

De Vries Duality for Compactifications and Completely Regular Spaces

Guram Bezhanishvili Patrick Morandi Bruce Olberding

New Mexico State University, Las Cruces, New Mexico, USA

SYSMICS Workshop on Duality in Algebra and Logic
14 September 2018

There are several dualities between the category **KHaus** of compact Hausdorff spaces and continuous maps and a category of algebraic structures:

- Gelfand-Neumark duality with (complex) C^* -algebras,
- Stone's duality with real C^* -algebras,
- Kakutani-Yosida duality involving vector lattices,
- Isbell duality with compact regular frames.

We will focus on de Vries duality, which focuses on regular open sets.

De Vries Duality

Let X be a compact Hausdorff space. An open set U is **regular open** if it is the interior of its closure. The set $\mathcal{RO}(X)$ of regular open subsets of X forms a complete Boolean algebra via

- $\bigvee_i U_i = \text{int}(\text{cl}(\bigcup_i U_i))$,
- $\bigwedge_i U_i = \text{int}(\bigcap_i U_i)$,
- $\neg U = \text{int}(X \setminus U)$;

and the canonical proximity relation is given by

$$U \prec V \text{ iff } \text{cl}(U) \subseteq V.$$

Definition.

- ① A binary relation \prec on a Boolean algebra A is a **de Vries proximity** provided it satisfies the following axioms:

(DV1) $1 \prec 1$.

(DV2) $a \prec b$ implies $a \leq b$.

(DV3) $a \leq b \prec c \leq d$ implies $a \prec d$.

(DV4) $a \prec b, c$ implies $a \prec b \wedge c$.

(DV5) $a \prec b$ implies $\neg b \prec \neg a$.

(DV6) $a \prec b$ implies there is c such that $a \prec c \prec b$.

(DV7) $b \neq 0$ implies there is $a \neq 0$ such that $a \prec b$.

- ② A **de Vries algebra** is a pair (A, \prec) , where A is a complete Boolean algebra and \prec is a de Vries proximity on A .

If $f : X \rightarrow Y$ is a continuous map between compact Hausdorff spaces, then there is a well-defined map from $\mathcal{RO}(Y)$ to $\mathcal{RO}(X)$ given by $U \mapsto \text{int}(\text{cl}(f^{-1}(U)))$.

Definition.

A map $\rho : A \rightarrow B$ between de Vries algebras is a **de Vries morphism** provided

$$(M1) \quad \rho(0) = 0.$$

$$(M2) \quad \rho(a \wedge b) = \rho(a) \wedge \rho(b).$$

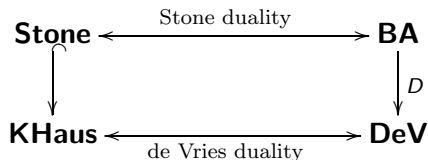
$$(M3) \quad a \prec b \text{ implies } \neg\rho(\neg a) \prec \rho(b).$$

$$(M4) \quad \rho(b) = \bigvee \{\rho(a) \mid a \prec b\}.$$

De Vries algebras and de Vries morphisms form a category **DeV**.

Relation with Stone Duality

De Vries duality extends Stone duality in the following sense.



The functor D sends a Boolean algebra B to the pair $(D(B), \prec)$, where $D(B)$ is the MacNeille completion of B , and \prec is defined by $u \prec v$ if there is $c \in B$ with $u \leq c \leq v$.

There is a functor $\mathbf{KHaus} \rightarrow \mathbf{DeV}$ which sends X to $(\mathcal{RO}(X), \prec)$. If $f : X \rightarrow Y$ is continuous, then f is sent to the map $f^* : \mathcal{RO}(Y) \rightarrow \mathcal{RO}(X)$, given by $f^*(U) = \text{int}(\text{cl}(f^{-1}(U)))$.

Theorem. (de Vries) There is a dual equivalence of categories between \mathbf{KHaus} and \mathbf{DeV} .

The functor in the opposite direction sends a de Vries algebra B to a compact Hausdorff space, called the **de Vries dual** of B . The space is built using a notion of a filter maximal with respect to a proximity condition.

Our goal is to extend de Vries duality to the category of completely regular spaces. The major difficulty is how to recover such a space from some sort of de Vries structures. De Vries duality yields a compact Hausdorff space, so we cannot recover the space in this way.

We do this by means of working with a category of compactifications of such spaces and a category of certain extensions of de Vries algebras.

We may produce a compactification of our space, however, from the dual of a de Vries algebra.

Compactifications

We recall that a **compactification** of a completely regular space X is a pair (Y, e) , where Y is a compact Hausdorff space and $e : X \rightarrow Y$ is an embedding such that the image $e(X)$ is dense in Y .

Suppose that $e : X \rightarrow Y$ and $e' : X \rightarrow Y'$ are compactifications. Write $e \leq e'$ provided there is a continuous map $f : Y' \rightarrow Y$ with $f \circ e' = e$.

$$\begin{array}{ccc} X & \xrightarrow{e'} & Y' \\ \text{Id} \downarrow & & \downarrow f \\ X & \xrightarrow{e} & Y \end{array}$$

The relation \leq is reflexive and transitive. Two compactifications e and e' are **equivalent** if $e \leq e'$ and $e' \leq e$. The equivalence classes of compactifications of X form a poset whose largest element is the Stone-Ćech compactification $s : X \rightarrow \beta X$.

In the classical setting, one considers compactifications of a fixed base space X . For our purposes we need to vary the base space.

We define a category **Comp** whose objects are compactifications $e : X \rightarrow Y$.

Morphisms in **Comp** are pairs (f, g) of continuous maps such that the following diagram commutes.

$$\begin{array}{ccc}
 X & \xrightarrow{e} & Y \\
 f \downarrow & & \downarrow g \\
 X' & \xrightarrow{e'} & Y'
 \end{array}$$

The composition of two morphisms (f_1, g_1) and (f_2, g_2) is defined to be $(f_2 \circ f_1, g_2 \circ g_1)$.

$$\begin{array}{ccc}
 X_1 & \xrightarrow{e_1} & Y_1 \\
 \downarrow f_1 & & \downarrow g_1 \\
 X_2 & \xrightarrow{e_2} & Y_2 \\
 \downarrow f_2 & & \downarrow g_2 \\
 X_3 & \xrightarrow{e_3} & Y_3
 \end{array}$$

$f_2 \circ f_1$ (left curved arrow) $g_2 \circ g_1$ (right curved arrow)

If X is a completely regular space, then $\mathcal{RO}(X)$ is a complete Boolean algebra. One might hope to recover X from $\mathcal{RO}(X)$ or from a de Vries structure $(\mathcal{RO}(X), \prec)$.

Proposition. If $e : X \rightarrow Y$ is a compactification, then $e^{-1} : \mathcal{RO}(Y) \rightarrow \mathcal{RO}(X)$ is a Boolean isomorphism.

Thus, there is a proximity \prec on $\mathcal{RO}(X)$ induced by e^{-1} , but the pair $(\mathcal{RO}(X), \prec)$ recovers Y from de Vries duality.

No matter what de Vries algebra we get from X , de Vries duality will return a compact Hausdorff space, and not X in general. We need more information in order to recover X .

Our recent work on canonical extensions of vector lattices helped us to come up with the notion of a de Vries extension.

For a set X the power set $\wp(X)$ is a complete Boolean algebra, and with the subset relation, $(\wp(X), \subseteq)$ is a de Vries algebra.

Proposition. Let $e : X \rightarrow Y$ be a compactification. Then the map $e^{-1} : \mathcal{RO}(Y) \rightarrow \wp(X)$ is a 1-1 de Vries morphism such that each element of $\wp(X)$ is a join of meets from the image.

If B is a complete Boolean algebra, then \leq is a proximity on B . The pair (B, \leq) is called an **extremally disconnected** de Vries algebra as its de Vries dual is an extremally disconnected space.

Definition. Let A be a de Vries algebra and (B, \leq) an atomic extremally disconnected de Vries algebra. A **de Vries extension** is a 1-1 de Vries morphism $\alpha : A \rightarrow B$ such that each element of B is a join of meets from $\alpha[A]$.

There is a category **DeVe** whose objects are de Vries extensions.

Morphisms in **DeVe** are pairs (ρ, σ) , where ρ is a de Vries morphism, σ is a complete Boolean homomorphism, and the following diagram commutes.

$$\begin{array}{ccc}
 A & \xrightarrow{\alpha} & B \\
 \rho \downarrow & & \downarrow \sigma \\
 A' & \xrightarrow{\alpha'} & B'
 \end{array}$$

Composition in **DeVe** (where \star is composition in **DeV**):

$$\begin{array}{ccc}
 A_1 & \xrightarrow{\alpha_1} & B_1 \\
 \rho_1 \downarrow & & \downarrow \sigma_1 \\
 A_2 & \xrightarrow{\alpha_2} & B_2 \\
 \rho_2 \downarrow & & \downarrow \sigma_2 \\
 A_3 & \xrightarrow{\alpha_3} & B_3
 \end{array}$$

$\rho_2 \star \rho_1$ is indicated by a curved arrow on the left side, and $\sigma_2 \circ \sigma_1$ is indicated by a curved arrow on the right side.

Dual Equivalence of **Comp** and **DeVe**

We now describe functors $E : \mathbf{Comp} \rightarrow \mathbf{DeVe}$ and $C : \mathbf{DeVe} \rightarrow \mathbf{Comp}$.

If $e : X \rightarrow Y$ is an object in **Comp**, its image under E is $e^{-1} : \mathcal{RO}(Y) \rightarrow \wp(X)$.

For a morphism (f, g) in **Comp**

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ f \downarrow & & \downarrow g \\ X' & \xrightarrow{e'} & Y' \end{array}$$

its image is (g^*, f^{-1}) .

$$\begin{array}{ccc} \mathcal{RO}(Y') & \xrightarrow{(e')^{-1}} & \wp(X') \\ g^* \downarrow & & \downarrow f^{-1} \\ \mathcal{RO}(Y) & \xrightarrow{e^{-1}} & \wp(X) \end{array}$$

Proposition. $E : \mathbf{Comp} \rightarrow \mathbf{DeVe}$ is a contravariant functor.

The only real issue in the proof is to show that $f^{-1} \star (e')^{-1} = e^{-1} \star g^*$.

$$\begin{array}{ccc} \mathcal{RO}(Y') & \xrightarrow{(e')^{-1}} & \wp(X') \\ g^* \downarrow & & \downarrow f^{-1} \\ \mathcal{RO}(Y) & \xrightarrow{e^{-1}} & \wp(X) \end{array}$$

The functor $C : \mathbf{DeVe} \rightarrow \mathbf{Comp}$

If B is a de Vries algebra, we denote its de Vries dual by Y_B . If $\alpha : A \rightarrow B$ is a de Vries extension, its de Vries dual $\alpha^* : Y_B \rightarrow Y_A$ isn't a compactification.

If B is an extremally disconnected de Vries algebra, the proximity is \leq and the de Vries dual Y_B is then the Stone space of B .

Definition. For an atomic extremally disconnected de Vries algebra B , let X_B be the set of isolated points of Y_B .

Isolated points of Y_B have the form $\uparrow b$ for some atom $b \in B$.

Lemma. Let A and B be de Vries algebras, with B atomic and extremally disconnected, and let $\alpha : A \rightarrow B$ be a 1-1 de Vries morphism. Then α is a de Vries extension iff the restriction of $\alpha_* : Y_B \rightarrow Y_A$ to X_B is 1-1.

Definition. For a de Vries extension $\alpha : A \rightarrow B$, define a topology on X_B as the least topology making $\alpha_* : X_B \rightarrow Y_A$ continuous.

Theorem. If $\alpha : A \rightarrow B$ is a de Vries extension, then X_B is completely regular and $\alpha_* : X_B \rightarrow Y_A$ is a compactification.

Defining the functor C

If $\alpha : A \rightarrow B$ is an object in **DeVe**, we define its image under C to be the compactification $\alpha_* : X_B \rightarrow Y_A$.

For a morphism (ρ, σ) in **DeVe**

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \rho \downarrow & & \downarrow \sigma \\ A' & \xrightarrow{\alpha'} & B' \end{array}$$

$C(\rho, \sigma) = (\sigma_+, \rho_*)$, where σ_+ is the Tarski dual of σ and ρ_* is the de Vries dual of ρ .

$$\begin{array}{ccc} X_{B'} & \xrightarrow{\alpha'_*} & Y_{A'} \\ \sigma_+ \downarrow & & \downarrow \rho_* \\ X_B & \xrightarrow{\alpha_*} & Y_A \end{array}$$

The Main Theorem

Theorem. The functors E and C yield a dual equivalence between **Comp** and **DeVe**.

Let **SComp** be the full subcategory of **Comp** consisting of Stone-Čech compactifications.

Since Stone-Čech is a functor, there is an equivalence between **CReg** and **SComp**.

By our main theorem, **CReg** is then dually equivalent to a full subcategory of **DeVe**.

Suppose we have a commutative diagram of compactifications of X .

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ & \searrow e' & \downarrow f \\ & & Y' \end{array}$$

E sends it to

$$\begin{array}{ccc} \mathcal{R}\mathcal{O}(Y') & \xrightarrow{(e')^{-1}} & \wp(X) \\ f^* \downarrow & \nearrow e^{-1} & \\ \mathcal{R}\mathcal{O}(Y) & & \end{array}$$

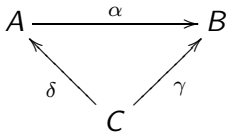
e^{-1} and $(e')^{-1}$ have the same image $\mathcal{R}\mathcal{O}(X)$.

Definition. We call two de Vries extensions $\alpha : A \rightarrow B$ and $\gamma : C \rightarrow B$ **compatible** if $\alpha[A] = \gamma[C]$.

Lemma. Two de Vries extensions $\alpha : A \rightarrow B$ and $\gamma : C \rightarrow B$ are compatible iff the topologies on X_B induced by the two extensions are equal.

Definition.

- 1 We say that a de Vries extension $\alpha : A \rightarrow B$ is **maximal** provided for every compatible de Vries extension $\gamma : C \rightarrow B$, there is a de Vries morphism $\delta : C \rightarrow A$ such that $\alpha \star \delta = \gamma$.



- 2 Let **MDeVe** be the full subcategory of **DeVe** consisting of maximal de Vries extensions.

Theorem. Let $e : X \rightarrow Y$ be a compactification and let $s : X \rightarrow \beta X$ be the Stone-Čech compactification of X . Then the following conditions are equivalent.

- 1 The de Vries extension $e^{-1} : \mathcal{RO}(Y) \rightarrow \wp(X)$ is maximal.
- 2 e is equivalent to s .

Theorem. There is a dual equivalence between **MDeVe** and **SComp**, and so there is a dual equivalence between **CReg** and **MDeVe**.

Therefore, **MDeVe** is a category of algebraic objects dually equivalent to the category of completely regular spaces.

These results can be used to characterize such topological properties as normality and local compactness in terms of de Vries extensions that are subject to additional axioms.

Thanks to the organizers for the invitation to speak at this workshop and thanks for your attention.