$\rightarrow$, $\multimap$ and $-\!*$

Tadeusz Litak (FAU Erlangen-Nuremberg)
Based mostly on a joint work with Albert Visser (Utrecht)

SYSMiCS, Orange, September 2018

Thanks and apologies to:

- the organizers,
- Wesley Fussner and
- the audience

This talk

- Basically an advertisement for
  Tadeusz Litak and Albert Visser, ***Lewis meets Brouwer: constructive strict implication***,
  Indagationes Mathematicae, A special issue "L.E.J. Brouwer, fifty years later", February 2018
  `https://arxiv.org/abs/1708.02143`
- … and some of our ongoing work
- … but also for Peter Jipsen and Tadeusz Litak, ***An algebraic glimpse at bunched implications and separation logic***, In: **Hiroakira Ono on Residuated Lattices and Substructural Logics**, Outstanding Contributions to Logic. To appear.
  JUST MAYBE DO NOT READ THIS PAPER YET

- The title might be slightly misleading
- What I have in mind is:
- comparing two very natural ways of extending IPC with another implication connective
- Also, one of them will get more attention than the other

- The obvious way (for this community …) of adding another implication to HAs…
- …via residuation/adjointness!
- Given a (commutative) monoid $(*, 1)$ on a complete HA distributing over $\bigvee$ …
- …we produce implication $-\!*$ of BI …
- … or $\backslash$, $/$ … of GBI in the noncommutative case

Motivation and applications

- Reasoning about shared mutable data structures

  mostly pointers/heap/allocation, but in last 10 yrs big on concurrency
- Generalizations of relation algebras

  weakening relations
- Complex algebras of (ordered, partial . . .) monoids, separation algebras, (generalized) effect algebras
- Ambient logic, trees and semistructured data
- Costs, logic programming and Petri nets
- (Regular) language models
- See our overview with Peter for more

  But perhaps wait a few days, please
- In the commutative setting, most of these models already described by Pym, O'Hearn, Yang (TCS'04)

- as it turns out, however, there are several convergent motivations for a very different beast ...
- intuitionistic strict implication $\strictif$!

- As we all know (or do we?) the following is the original syntax of modern modal logic :

$$\mathcal{L}_{\prec} \quad \varphi, \psi ::= \top \mid \bot \mid p \mid \varphi \to \psi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \prec \psi$$

- $\prec$ is the strict implication of Clarence Irving Lewis (1918,1932)

  who is not C.S. Lewis, David Lewis or Lewis Carroll
- $\Box\varphi$ is then definable as $\top \prec \varphi$
- Over CPC, the converse holds too ...
- ...i.e., $\varphi \prec \psi$ is $\Box(\varphi \to \psi)$, i.e., $\top \prec (\varphi \to \psi)$
- Truth of strict implication at w = truth of material implication in all possible worlds seen from w

- Lewis indeed wanted to have classical (involutive) negation
- In fact, he introduced $\strictif$ as defined using $\Diamond$

  somehow did not explicitly work with $\Box$ in the signature
- But perhaps this is why $\strictif$ slid into irrelevance …
- …which did not seem to make him happy
- He didn't even like the name "modal logic" …

There is a logic restricted to indicatives; the truth-value logic most impressively developed in "Principia Mathematica". But those who adhere to it usually have thought of it—so far as they understood what they were doing—as being the universal logic of propositions which is independent of mode. And when that universal logic was first formulated in exact terms, they failed to recognize it as the only logic which is independent of the mode in which propositions are entertained and dubbed it "modal logic".

- Curiously, Lewis seemed sympathetic towards non-classical systems (mostly the Łukasiewicz logic)
  - A detailed discussion in *Symbolic Logic*, 1932
  - A paper on "Alternative Systems of Logic", *The Monist*, same year
  - Both references analyze possible definitions of "truth-implications"/"implication-relations" available in finite, but not necessarily binary matrices.
- I found just one reference where he mentions (rather favourably) Brouwer and intuitionism …

[T]he mathematical logician Brouwer has maintained that the law of the Excluded Middle is not a valid principle at all. The issues of so difficult a question could not be discussed here; but let us suggest a point of view at least something like his. . . . The law of the Excluded Middle is not writ in the heavens: it but reflects our rather stubborn adherence to the simplest of all possible modes of division, and our predominant interest in concrete objects as opposed to abstract concepts. The reasons for the choice of our logical categories are not themselves reasons of logic any more than the reasons for choosing Cartesian, as against polar or Gaussian coördinates, are themselves principles of mathematics, or the reason for the radix 10 is of the essence of number.

"Alternative Systems of Logic", *The Monist*, 1932

- No indication he was aware of Kolmogorov, Heyting, Glivenko ...
- Maybe he should've followed up on that ...
- ...especially that there were more analogies between him and Brouwer
  - almost perfectly parallel life dates
  - wrote his 1910 PhD on *The Place of Intuition in Knowledge*
  - a solid background in/influence of idealism and Kant ...
- Anyway, there are several other ways in which one arrives at intuitionistic $\dashv\!3$

New incarnations of intuitionistic $\dashv\!\!3$

- Metatheory of arithmetic
  Σ^0_1-preservativity for a theory T extending HA:

  $$A \dashv\!\!3_T B \Leftrightarrow \forall \Sigma^0_1\text{-sentences } S\ (\ T \vdash S \to A \Rightarrow T \vdash S \to B)$$

  Albert working on this since 1985, later more contributions made also by Iemhoff, de Jongh, Zhou ...
- Functional programming
  Distinction between arrows of John Hughes and applicative functors/idioms of McBride/Patterson

  A series of papers by Lindley, Wadler, Yallop
- Proof theory of guarded (co)recursion

  Nakano and more recently Clouston&Goré
- Analysis of Kripke semantics

  generalizing defining conditions of profunctors/weakening relations

Idioms are oblivious, arrows are meticulous, monads are promiscuous

Sam Lindley, Philip Wadler and Jeremy Yallop

Laboratory for Foundations of Computer Science
The University of Edinburgh

Abstract

We revisit the connection between three notions of computation: Moggi's *monads*, Hughes's *arrows* and McBride and Paterson's *idioms* (also called *applicative functors*). We show that idioms are equivalent to arrows that satisfy the type isomorphism $A \leadsto B \simeq 1 \leadsto (A \to B)$ and that monads are equivalent to arrows that satisfy the type isomorphism $A \leadsto B \simeq A \to (1 \leadsto B)$. Further, idioms embed into arrows and arrows embed into monads.

Keywords: applicative functors, idioms, arrows, monads

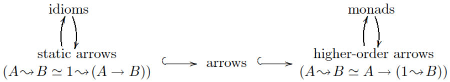


Fig. 1. Idioms, arrows and monads

$\leadsto$ here is our $\exists$

ENTCS 2011, proceedings of MSFP 2008

- Each of these could consume most of the talk …
- …and would interest only a section of the audience
- The body of the work in the metatheory of intuitionistic arithmetic is particularly spectacular …
- …and way too little known
- I can only give you a teaser
- …and Kripke semantics is ideal for this
- You need to read our paper with Albert for more

Kripke semantics for intuitionistic $\Box$:

- Nonempty set of worlds
- Two relations:
  - Intuitionistic partial order relation $\preceq$, drawn as $\rightarrow$;
  - Modal relation $\sqsubset$, drawn as $\rightsquigarrow$.
- Semantics for $\Box$: $w \Vdash \Box\varphi$ if for any $v \sqsupset w$, $v \Vdash \varphi$
- Semantics for $\strictif$:

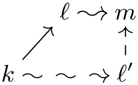
$$w \Vdash \varphi \strictif \psi \text{ if for any } v \sqsupset w, v \Vdash \varphi \text{ implies } v \Vdash \psi$$

- What is the minimal condition to guarantee persistence?
- That is, given A, B upward closed, is

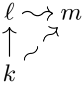
$$A \multimap B = \{w \mid \text{ for any } v \sqsupset w, v \in A \text{ implies } v \in B\}$$

upward closed?
- Is it it stronger than the one ensuring persistence for $\Box A$?

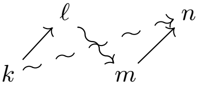
Four frame conditions (known since 1980's)



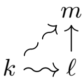
□-p
(persistence for □)



prefixing
(persistence for $\strictif$)



mix / brilliancy
profunctors/weakening rels.



postfixing

$\Longleftarrow$ both equivalent in presence of □-p , collapsing $\strictif$ to □

- brilliancy obtains naturally in, e.g., Stone-Jónsson-Tarski for □
- . . . but $\strictif$ can feel it! $\Longrightarrow$ collapse of $\strictif$ to □
- Over prefixing (or $\strictif$-frames) $\Box(\varphi \to \psi)$ implies $\varphi \strictif \psi$, but not the other way around

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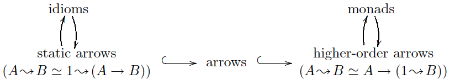


Fig. 1. Idioms, arrows and monads

$\leadsto$ here is our ∃

ENTCS 2011, proceedings of MSFP 2008

Many programs and libraries involve components that are "function-like", in that they take inputs and produce outputs, but are not simple functions from inputs to outputs ... [S]uch "notions of computation" defin[e] a common interface, called "arrows". ... Monads ... serve a similar purpose, but arrows are more general. In particular, they include notions of computation with static components, independent of the input, as well as computations that consume multiple inputs.

Ross Paterson

- I'd suggest calling FP arrows strong arrows
- They satisfy in addition the axiom $(\varphi \to \psi) \to \varphi \mathrel{-\!\!3} \psi$
- ...or, equivalently, $\mathsf{S_a}$ $\quad \varphi \to \Box\varphi$
- Why "equivalently"?

  After all, many $\Box$-principles not equivalent to $\mathrel{-\!\!3}$-counterparts

$$\begin{aligned} \varphi \to \psi &\leq \Box(\varphi \to \psi) \\ &\leq \varphi \mathrel{-\!\!3} \psi \end{aligned}$$

- This forces $\sqsubset$ to be contained in $\preceq$
- ...rather degenerate in the classical case ...

  only three consistent logics of (disjoint unions of) singleton(s)
- ...and yet intuitionistically you have a whole CS zoo

  type inhabitation of idioms, arrows, strong monads/PLL ...

  plus superintuitionistic logics as a degenerate case

  also recent attempts at "intuitionistic epistemic logics", esp. Artemov and Protopopescu, ignoring all references I've mentioned

Axioms and rules of iA^-:

Those of IPC plus:

$$\mathsf{Tra} \quad (\varphi \dashv\!\!3\, \psi) \wedge (\psi \dashv\!\!3\, \chi) \to \varphi \dashv\!\!3\, \chi$$

$$\mathsf{K_a} \quad (\varphi \dashv\!\!3\, \psi) \wedge (\varphi \dashv\!\!3\, \chi) \to \varphi \dashv\!\!3\, (\psi \wedge \chi)$$

$$\mathsf{N_a} \ \frac{\varphi \to \psi}{\varphi \dashv\!\!3\, \psi.}$$

Axioms and rules of the full minimal system iA:

All the axioms and rules of IPC and iA^- and

$$\mathsf{Di} \quad (\varphi \dashv\!\!3\, \chi) \wedge (\psi \dashv\!\!3\, \chi) \to (\varphi \vee \psi) \dashv\!\!3\, \chi.$$

Why Di can be problematic?

- Arithmetic: valid for HA, but not PA!

  As we will see, this is what makes interpretability nontrivial

- Functional programming: references above do not even study interaction with coproducts

$\Box$ and $\mathrel{-\!\!3}$ variants

$$\varphi \mathrel{-\!\!3} \psi \to (\varphi \to \psi) \lor \varphi \quad \Longrightarrow \quad \Box(\psi \to \varphi) \to (\psi \to \varphi) \lor \psi$$

$$\varphi \mathrel{-\!\!3} \Box\varphi \quad \Longrightarrow \quad \Box\varphi \to \Box\Box\varphi$$

$$\Box\varphi \mathrel{-\!\!3} \varphi \quad \Longrightarrow \quad \Box\Box\varphi \to \Box\varphi$$

$$(\Box\varphi \to \varphi) \mathrel{-\!\!3} \varphi \quad \Longrightarrow \quad \Box(\Box\varphi \to \varphi) \to \Box\varphi$$

$$(\varphi \mathrel{-\!\!3} \psi) \lor (\psi \mathrel{-\!\!3} \varphi) \quad \Longleftarrow \quad \Box(\varphi \to \psi) \lor \Box(\psi \to \varphi)$$

None of these implications can be reversed

Derivation exercises

Lots to be found in our paper, e.g., a generalization of $\mathsf{K_a}$:

$$\begin{aligned} \varphi \multimap (\psi \to \chi) &\vdash (\varphi \wedge \psi) \multimap (\psi \wedge (\psi \to \chi)) && \text{by } \mathsf{N_a} \text{ and } \mathsf{K_a} \\ &\vdash (\varphi \wedge \psi) \multimap \chi && \text{by monotonicity of } \multimap \end{aligned}$$

Another curious one:

$$\begin{aligned} \psi \multimap \chi &\vdash_{\mathsf{iA}} \psi \multimap (\psi \to \chi) \wedge \neg\psi \multimap (\psi \to \chi) && \text{by } \mathsf{Tra} \text{ and } \mathsf{N_a} \\ &\vdash_{\mathsf{iA}} (\psi \vee \neg\psi) \multimap (\psi \to \chi) && \text{by } \mathsf{Di} \end{aligned}$$

We thus get

$$\psi \multimap \chi \dashv\vdash_{\mathsf{iA}} (\psi \vee \neg\psi) \multimap (\psi \to \chi)$$

- The validity of

$$(p \multimap q) \leftrightarrow \ (p \vee \neg p) \multimap (p \to q)$$

implies that Col, i.e., $\Box$-collapse

$$(p \multimap q) \leftrightarrow \ \Box(p \to q)$$

is valid over classical logic
- Note no other classical tautology in one variable would do!

$$p \multimap q \nvdash (\neg\neg p \to p) \multimap (p \to q)$$

- Completeness results for many such systems published by Iemhoff et al

  Her 2001 PhD, 2003 MLQ, 2005 SL with de Jongh and Zhou

  Also Zhou's ILLC MSc in 2003
- In our paper, we announce more such completeness and correspondence results

  based on on a suitable extension of Gödel-McKinsey-Tarski and

  Wolter-Zakharyaschev for ordinary intuitionistic modal logics

  Details to be published separately
- As we have Alexandra & co. here, a few words on this
- But we better discuss some algebraic aspects first

  Absent in the paper with Albert
- And this will be a good excuse to return to comparison with (G)BI

In the absence of $\strictif$

- Weak logics with strict implication and weak Heyting algebras

  Corsi 1987, Došen 1993, Celani and Jansana 2001, 2005

  Related system: Visser 1981, Epstein and Horn 1976

- Classical semantics, i.e., $\preceq$ discrete ...

  but in the absence of $\to$, how much of a difference?

- Problematic even from the point of view of algebraizability
- Došen proposed a Hilbert-style system, but it does not capture local consequence ...
- ... and the deductive systems capturing either relation in $\mathcal{L}w_{\strictif}$ are not even protoalgebraic

- But of course things get fixed if both implications are combined
- iA has proper algebraic semantics:

Definition

A (constructive) Lewis's algebra or iA-algebra:

$$\mathfrak{A} := (A, \wedge, \vee, \dashv, \to, \bot, \top),$$

where

- $(A, \wedge, \vee, \to, \bot, \top)$ is a Heyting algebra and
- $(A, \wedge, \vee, \dashv, \bot, \top)$ is a weakly Heyting algebra (Celani, Jansana), i.e.,

C1 $a \dashv b \wedge a \dashv c = a \dashv (b \wedge c)$,
C2 $a \dashv c \wedge b \dashv c = (a \vee b) \dashv c$,
C3 $a \dashv b \wedge b \dashv c \leq a \dashv c$,
C4 $a \dashv a = \top$.

- An interesting exercise, which I guess should be automatic:
- Could one obtain iA by fibring or dovetailing IPC with the minimal weak logic with strict implication?
- This brings us back to BI with its $*$ and $-\!*$

  Fibring/dovetailing $\mathsf{FL_e}$ with IPC, cf. Gabbay's intro to Pym's book
- How closely iA and BI are related?
- As it turns out, not quite, unless you want to add some powerful axioms

Axiomatization(s) of (G)BI

- A Hilbert-style system for GBI: extend IPC with

$$(\varphi \cdot \psi) \cdot \chi \leftrightarrow \varphi \cdot (\psi \cdot \chi) \qquad 1 \cdot \varphi \leftrightarrow \varphi \qquad \varphi \cdot 1 \leftrightarrow \varphi$$

  and the bidirectional residuation rules $\dfrac{\varphi \cdot \psi \to \chi}{\psi \to \varphi \backslash \chi}$ $\quad$ $\dfrac{\varphi \cdot \psi \to \chi}{\varphi \to \chi / \psi}$.

- A Hilbert system for BI obtained by adding $\varphi * \psi \to \psi * \varphi$, omitting $/$, and replacing $\cdot$ and $\backslash$ by $*$ and $-\!*$, respectively.
  Similar one in Pym's monograph

- Our overview proposes an equivalent Hilbert-style axiomatization of BI over IPC:

$$(\varphi * \psi) * \chi \leftrightarrow \varphi * (\psi * \chi) \qquad \varphi * \psi \to \psi * \varphi \qquad \varphi * 1 \leftrightarrow \varphi \qquad \varphi * (\varphi -\!* \psi) \to \psi$$

$$\varphi -\!* (\psi -\!* \chi) \leftrightarrow \varphi * \psi -\!* \chi \qquad \frac{\varphi \to \psi}{\varphi * \chi \to \psi * \chi} \qquad \frac{\varphi \to \psi}{1 \to \varphi -\!* \psi}.$$

Going in different directions

- $(\top \mathrel{-\!\!*} \varphi) \to (\top \mathrel{-\!\!*} (\top \mathrel{-\!\!*} \varphi))$ holds in BI
- Its $\mathrel{-\!\!\!\!\exists}$-counterpart is $\mathsf{4}$: $\Box\varphi \to \Box\Box\varphi$ $\Longrightarrow$ not universally valid in $\mathrel{-\!\!\!\!\exists}$-frames
- $\mathsf{4}$ of course valid whenever S is
- But then again the $\mathrel{-\!\!*}$-counterpart of S, i.e., $(\varphi \to \psi) \to \varphi \mathrel{-\!\!*} \psi$ is not a theorem of BI
- In fact, its addition would yield weakening ...
- ...just like closing under the $\mathrel{-\!\!*}$-variant of $\mathsf{N_a}$
- On the other hand, there are theorems of BI such as $\varphi \mathrel{-\!\!*} (\psi \mathrel{-\!\!*} \chi) \to \psi \mathrel{-\!\!*} (\varphi \mathrel{-\!\!*} \chi)$ whose $\mathrel{-\!\!\!\!\exists}$-counterpart is not valid even in S ...

Systematic completeness/correspondence results ...

- ... by reducing to a classical (multi-)modal language?
- For $\mathcal{L}_\Box$, methodology developed by Wolter & Zakharyashev in the late 1990's
- Correspondence language:

$$\mathcal{L}i_{[\mathsf{i}\,\mathsf{m}]} \quad \varphi, \psi ::= \top \mid \bot \mid p \mid \varphi \to \psi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid [\mathsf{i}]\varphi \mid [\mathsf{m}]\varphi$$

- Gödel-(McKinsey-Tarski) translation for $\mathcal{L}_\Box$:

$$t_\Box(\Box\varphi) := [\mathsf{i}][\mathsf{m}](t\varphi)$$

and $[\mathsf{i}]$ in front of every subformula

- $t_\Box$ embeds faithfully every $\mathsf{i}_\Box$-logic into a whole cluster of extensions of BM …
- …the latter being the logic with $\mathsf{S4}$ axioms for $[\,\mathsf{i}\,]$
- Each such cluster has a maximal element, obtained with the help of the Grzegorczyk axiom and

$$\mathsf{mix} \qquad [\mathsf{m}]\varphi \to [\,\mathsf{i}\,][\mathsf{m}][\,\mathsf{i}\,]\varphi$$

- The translation reflects decidability, completeness, fmp. Above mix, it also reflects canonicity
- …enough to find one $\mathcal{L}i_{[\mathsf{i}\,\mathsf{m}]}$-counterpart with the desired property!
- …can use the Sahlqvist algorithm, SQEMA …

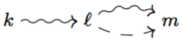
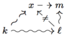
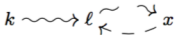
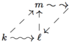
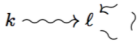
- Extending the Gödel-(McKinsey-Tarski) translation to $\mathcal{L}_{\multimap}$

$$t(\varphi \multimap \psi) := [\mathsf{i}][\mathsf{m}](t\varphi \to t\psi)$$

- The only change one seems to need in the preceding slide is (obviously) replacing mix with

$$\mathsf{lewis} \quad [\mathsf{m}]\varphi \to [\mathsf{i}][\mathsf{m}]\varphi$$

- Apart from this, everything seems to work

Box	brilliant	$k \sqsubset \ell \preceq m \Rightarrow k \sqsubset m$	
$4_{\Box}$	semi-transitive	$k \sqsubset \ell \sqsubset m \Rightarrow$ $\exists x. k \sqsubset x \preceq m$	
4_{a}	gathering	$k \sqsubset \ell \sqsubset m \Rightarrow \ell \preceq m$	
$L_{\Box}$		$\sqsubset$-Noetherian (conversely well-founded) and semi-transitive	
W_{a}	supergathering	on finite frames: $k \sqsubset \ell \sqsubset m \Rightarrow$ $\exists x \sqsupset k.(\ell \prec x \preceq m)$	
M_{a}	Montagna	$k \sqsubset \ell \preceq m \Rightarrow \exists x \sqsupset k.(\ell \preceq x \preceq m \;\&\; x{\uparrow_{\sqsubset\preceq}} \subseteq m{\uparrow_{\sqsubset\preceq}})$	
$S_{\Box}$	strong	$k \sqsubset \ell \Rightarrow k \preceq \ell$	
CB_{a}	$\sqsubset$-dominated	$k \prec \ell \Rightarrow k \sqsubset \ell$	
$CB_{\Box}$	weakly $\sqsubset$-dominated	$k \prec \ell \Rightarrow \exists m \sqsupset k. m \preceq \ell$	
Lin_{a}	weakly semi-linear	$k \sqsubset \ell \;\&\; k \sqsubset m \Rightarrow (m \preceq \ell \text{ OR } \ell \preceq m)$	
$Lin_{\Box}$	strongly semi-linear	$k \sqsubset \ell \preceq \ell' \;\&\; k \sqsubset m \preceq m' \Rightarrow (m' \preceq \ell' \text{ OR } \ell' \preceq m')$	
$C4_{\Box}$	semi-dense	$k \sqsubset \ell \Rightarrow \exists x \preceq \ell. \exists y \sqsupset k. y \sqsubset x$	
$C4_{\mathsf{a}}$	pre-reflexive	$k \sqsubset \ell \Rightarrow \exists x \sqsupset \ell. x \preceq \ell$	
Hug	semi-nucleic	$k \sqsubset \ell \Rightarrow \exists m \succeq k.$ $\exists m' \sqsupset m.\ \ell \preceq m.m' \preceq \ell$	
App_{a}	almost reflexive	$k \sqsubset \ell \Rightarrow \ell \sqsubset \ell$	

More on intuitionistic vs. classical

	int $\Longrightarrow$ cl	cl $\Longrightarrow$ imp
bunched	GMT: Ishtiaq & O'Hearn, POPL'01 Larchey-Wendling & Galmiche MSCS'09	no recursive translation (undecidability)
modal	GMT: see preceding slides	our FSCD'17 paper ($\neg\neg$-translations, $\Box$ only)

- Finally, a few words on preservativity
- Let us first recall the simpler idea of provability …
- …or even more generally, that of arithmetical interpretation of a propositional logic

- Extend $\mathcal{L}$ to $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_k}$. with operators $\circledcirc_0, \ldots, \circledcirc_k$
  where $\circledcirc_i$ has arity n_i
- F assigns to every $\circledcirc_i$ an arithmetical formula $A(v_0, \ldots, v_{n_i-1})$
  where all free variables are among the variables shown
- We write $\circledcirc_{i,F}(B_0, \ldots, B_{n_i-1})$ for $F(\circledcirc_i)(\underline{\ulcorner B_0 \urcorner}, \ldots, \underline{\ulcorner B_{n_i-1} \urcorner})$
  Here $\underline{\ulcorner C \urcorner}$ is the numeral of the Gödel number of C
- f maps $Vars$ to arithmetical sentences. Define $(\varphi)^f_F$:
  - $(p)^f_F := f(p)$
  - $(\cdot)^f_F$ commutes with the propositional connectives
  - $(\circledcirc_i(\varphi_0, \ldots, \varphi_{n_i-1}))^f_F := \circledcirc_F((\varphi_0)^f_F, \ldots, (\varphi_{n_i-1})^f_F)$

- Let T be an arithmetical theory

  An extension of i-EA, the intuitionistic version of Elementary Arithmetic, in the arithmetical language

- A modal formula in $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_k}$ is T-valid w.r.t. F iff, for all assignments f of arithmetical sentences to $Vars$, we have $T \vdash (\varphi)^f_F$.
- Write $\Lambda_{T,F}$ for the set of $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_k}$-formulas that are T-valid w.r.t. F.
- Of course, $\Lambda_{T,F}$ interesting only for well-chosen F

- First, consider a single unary $\circledcirc = \Box$ and any arithmetical theory T ...
- ... which comes equipped with a $\Delta_0(\mathsf{exp})$-predicate α_T encoding its axiom set.
- Let provability in T be arithmetised by prov_T.
- Set $\mathsf{F}_{0,T}(\Box) := \mathsf{prov}_\mathsf{T}(v_0)$. Let $\Lambda^*_T := \Lambda_{T,\mathsf{F}_{0,T}}$.
- Intuitionistic Löb's logic i-GL is given by the following axioms over IPC.

  N $\vdash \varphi \;\Rightarrow \vdash \Box\varphi$
  K $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  L $\vdash \Box(\Box\varphi \to \varphi) \to \Box\varphi$

The theory GL is obtained by extending i-GL with classical logic

If T is a Σ^1_0-sound classical theory, then $\Lambda^*_T = \mathsf{GL}$ (Solovay)

In contrast, the logic i-GL is not complete for HA:

- $\vdash \Box\neg\neg\Box\varphi \to \Box\Box\varphi$.
- $\vdash \Box(\neg\neg\Box\varphi \to \Box\varphi) \to \Box\Box\varphi$
- $\vdash \Box(\varphi \vee \psi) \to \Box(\varphi \vee \Box\psi)$.

Still unknown what the ultimate axiomatization is

- Many possible interpretations of a binary connective

  not all of them producing Lewis' arrows!
- Interpretability
- Π^0_1-conservativity
- Σ^0_1-preservativity

  classically, the last two intertranslatable, like $\Box$ and $\Diamond$

- The notion of Σ^0_1-preservativity for a theory T (Visser 1985) is defined as follows:
- $A \dashv_T B$ iff, for all Σ^0_1-sentences S, if $T \vdash S \to A$, then $T \vdash S \to B$
- Alternatively:

Theorem

$A \dashv_T B$ iff, for all n, $T \vdash \Box_{T,n} A \to B$

- This does yields Lewis' arrow …
- …with interesting additional axioms

Examples of valid principles

4_a $\vdash \varphi \mathrel{-\!\!3} \Box\varphi$

L_a $(\Box\varphi \to \varphi) \mathrel{-\!\!3} \varphi$

W_a $(\varphi \land \Box\psi) \mathrel{-\!\!3} \psi \to \varphi \mathrel{-\!\!3} \psi$

W'_a $\varphi \mathrel{-\!\!3} \psi \to (\Box\psi \to \varphi) \mathrel{-\!\!3} \psi$

M_a $\varphi \mathrel{-\!\!3} \psi \to (\Box\chi \to \varphi) \mathrel{-\!\!3} (\Box\chi \to \psi)$

M'_a $(\varphi \land \Box\chi) \mathrel{-\!\!3} \psi \to \varphi \mathrel{-\!\!3} (\Box\chi \to \psi)$

- Still no ultimate axiomatization... but perhaps better candidates and better insights than for $\Box$ only, see our paper

Additional axioms in well-behaved/pathological theories
E.g., in presence of The Completeness Principle for a theory T:

S_a $(\varphi \to \psi) \to \varphi \mathrel{-\!\!3} \psi$, i.e., S'_a: $\varphi \mathrel{-\!\!3} \psi \to \varphi \to \Box\psi$

- Our present work includes computation of fixpoints of modalized formulas
- (below S, more interesting than in the presence of $\Box$ only!)
- ...encoding of fixpoints of positive formulas and retraction of μ-calculus

Take-home message

- Go intuitionistic if you can
- Study fusions/fibrings/combinations of logics
- Don't worry if your signature grows bigger as a result
- CS can suggest a lot of models to play with ...
- ... but so can arithmetic
- Interplay of "intrinsic/extrinsic" (or is it "implicit/explicit"?) perspectives still not fully explored