Possibilities for Boolean, Heyting, and modal algebras

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What this talk is about

An alternative to the standard representation theory for Boolean, Heyting, and modal algebras from Stone (1934, 1937) and Jónsson and Tarski (1951).

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While the standard theory leads to the well-known "possible world semantics" in logic, the alternative theory forms the basis of the "possibility semantics" in logic.

Papers on which this talk is based

W. H. Holliday, "Possibility Frames and Forcing for Modal Logic," UC Berkeley Working Paper in Logic and the Methodology of Science (available online).

G. Bezhanishvili and W. H. Holliday, "Locales, Nuclei, and Dragalin Frames," *Advances in Modal Logic*, 2016.

G. Bezhanishvili and W. H. Holliday, "A Semantic Hierarchy for Intuitionistic Logic," forthcoming in a special issue of *Indagationes Mathematicae* on L.E.J. Brouwer: Fifty Years Later (available online).

N. Bezhanishvili and W. H. Holliday, "Choice-Free Stone Duality," UC Berkeley Working Paper in Logic and the Methodology of Science (available online).

Other work related to this program

J. van Benthem, N. Bezhanishvili, and W. H. Holliday, "A Bimodal Perspective on Possibility Semantics," *Journal of Logic and Computation*, 2016.

M. Harrison-Trainor, "A Representation Theorem for Possibility Models" and "First-Order Possibility Models and Finitary Completeness Proofs," under review.

W. H. Holliday, "Partiality and Adjointness in Modal Logic," *Advances in Modal Logic*, 2014.

W. H. Holliday, "Algebraic Semantics for S5 with Propositional Quantifiers," forthcoming in *Notre Dame Journal of Formal Logic*, 2017.

W. H. Holliday and T. Litak, "Complete Additivity and Modal Incompleteness," forthcoming in *Review of Symbolic Logic*, 2018.

G. Massas, *Possibility spaces, Q-completions and Rasiowa-Sikorski lemmas for non-classical logics*, ILLC Master of Logic Thesis, 2016.

K. Yamamoto, "Results in Modal Correspondence Theory for Possibility Semantics," *Journal of Logic and Computation*, 2017.

Z. Zhao, "Algorithmic Correspondence and Canonicity for Possibility Semantics," arXiv, 2016.

Standard theory

Let's start with the standard representation theory for Boolean, Heyting, and modal algebras from Stone (1934, 1937) and Jónsson and Tarski (1951)

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complete and atomic MA, completely multiplicative $\Box$	$\Rightarrow$ *atoms with relation aRb iff $a \nleq \Box\neg b$* $\Leftarrow$ *subsets with operation $\Box_R$ as above*	set with relation ("Kripke frame")

Two reasons to go beyond the standard theory

1. The relational structures of the standard theory are concrete and intuitive, but they only allow us to represent atomic/J^∞-generated MAs/HAs.

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For point 1, I will provide some motivation from the point of view of logic.

Superintuitionistic logics

A **superintuitionistic logic** is any set of formulas of the language of propositional logic that contains the axioms of the intuitionistic propositional calculus (IPC) and is closed under uniform substitution and modus ponens.

Superintuitionistic logics ordered by inclusion form a lattice that is dually isomorphic to the lattice of **varieties of Heyting algebras**.

There are continuum-many superintuitionistic logics. Some examples:

$$\begin{aligned}\text{Logic of Weak Excluded Middle} &= \text{IPC} + \neg p \vee \neg\neg p;\\ \text{Gödel-Dummet Logic} &= \text{IPC} + (p \to q) \vee (q \to p);\\ \text{Classical Logic} &= \text{IPC} + p \vee \neg p.\end{aligned}$$

Modal logics

The modal language adds to the propositional language a unary connective $\Box$.

A **modal logic** is any set of formulas of the modal language that contains all classical tautologies and the axiom $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$ and is closed under uniform substitution, modus ponens, and prefixing $\Box$.

Modal logics ordered by inclusion form a lattice that is dually isomorphic to the lattice of **varieties of modal algebras**.

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$$\begin{aligned}
\mathsf{K} &= \text{the minimal modal logic;} \\
\mathsf{S4} &= \mathsf{K} + \{\Box p \to p, \Box p \to \Box\Box p\}; \\
\text{Gödel-Löb Logic} &= \mathsf{K} + \{\Box(\Box p \to p) \to \Box p\}.
\end{aligned}$$

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The research program I will describe already led to the solution of this problem.

Incompleteness with richer languages

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In a complete MA, we can interpret $\forall$ and $\exists$ with meets and joins:

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\begin{aligned}
v(\forall p\varphi) &= \bigwedge\{v'(\varphi) \mid v' \text{ a valuation differing from } v \text{ at most at } p\}.\\
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In a complete BA, we can simply interpret $\Box$ by:

$$v(\Box\varphi) = \begin{cases} 1 & \text{if } v(\varphi) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Theorem (H. 2017)

The set of formulas valid in all complete BAs is axiomatized by the logic S5Π*, which adds to the modal logic S5 the following axioms and rule:*

- $\forall$*-distribution:* $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$.
- $\forall$*-instantiation:* $\forall p\varphi \to \varphi^p_\psi$ *where* ψ *is free for p in* φ;
- *Vacuous-*$\forall$*:* $\varphi \to \forall p\varphi$ *where p is not free in* φ.
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By contrast, if we restrict to atomic cBAs (as in possible world semantics) one obtains additional validities not derivable in S5Π, such as:

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My student Yifeng Ding is pushing further with the program of interpreting propositionally quantified modal logics in complete (not necessarily atomic) MAs.

Chronological staring point

The starting point of my work on this project was L. Humberstone's 1981 paper

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While Humberstone motivated the semantics with philosophical considerations, I'll give a different, mathematical motivation.

Mathematical starting point

Stone and Tarski observed that the **regular opens** of any topological space X, i.e., those opens such that $U = \mathrm{int}(\mathrm{cl}(U))$, form a complete BA with

$$\begin{array}{rcl} \neg U & = & \mathrm{int}(X \setminus U) \\ \bigwedge \{U_i \mid i \in I\} & = & \mathrm{int}(\bigcap \{U_i \mid i \in I\}) \\ \bigvee \{U_i \mid i \in I\} & = & \mathrm{int}(\mathrm{cl}(\bigcup \{U_i \mid i \in I\})). \end{array}$$

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In fact, any complete BA arises (isomorphically) in this way from an Alexandroff space, i.e., as the regular opens in the downset/upset topology of a poset.

The regular open algebra of a poset

In the case of upsets of a poset, the regular opens are the U such that

$$U = \{x \in X \mid \forall y \geqslant x \, \exists z \geqslant y : z \in U\},$$

which is equivalent to:

- persistence: if $x \in U$ and $x \leqslant y$, then $y \in U$, and
- refinability: if $x \notin U$, then $\exists y \geqslant x$: $y \in \neg U$.

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The BA operations are given by:

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As Takeuti and Zaring (*Axiomatic Set Theory*, p. 1) explain:

> One feature [of the theory developed in this book] is that it establishes a relationship between Cohen's method of forcing and Scott-Solovay's method of Boolean valued models. The key to this theory is found in a rather simple correspondence between partial order structures and complete Boolean algebras. . . . With each partial order structure **P**, we associate the complete Boolean algebra of regular open sets determined by the order topology on **P**. With each Boolean algebra **B**, we associate the partial order structure whose universe is that of **B** minus the zero element and whose order is the natural order on **B**.

Mathematical starting point

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algebras		represented by
complete BA	$\xRightarrow{\textit{nonzero elements with restricted reverse order}}$ $\xLeftarrow[\textit{regular opens in upset topology}]{}$	poset

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Possibility semantics for intuitionistic logic generalizes the idea to HAs.

Possibility frames

A (full) possibility frame is a pair (X, R) where X is a poset, R is a binary relation on X, and the operation $\Box_R$ defined by

$$\Box_R U = \{x \in X \mid R(x) \subseteq U\}$$

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Proposition (H. 2015)

The class of possibility frames is definable in the first-order language of R and $\leqslant$.

Proposition (H. 2015)

For any possibility frame (X, R_0), *there is a possibility frame* (X, R) *such that* $\Box_{R_0} = \Box_R$ *and* (X, R) *satisfies:*

- $\boldsymbol{R}$⇔**win**: xRy *iff* $\forall y' \geqslant y\ \exists x' \geqslant x\ \forall x'' \geqslant x'\ \exists y'' \geqslant y'$: $x''Ry''$.

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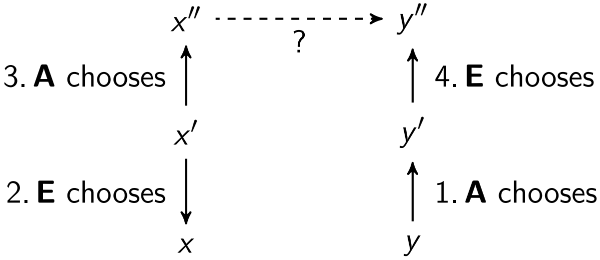
This has a natural game-theoretic interpretation: xRy iff player **E** has a winning strategy in the accessibility game starting from (x, y).



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Possibility semantics for modal logic extends this idea to MAs.

Extending the regular open representation

algebras		represented by
complete MA with completely multiplicative $\Box$	$\Longrightarrow$ *nonzero elements with restricted reverse order and R defined as below* $\Longleftarrow$ *regular opens in upset topology with* $\Box_R$	possibility frame

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Going from a complete and completely multiplicative MA to a possibility frame in this way and then taking the regular opens of that possibility frame with the operation $\Box_R$ gives you back an isomorphic copy of your original MA.

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This is based on an important fact about complete multiplicativity...

Complete multiplicativity

Complete multiplicativity says that $\Box$ distributes over the meet of **any set of elements** that has a meet: $\Box \bigwedge\{a_i \mid i \in I\} = \bigwedge\{\Box a_i \mid i \in I\}$.

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Surprisingly, this ostensibly second-order condition is in fact first-order.

Theorem (H. and Litak 2015)

The operation $\Box$ in an MA is completely multiplicative iff:

$$\textit{if } x \nleq \Box\neg y, \textit{ then } \exists \textit{ nonzero } y' \leq y \textit{ such that } xRy',$$

where xRy' means as before that $\forall$ nonzero $y'' \leq y'$: $x \nleq \Box\neg y''$.

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All of the above could be stated in terms of the complete additivity of $\Diamond$.

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H. Andréka, Z. Gyenis, and I. Németi, who learned of our result above from S. Givant, generalized it to arbitrary posets with completely additive operators.

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Instead of going into the details of this, let's now assume completely multiplicativity and consider atomicity...

Contrasts I: duality without atomicity

Let's contrast Kripke frames and possibility frames.

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Theorem (Thomason 1975)

The category of **complete and atomic** *BAs with a completely multiplicative* $\Box$ *and complete Boolean homomorphisms preserving* $\Box$ *is dually equivalent to the category of Kripke frames and p-morphisms.*

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Theorem (H. 2015)

The category of **complete** *BAs with a completely multiplicative* $\Box$ *and complete Boolean homomorphisms preserving* $\Box$ *is dually equivalent to a reflective subcategory of the category of possibility frames and p-morphisms.*

Contrasts II: Kripke incompleteness

Combining the preceding duality with an incompleteness theorem of Litak 2004, some extra construction (for the "continuum-many" part), and Thomason's simulation of polymodal logics by unimodal logics, we obtain:

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There are continuum-many unimodal logics that are Kripke frame incomplete but possibility frame complete.

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Remark. For non-normal modal logic, we can use "neighborhood possibility frames" to prove consistency of *very simple and philosophically motivated* logics that are not sound with respect any atomic Boolean algebra expansion. E.g., take an S5 $\Diamond$ and a congruential O with the axiom:

$$\Diamond p \to (\Diamond(p \wedge Op) \wedge \Diamond(p \wedge \neg Op)).$$

Similarities I: Sahlqvist correspondence theorem

Theorem (Sahlqvist 1973)

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Theorem (Yamamoto 2016)

Any class of possibility frames defined by a Sahlqvist modal formula is also definable by a formula in the first-order language of R and $\leqslant$.

Further results on correspondence and canonicity have been obtained by Z. Zhao.

Similarities II: Goldblatt-Thomason theorem

Theorem (Goldblatt and Thomason 1975)

If a class F of Kripke frames is closed under elementary equivalence, then F is definable by modal formulas iff F is closed under

- *surjective p-morphisms, generated subframes, and disjoint unions,*

while the complement of F is closed under ultrafilter extensions.

Theorem (H. 2015)

If a class F of possibility frames is closed under elementary equivalence, then F is definable by modal formulas iff F is closed under

- *dense possibility morphisms, selective subframes, and disjoint unions,*

while its complement is closed under filter extensions.

Representation of arbitrary MAs

For the representation of arbitrary MAs, there have been two closely related approaches in the modal logic literature: descriptive frames and modal spaces.

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Let's consider descriptive frames.

General frames

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Conversely, each MA A gives rise to a general frame A_+:

- the set of ultrafilters of A with
- the relation R defined by uRu' iff $\{a \in A \mid \Box a \in u\} \subseteq u'$ and
- the distinguished collection of sets $\widehat{a} = \{u \in \mathit{UltFilt}(A) \mid a \in u\}$ for $a \in A$.

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Then $(A_+)^+$ is isomorphic to A.

Those F for which $(F^+)_+$ is isomorphic to F are the **descriptive frames**, which can be characterized by several nice properties.

General possibility frames

A **general possibility frame** F is a possibility frame plus a distinguished modal subalgebra of the full regular open algebra with $\Box_R$.

Each such F give rise to an MA $F^\star$ via the distinguished subalgebra.

Conversely, each MA A gives rise to a general possibility frame $A_\star$:

- the set of proper filters of A with $\sqsubseteq$ as the inclusion order,
- the relation R defined by uRu' iff $\{a \in A \mid \Box a \in u\} \subseteq u'$, and
- the distinguished collection of sets $\widehat{a} = \{u \in \mathit{PropFilt}(A) \mid a \in u\}$ for $a \in A$.

Then $(A_\star)^\star$ is isomorphic to A.

Those F for which $(F^\star)_\star$ is isomorphic to F are the **filter-descriptive frames**, which can be characterized by several nice properties.

Choice-free duality

Theorem (Goldblatt 1974)

(ZF + Prime Ideal Theorem) The category of Boolean algebras with a multiplicative $\Box$ *and Boolean homomorphisms preserving* $\Box$ *is dually equivalent to the category of "descriptive" general frames with p-morphisms.*

Theorem (H. 2015)

(ZF) The category of Boolean algebras with a multiplicative $\Box$ *and Boolean homomorphisms preserving* $\Box$ *is dually equivalent to the category of "filter-descriptive" general possibility frames with p-morphisms.*

Constructive canonical extension

Following Gehrke and Harding, an MA B is a canonical extension of an MA A iff:

1. B is complete with completely multiplicative $\Box$, and there is a MA-embedding e of A into B;
2. every element of B is a join of meets of e-images of elements of A;
3. for any sets X, Y of elements of A, if $\bigwedge^B e[X] \leq^B \bigvee^B e[X]$, then there are finite $X' \subseteq X$ and $Y' \subseteq Y$ such that $\bigwedge X' \leq \bigvee Y'$.

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Theorem (Jonsson and Tarski 1951)

(ZF + Prime Ideal Theorem) For any modal algebra A, the powerset algebra of A_+ with $\Box_R$ is a canonical extension of A.

Theorem

(ZF) For any modal algebra A, the full regular open algebra of $A_\star$ with $\Box_R$ is a canonical extension of A.

Modal spaces

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Rather than discussing modal spaces, let's just focus on the Boolean part: BAs represented by Stone spaces.

Stone spaces and spectral spaces

A space X is a **Stone space** if X is a zero-dimensional compact Hausdorff space.

A space X is a **spectral space** if X is compact, T_0, coherent (the compact open sets of X are closed under intersection and form a base for the topology of X), and sober (every completely prime filter in $\Omega(X)$ is $\Omega(x)$ for some $x \in X$).

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Theorem (Stone 1936)

(ZF + PIT) Any BA A is isomorphic to the BA of clopens of a Stone space: $\mathit{UltFilt}(A)$ with the topology generated by basic opens $\hat{a} = \{u \in \mathit{UltFilt}(A) \mid a \in u\}$ for $a \in A$.

Theorem (Stone 1938)

(ZF + PIT) Any DL L is isomorphic to the DL of compact opens of a spectral space: $\mathit{PrimeFilt}(L)$ with the topology generated by basic opens $\hat{a} = \{u \in \mathit{PrimeFilt}(A) \mid a \in u\}$ for $a \in L$.

"Choice-free Stone duality"

Theorem (N. Bezhanishvili and H. 2016)

(ZF) Any BA A is isomorphic to the BA of compact open regular open *sets* (*with operations defined as in the regular open algebra*) *of a* **UV-space** (*see below*)*:* $PropFilt(A)$ *with the topology generated by basic opens* $\widehat{a} = \{u \in PropFilt(A) \mid a \in u\}$ *for* $a \in A$*.*

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A **UV-space** is a T_0 space X satisfying the following (implying X is spectral):

1. $\mathsf{CO}\mathcal{RO}(X)$ is closed under $\cap$ and regular open negation;
2. $x \not\leqslant y \Rightarrow$ there is a $U \in \mathsf{CO}\mathcal{RO}(X)$ s.t. $x \in U$ and $y \notin U$;
3. every proper filter in $\mathsf{CO}\mathcal{RO}(X)$ is $\mathsf{CO}\mathcal{RO}(x)$ for some $x \in X$.

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They are so named because they also arise as the hyperspace of nonempty closed sets of a Stone space Y endowed with the **upper Vietoris** topology, generated by basic opens $[U] = \{F \in \mathrm{F}(Y) \mid F \subseteq U\}$ for $U \in \mathrm{Clop}(Y)$.

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A **UV-map** is a spectral map that is also a p-morphism with respect to the specialization orders.

Theorem (N. Bezhanishvili and H. 2016)

(ZF) The category of BAs with BA homomorphisms is dually equivalent to the category of UV-spaces with UV-maps.

Generalizing to HAs

So far everything has been based on the BA of regular open subsets of a poset.

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This can be seen as a special case of something more general. . .

Nuclei

Regular that a regular open set is a fixpoint of the operation $\mathrm{int}(\mathrm{cl}(\cdot))$ on the open sets of a space. Thinking in terms of the cHA of open sets, this is the operation $\neg\neg$ of double negation (and the fact that the fixpoints of double negation form a BA gives an algebraic proof of Glivenko's theorem).

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The operation $\neg\neg$ is an example of a *nucleus* on an HA.

A **nucleus** on an HA H is a function $j : H \to H$ satisfying:

1. $a \leq ja$ (inflationarity);
2. $jja \leq ja$ (idempotence);
3. $j(a \wedge b) = ja \wedge jb$ (multiplicativity).

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For any HA H and nucleus j on H, let $H_j = \{a \in H \mid ja = a\}$.

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- $a \rightarrow_j b = a \rightarrow b$;
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If H is a complete, so is H_j, where $\bigwedge_j S = \bigwedge S$ and $\bigvee_j S = j(\bigvee S)$.

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In the case $j = \neg\neg$, we have that H_j is a BA.

Representing cHAs as fixpoints of a nucleus on upsets

Dragalin showed that every cHA can be represented using a triple $(S, \leqslant, j)$ where $(S, \leqslant)$ is a poset and j is a nucleus on $\mathrm{Up}(S, \leqslant)$.

Theorem (Dragalin 1981)

Every cHA is isomorphic to the algebra of fixpoints of a nucleus on the upsets of a poset.

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But we would like to replace the operation j with something more concrete...

Intuitionistic possibility frames

An intuitionistic possibility frame is a triple $(S, \leqslant_1, \leqslant_2)$ where $\leqslant_1$ and $\leqslant_2$ are preorders on S such that $\leqslant_2$ is a subrelation of $\leqslant_1$.[1]

[1] In Bezhanishvili and H. 2016, these are *normal FM-frames* after Fairtlough and Mendler 1997.

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Proposition (Fairtlough and Mendler 1997)

For any such $(S, \leqslant_1, \leqslant_2)$*, the operation* $\Box_1\Diamond_2$ *given by*

$$\Box_1\Diamond_2 U = \{x \in S \mid \forall y \geqslant_1 x \, \exists z \geqslant_2 y : z \in U\}$$

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This approach is related to **Urquhart**'s representation of lattices using doubly-ordered sets—see "Representations of complete lattices and the Funayama embedding" by Bezhanishvili, Gabelaia, H., and Jibladze.

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Guillaume Massas (now at UC Berkeley) gave a different proof in his ILLC thesis.

Intuitionistic possibility frames

Recall that a Kripke frame (poset) $(S, \leqslant)$ can represent only the very special J^{∞}-generated complete Heyting algebras via their algebras $\mathrm{Up}(S, \leqslant)$ of upsets.

By contrast, intuitionistic possibility frames $(S, \leqslant_1, \leqslant_2)$ can be used to represent ALL complete Heyting algebras.

Theorem (Bezhanishvili and H. 2016)

Every complete Heyting algebra is isomorphic to the algebra of $\Box_1\Diamond_2$-fixpoints of some intuitionistic possibility frame.

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Both involve essentially the following construction from a cHA H:

$$S = \{\langle a, b\rangle \in H^2 \mid a \nleqslant b\}$$

$$\langle a, b\rangle \leqslant_1 \langle c, d\rangle \Leftrightarrow a \geqslant c, \quad \langle a, b\rangle \leqslant_2 \langle c, d\rangle \Leftrightarrow a \geqslant c \,\&\, b \leqslant d.$$

Back to Kuznetsov's problem

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Example: one way to solve Kuznetsov's problem (is every si-logic the logic of some class of topological spaces?) in the negative would be to show that there are si-logics that are not the logic of any class of intuitionistic possibility frames.

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To be continued. . .

What about choice-free representation of all HAs?

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To be continued. . .

Alternative theory

algebras		represented by
BA	$\Rightarrow$ *proper filters with* $\widehat{a}$*-generated topology* $\Leftarrow$ *compact open regular opens*	UV-space
complete BA	$\Rightarrow$ *nonzero elements with restricted reverse order* $\Leftarrow$ *regular open upsets*	poset
complete HA	$\Rightarrow$ $\{\langle a,b\rangle \in H^2 \mid a \nleqslant b\}$, $\langle a,b\rangle \leqslant_1 \langle c,d\rangle \Leftrightarrow a \geqslant c$ $\langle a,b\rangle \leqslant_2 \langle c,d\rangle \Leftrightarrow a \geqslant c \;\&\; b \leqslant d$ $\Leftarrow$ $\Box_1 \Diamond_2$*-fixpoints*	intuitionistic possibility frame
MA	$\Rightarrow$ *as in BA case with* uRu' *iff* $\{a \mid \Box a \in u\} \subseteq u'$ $\Leftarrow$ *as in BA case with* $\Box_R U = \{x \mid R(x) \subseteq U\}$	modal UV-space
complete MA with completely multiplicative $\Box$	$\Rightarrow$ *as in cBA case with* aRb *iff* $\forall$ *nonzero* $b' \leqslant b$: $a \nleqslant \Box \neg b'$ $\Leftarrow$ *as in cBA case with* $\Box_R$ *as above*	possibility frame