdash \boxed{np}}}{\boxed{(np\backslash s)/np} \vdash (np \cdot\backslash\cdot s) \cdot/\cdot np}(/L)$$

$$\dfrac{\boxed{(np\backslash s)/np} \vdash (np \cdot\backslash\cdot s) \cdot/\cdot np}{(np\backslash s)/np \vdash (np \cdot\backslash\cdot s) \cdot/\cdot np}\leftharpoonup$$

$$\dfrac{\dfrac{(np\backslash s)/np \vdash (np \cdot\backslash\cdot s) \cdot/\cdot np}{\boxed{np} \vdash s \cdot/\cdot (((np\backslash s)/np) \cdot\otimes\cdot np)}\leftharpoondown \quad \dfrac{\text{student}}{n \vdash \boxed{n}}}{\boxed{np/n} \vdash (s \cdot/\cdot (((np\backslash s)/np) \cdot\otimes\cdot np)) \cdot/\cdot n}(/L)$$

$$\dfrac{\boxed{np/n} \vdash (s \cdot/\cdot (((np\backslash s)/np) \cdot\otimes\cdot np)) \cdot/\cdot n}{np/n \vdash (s \cdot/\cdot (((np\backslash s)/np) \cdot\otimes\cdot np)) \cdot/\cdot n}\leftharpoonup$$

$$\dfrac{\dfrac{np/n \vdash (s \cdot/\cdot (((np\backslash s)/np) \cdot\otimes\cdot np)) \cdot/\cdot n}{\boxed{np} \vdash ((np\backslash s)/np) \cdot\backslash\cdot (((np/n) \cdot\otimes\cdot n) \cdot\backslash\cdot s)}\leftharpoondown \quad \dfrac{\text{teacher}}{n \vdash \boxed{n}}}{\boxed{np/n} \vdash (((np\backslash s)/np) \cdot\backslash\cdot (((np/n) \cdot\otimes\cdot n) \cdot\backslash\cdot s)) \cdot/\cdot n}(/L)$$

$$\dfrac{\dfrac{\boxed{np/n} \vdash (((np\backslash s)/np) \cdot\backslash\cdot (((np/n) \cdot\otimes\cdot n) \cdot\backslash\cdot s)) \cdot/\cdot n}{np/n \vdash (((np\backslash s)/np) \cdot\backslash\cdot (((np/n) \cdot\otimes\cdot n) \cdot\backslash\cdot s)) \cdot/\cdot n}\leftharpoonup}{((np/n) \cdot\otimes\cdot n) \cdot\otimes\cdot (((np\backslash s)/np) \cdot\otimes\cdot ((np/n) \cdot\otimes\cdot n)) \vdash \boxed{s}}\rightharpoonup$$

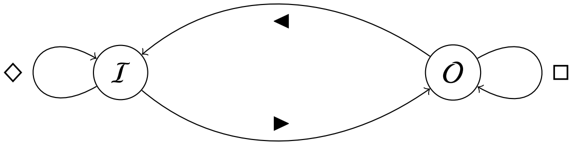
$$\mu\alpha_0.\langle\ \text{some} \upharpoonleft (\tilde{\mu} y_0.\langle\ \text{every} \upharpoonleft (\tilde{\mu} z_0.\langle\ \text{likes} \upharpoonleft ((z_0 \setminus \alpha_0) / y_0)\ \rangle / \text{student})\ \rangle / \text{teacher})\ \rangle$$

$$\lambda\alpha_0.(\lceil\text{some}\rceil\ \langle\lambda y_0.(\lceil\text{every}\rceil\ \langle\lambda z_0.(\lceil\text{likes}\rceil\ \langle\langle z_0, \alpha_0\rangle, y_0\rangle), \lceil\text{student}\rceil\rangle), \lceil\text{teacher}\rceil\rangle)$$

$$\lambda\alpha_0.(\exists\ \lambda z_1.((\wedge\ (\textsc{teacher}\ z_1))\ (\forall\ \lambda y_2.((\Rightarrow\ (\textsc{student}\ y_2))\ (\alpha_0\ ((\textsc{likes}\ z_1)\ y_2))))))$$

Focusing via display

Capturing focusing via “polarities”



- $\mathbb{G} = (G, \leq, \cdot)$

  $\mathbb{G} = (G, \leq, \cdot, +, /_{\cdot}, \backslash_{\cdot}, /_{+}, \backslash_{+})$
- $\mathcal{I} = (\mathcal{P}^{\updownarrow}(G), \subseteq, \otimes, \oslash, \oslash, \Diamond, \blacktriangleleft)$
- $\mathcal{O} = (\mathcal{P}^{\updownarrow}(G), \subseteq, \oplus, /, \backslash, \Box, \blacktriangleright)$

The language of multi-type Lambek-Grishin calculus

$$\mathcal{I}\begin{cases} A ::= a \mid \blacktriangleleft P \mid A \otimes A \mid P \obslash A \mid A \oslash P \\ \Sigma ::= A \mid \hat{\Diamond}\,\Sigma \mid \Sigma \,\hat{\otimes}\, \Sigma \mid \Gamma \,\hat{\obslash}\, \Sigma \mid \Sigma \,\hat{\oslash}\, \Gamma \end{cases}$$

$$\mathcal{O}\begin{cases} P ::= p \mid \blacktriangleright A \mid P \oplus P \mid P \,/\, A \mid A \,\backslash\, P \\ \Gamma ::= P \mid \check{\Box}\,\Gamma \mid \Gamma \,\check{\oplus}\, \Gamma \mid \Sigma \,\check{\backslash}\, \Gamma \mid \Gamma \,\check{/}\, \Sigma \end{cases}$$

Multi-type focused Lambek-Grishin calculus m.fLG

m.**fLG** includes three turnstiles ($\Sigma \vdash_{\mathcal{IO}} \Gamma$, $\Sigma \vdash_{\mathcal{II}} A$, and $P \vdash_{\mathcal{OO}} \Gamma$) and the following rules:

- Coaxiom / Axiom

$$\text{Coax}\,\frac{}{p \vdash_{\mathcal{OO}} \check{\Box}\, p} \qquad \frac{}{\hat{\Diamond}\, a \vdash_{\mathcal{II}} a}\,\text{Ax}$$

- Defocusing / Focusing

$$\leftharpoonup\,\frac{P \vdash_{\mathcal{OO}} \check{\Box}\,\Gamma}{\blacktriangleleft P \vdash_{\mathcal{IO}} \Gamma} \qquad \frac{\hat{\Diamond}\,\Sigma \vdash_{\mathcal{II}} A}{\Sigma \vdash_{\mathcal{IO}} \blacktriangleright A}\,\rightharpoondown$$

$$\leftharpoondown\,\frac{A \vdash_{\mathcal{IO}} \Gamma}{\blacktriangleright A \vdash_{\mathcal{OO}} \check{\Box}\,\Gamma} \qquad \frac{\Sigma \vdash_{\mathcal{IO}} P}{\hat{\Diamond}\,\Sigma \vdash_{\mathcal{II}} \blacktriangleleft P}\,\rightharpoonup$$

- Display rules

$$\text{rp}\ \frac{\Pi \vdash_{\mathcal{IO}} \Sigma \check{\backslash} \Gamma}{\dfrac{\Sigma \hat{\otimes} \Pi \vdash_{\mathcal{IO}} \Gamma}{\Sigma \vdash_{\mathcal{IO}} \Gamma \check{/} \Pi}\ \text{drp}} \qquad \frac{\Sigma \hat{\oslash} \Delta \vdash_{\mathcal{IO}} \Gamma}{\dfrac{\Sigma \vdash_{\mathcal{IO}} \Gamma \check{\oplus} \Delta}{\Gamma \hat{\obslash} \Sigma \vdash_{\mathcal{IO}} \Delta}\ \text{drp}}\ \text{rp}$$

- Grishin rules

$$\text{G1}\ \frac{\Sigma \hat{\otimes} \Pi \vdash_{\mathcal{IO}} \Gamma \check{\oplus} \Delta}{\Gamma \hat{\obslash} \Sigma \vdash_{\mathcal{IO}} \Delta \check{/} \Pi} \qquad \frac{\Sigma \hat{\otimes} \Pi \vdash_{\mathcal{IO}} \Gamma \check{\oplus} \Delta}{\Gamma \hat{\obslash} \Pi \vdash_{\mathcal{IO}} \Sigma \check{\backslash} \Delta}\ \text{G2}$$

$$\text{G3}\ \frac{\Sigma \hat{\otimes} \Pi \vdash_{\mathcal{IO}} \Gamma \check{\oplus} \Delta}{\Pi \hat{\oslash} \Delta \vdash_{\mathcal{IO}} \Sigma \check{\backslash} \Gamma} \qquad \frac{\Sigma \hat{\otimes} \Pi \vdash_{\mathcal{IO}} \Gamma \check{\oplus} \Delta}{\Sigma \hat{\oslash} \Delta \vdash_{\mathcal{IO}} \Gamma \check{/} \Pi}\ \text{G4}$$

- Logical rules

$$\otimes_L\ \frac{A \mathbin{\hat{\otimes}} B \vdash_{\mathcal{IO}} \Gamma}{A \otimes B \vdash_{\mathcal{IO}} \Gamma} \qquad \frac{\hat{\Diamond}\Sigma \vdash_{\mathcal{OO}} A \qquad \hat{\Diamond}\Pi \vdash_{\mathcal{OO}} B}{\hat{\Diamond}(\Sigma \mathbin{\hat{\otimes}} \Pi) \vdash_{\mathcal{OO}} A \otimes B}\ \otimes_R$$

$$\oslash_L\ \frac{A \mathbin{\hat{\oslash}} P \vdash_{\mathcal{IO}} \Gamma}{A \oslash P \vdash_{\mathcal{IO}} \Gamma} \qquad \frac{\hat{\Diamond}\Sigma \vdash_{\mathcal{OO}} A \qquad P \vdash_{\mathcal{II}} \check{\Box}\Gamma}{\hat{\Diamond}(\Sigma \mathbin{\hat{\oslash}} \Gamma) \vdash_{\mathcal{OO}} A \oslash P}\ \oslash_R$$

$$\obslash_L\ \frac{P \mathbin{\hat{\obslash}} A \vdash_{\mathcal{IO}} \Gamma}{P \obslash A \vdash_{\mathcal{IO}} \Gamma} \qquad \frac{\hat{\Diamond}\Sigma \vdash_{\mathcal{OO}} A \qquad P \vdash_{\mathcal{II}} \check{\Box}\Gamma}{\hat{\Diamond}(\Gamma \mathbin{\hat{\obslash}} \Sigma) \vdash_{\mathcal{OO}} P \obslash A}\ \obslash_R$$

$$\oplus_L\ \frac{P \vdash_{\mathcal{II}} \check{\Box}\Sigma \qquad Q \vdash_{\mathcal{II}} \check{\Box}\Pi}{P \oplus Q \vdash_{\mathcal{II}} \check{\Box}(\Sigma \mathbin{\check{\oplus}} \Pi)} \qquad \frac{\Sigma \vdash_{\mathcal{IO}} P \mathbin{\check{\oplus}} Q}{\Sigma \vdash_{\mathcal{IO}} P \oplus Q}\ \oplus_R$$

$$\backslash_L\ \frac{\hat{\Diamond}\Sigma \vdash_{\mathcal{OO}} A \qquad P \vdash_{\mathcal{II}} \check{\Box}\Gamma}{A \backslash P \vdash_{\mathcal{II}} \check{\Box}(\Sigma \mathbin{\check{\backslash}} \Gamma)} \qquad \frac{\Sigma \vdash_{\mathcal{IO}} A \mathbin{\check{\backslash}} P}{\Sigma \vdash_{\mathcal{IO}} A \backslash P}\ \backslash_R$$

$$/_L\ \frac{\hat{\Diamond}\Sigma \vdash_{\mathcal{II}} A \qquad P \vdash_{\mathcal{II}} \check{\Box}\Gamma}{P / A \vdash_{\mathcal{OO}} \check{\Box}(\Gamma \mathbin{\check{/}} \Sigma)} \qquad \frac{\Sigma \vdash_{\mathcal{IO}} P \mathbin{\check{/}} A}{\Sigma \vdash_{\mathcal{IO}} P / A}\ /_R$$

A modal language encoding proofs

$$
\leftharpoonup\ \frac{q \vdash_{\mathcal{OO}} \check{\Box} q}{\blacktriangleleft q \vdash_{\mathcal{IO}} q}
\qquad
\leftharpoonup\ \frac{r \vdash_{\mathcal{OO}} \check{\Box} r}{\blacktriangleleft r \vdash_{\mathcal{IO}} r}
$$

$$
\frac{\blacktriangleleft q \vdash_{\mathcal{IO}} q}{\hat{\Diamond} \blacktriangleleft q \vdash_{\mathcal{II}} \blacktriangleleft q}\ \rightharpoondown
\qquad
\frac{\blacktriangleleft r \vdash_{\mathcal{IO}} r}{\hat{\Diamond} \blacktriangleleft r \vdash_{\mathcal{II}} \blacktriangleleft r}\ \rightharpoondown
$$

$$
\frac{\hat{\Diamond} \blacktriangleleft q \vdash_{\mathcal{II}} \blacktriangleleft q \qquad \hat{\Diamond} \blacktriangleleft r \vdash_{\mathcal{II}} \blacktriangleleft r}{\hat{\Diamond} (\blacktriangleleft q \otimes \blacktriangleleft r) \vdash_{\mathcal{II}} \blacktriangleleft q \otimes \blacktriangleleft r}\ \otimes_R
$$

$$
\frac{\hat{\Diamond} (\blacktriangleleft q \otimes \blacktriangleleft r) \vdash_{\mathcal{II}} \blacktriangleleft q \otimes \blacktriangleleft r}{\blacktriangleleft q \otimes \blacktriangleleft r \vdash_{\mathcal{IO}} \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}\ \rightharpoonup
$$

$$
\leftharpoondown\ \frac{\blacktriangleleft q \otimes \blacktriangleleft r \vdash_{\mathcal{IO}} \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}{\blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r) \vdash_{\mathcal{OO}} \check{\Box} \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}
$$

$$
\backslash_L\ \frac{\hat{\Diamond} a \vdash_{\mathcal{II}} a \qquad \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r) \vdash_{\mathcal{OO}} \check{\Box} \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}{a \backslash \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r) \vdash_{\mathcal{OO}} \check{\Box} (a \,\check{\backslash}\, \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r))}
$$

$$
\leftharpoonup\ \frac{a \backslash \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r) \vdash_{\mathcal{OO}} \check{\Box} (a \,\check{\backslash}\, \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r))}{\blacktriangleleft (a \backslash \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)) \vdash_{\mathcal{IO}} a \,\check{\backslash}\, \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}
$$

$$
\text{rp}\ \frac{\blacktriangleleft (a \backslash \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)) \vdash_{\mathcal{IO}} a \,\check{\backslash}\, \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}{a \,\hat{\otimes}\, \blacktriangleleft (a \backslash \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)) \vdash_{\mathcal{IO}} \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}
$$

$$
\frac{a \,\hat{\otimes}\, \blacktriangleleft (a \backslash \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)) \vdash_{\mathcal{IO}} \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}{a \otimes \blacktriangleleft (a \backslash \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)) \vdash_{\mathcal{IO}} \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}
$$

$$
\leftharpoondown\ \frac{a \otimes \blacktriangleleft (a \backslash \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)) \vdash_{\mathcal{IO}} \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r)}{\blacktriangleleft (a \otimes \Box (a \backslash \blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r))) \vdash_{\mathcal{II}} \check{\Box} (\blacktriangleright (\blacktriangleleft q \otimes \blacktriangleleft r))}
$$

$$
\Updownarrow \tau
$$

$$
\boxed{a \otimes (a \backslash (q \otimes r))} \vdash q \otimes r
$$

Conclusions and future work

- fLG + structural control operators is an appropriate system for linguistic analysis.
- The multi-type approach provides the natural framework for
  - design of modular focused calculi
  - semantic analysis of focused proofs

Future works:

- The operators of fLG are inherently polymorphic: exploiting this feature via heterogeneous modalities is technically feasible and could lead to new insights.
- Modular game theoretic semantics for focused proofs.