Semantic and Syntactic Consequence

to Roberto Cignoli
in memoriam

DANIELE MUNDICI
Department of Mathematics and Computer Science, University of Florence
mundici@math.unifi.it

syntactic tautologies in boolean logic

DEFINITION A boolean *syntactic tautology* is a formula that can be derived from the following *basic tautologies* via substitution and modus ponens:

$x \to (y \to x)$

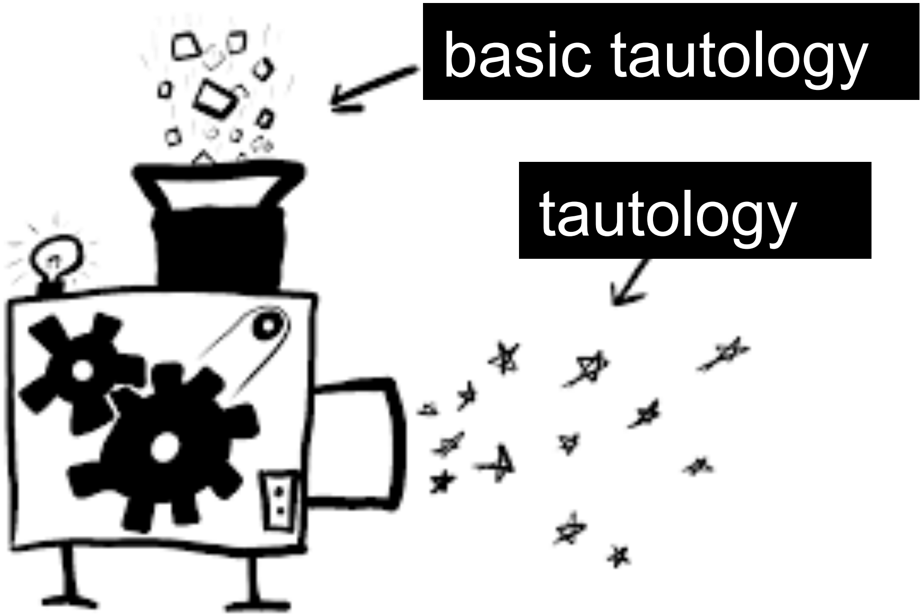
$(x \to y) \to ((y \to z) \to (x \to z))$

$((x \to y) \to y) \to ((y \to x) \to x)$

$(\neg y \to \neg x) \to (x \to y)$

$(\neg y \to y) \to y$ (Consequentia Mirabilis)

syntactic tautologies in wide generality



DEFINITION A formula f is a *syntactic tautology* if it is obtainable (typically from some *basic tautologies*) by some mechanical procedure.

semantic tautologies in boolean logic

DEFINITION A boolean *semantic tautology* is a formula f such that every **valuation** into the two element boolean algebra $\{0,1\}$ (the *matrix* of boolean logic) evaluates f to 1.

semantic tautologies in wide generality

homomorphism

DEFINITION A formula f is a *semantic tautology* of L if every **valuation** gives value 1 to f.

semantic tautologies in wide generality

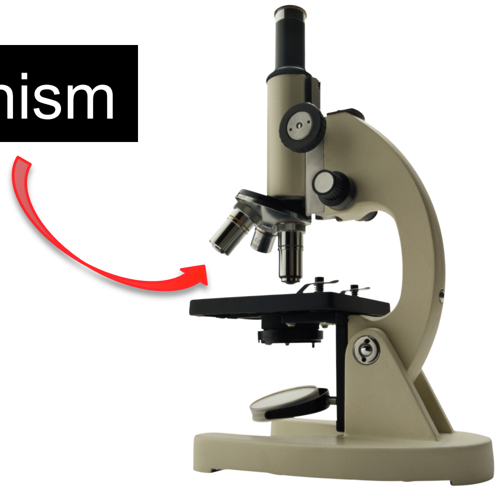
homomorphism

DEFINITION A formula f is a *semantic tautology* of L if every **valuation** gives value 1 to f.

we assume there are algebras of the same type (*matrices*), in such a way that ***valuations*** are homomorphisms of the term algebra of L into a matrix; each matrix has just *one constant* 1 designated for "true"

semantic tautologies in wide generality

homomorphism



DEFINITION A formula f is a *semantic tautology* of L if every **valuation** gives value 1 to f.

we assume there are algebras of the same type (*matrices*), in such a way that ***valuations*** are homomorphisms of the term algebra of L into a matrix; each matrix has just *one constant* 1 designated for "true"

DESIDERATUM *Syntactic and semantic tautologies must coincide.*

completeness theorem for boolean tautologies

THEOREM (fulfilling the Desideratum) *Boolean syntactic and semantic tautologies coincide*.

completeness theorem for boolean tautologies

THEOREM (fulfilling the Desideratum) *Boolean syntactic and semantic tautologies coincide*.

Next, taking inspiration from Algebra, let us consider the main theme of this talk, namely
CONSEQUENCE

ALGEBRA: two ways to generate an ideal

VIA RULES: the ideal generated by a set F of elements of an algebra A is the set of elements obtainable from F via finitely many applications of certain operations and relations of A.
(*Depends on the definition of these operations and relations*)

ALGEBRA: two ways to generate an ideal

VIA RULES: the ideal generated by a set F of elements of an algebra A is the set of elements obtainable from F via finitely many applications of certain operations and relations of A.
(*Depends on the definition of these operations and relations*)

VIA SPECIAL IDEALS: an element g belongs to the ideal generated by a set F of elements of an algebra A if for every "irreducible" ideal I such that $f/I=0$ for all f in F, we also have $g/I=0$.
(*Depends on the definition of "irreducible ideal", or "irreducible congruence"*)

LOGIC: two definitions of “deductive closure”

VIA RULES: ψ belongs to the deductive closure of a set of formulas if it is obtainable from a subset of F via finitely many “mechanical” manipulations

(typically, involving **time-honored “rules”** such as “modus ponens”, [if each matrix has an operation → obeying the following *minimum requirement for an implication*: $(x \to (y \to z)) = (y \to (x \to z))$]

LOGIC: two definitions of "deductive closure"

VIA RULES: ψ belongs to the deductive closure of a set of formulas if it is obtainable from a subset of F via finitely many "mechanical" manipulations

(typically, involving **time-honored "rules"** such as "modus ponens", [if each matrix has an operation → obeying the following *minimum requirement for an implication*: $(x\rightarrow (y\rightarrow z)) = (y\rightarrow (x\rightarrow z))$]

VIA VALUATIONS/MODELS ψ belongs to the deductive closure of a set Θ of formulas if every valuation assigning 1 to every element θ of Θ also assigns 1 to ψ.

LOGIC: two definitions of "deductive closure"

VIA RULES: ψ belongs to the deductive closure of a set of formulas if it is obtainable from a subset of F via finitely many "mechanical" manipulations

(typically, involving **time-honored "rules"** such as "modus ponens", [if each matrix has an operation $\rightarrow$ obeying the following *minimum requirement for an implication*: $(x\rightarrow (y\rightarrow z)) = (y\rightarrow (x\rightarrow z))$]

VIA VALUATIONS/MODELS ψ belongs to the deductive closure of a set Θ of formulas if every valuation assigning 1 to every element θ of Θ also assigns 1 to ψ.

rules are 2500 years older than valuations

Semantic consequence (first half of the XXth century)

The sentence X follows logically *from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.*†

† "After the original of this paper had appeared in print, H. Scholz in his article 'Die Wissenschaftslehre Bolzanos, Eine Jahrhundert-Betrachtung', Abhandlungen der Fries'schen Schule, new series, vol. 6, pp. 399-472 (see in particular p. 472, footnote 58) pointed out a far-reaching analogy between this definition of consequence and the one suggested by B. Bolzano about a hundred years earlier."

[Note added by Tarski in English in A. Tarski,
"Logic, Semantics, Metamathematics", Oxford (1956) p. 417]

Semantic consequence (first half of the XXth century)

The sentence X follows logically *from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.*†

Semantic consequence (first half of the XXth century)

The sentence X follows logically *from the sentences of the class K if and only if every* $\boxed{\textit{model}}$ *of the class K is also a model of the sentence X.*†

In first-order logic, “models” have a well known definition. In our propositional framework, “models” are valuations:

Semantic consequence (first half of the XXth century)

The sentence X follows logically *from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.*†

In first-order logic, “models” have a well known definition. In our propositional framework, “models” are valuations:

REFORMULATION (for logics given by matrices) A formula ψ is a *semantic consequence* of a set Θ of formulas if every valuation that gives value 1 to all formulas of Θ also gives value 1 to ψ.

The boolean case

Syntactic consequence in boolean logic

DEFINITION In boolean logic, a formula ψ is a *syntactic consequence* of a set Θ of formulas if it can be derived from the syntactic tautologies and Θ via finitely many applications of "modus ponens"

Semantic consequence in boolean logic

DEFINITION In boolean logic, a formula ψ is a *(Bolzano-Tarski) semantic consequence* of a set Θ of formulas if every valuation that gives value 1 to all formulas of Θ also gives value 1 to ψ.

Semantic consequence in boolean logic

DEFINITION In boolean logic, a formula ψ is a *(Bolzano-Tarski) semantic consequence* of a set Θ of formulas if every valuation that gives value 1 to all formulas of Θ also gives value 1 to ψ.

THEOREM *In boolean logic syntactic consequence coincides with Bolzano-Tarski semantic sequence*.

This theorem is misldeading

This completeness theorem depends on the following facts:

—“valuations” are quotients by (dual) maximal ideals (filters) of the free boolean algebra

—maximal ideals = meet irreducible ideals = prime ideals.

In algebraic logic, as well as in ring theory, these identities are exceptional.

Thus e.g., in Łukasiewicz logic, adoption of the Bolzano-Tarski paradigm leads to a gap between syntactic and semantic consequence.

Łukasiewicz logic (presented as in D.M., 2018)

THEOREM *Any algebra* $([0,1], 0, \neg, \rightarrow)$ *with a* ***continuous*** *binary operation* $\rightarrow$ *having the properties* $x\rightarrow(y\rightarrow z)=x\rightarrow(y\rightarrow z)$ *and* $y\rightarrow z=1$ *iff* $y\leq z$, *necessarily satisfies the following equations:*

$$1=x\rightarrow(y\rightarrow x)$$
$$1=(x\rightarrow y)\rightarrow((y\rightarrow z)\rightarrow(x\rightarrow z))$$
$$1=((x\rightarrow y)\rightarrow y)\rightarrow((y\rightarrow x)\rightarrow x)$$
$$1=(\neg y\rightarrow \neg x)\rightarrow(x\rightarrow y)$$

Łukasiewicz

Łukasiewicz logic (presented as in D.M., 2018)

THEOREM *Any algebra* $([0,1], 0, \neg, \rightarrow)$ *with a* ***continuous*** *binary operation* $\rightarrow$ *having the properties* $x\rightarrow(y\rightarrow z)=x\rightarrow(y\rightarrow z)$ *and* $y\rightarrow z = 1$ *iff* $y\leq z$, *necessarily satisfies the following equations:*

$$1=x\rightarrow(y\rightarrow x)$$
$$1=(x\rightarrow y)\rightarrow((y\rightarrow z)\rightarrow(x\rightarrow z))$$
$$1=((x\rightarrow y)\rightarrow y)\rightarrow((y\rightarrow x)\rightarrow x)$$
$$1=(\neg y\rightarrow \neg x)\rightarrow(x\rightarrow y)$$

Łukasiewicz

$$1=x\rightarrow(y\rightarrow x)$$
$$1=(x\rightarrow y)\rightarrow((y\rightarrow z)\rightarrow(x\rightarrow z))$$
$$1=((x\rightarrow y)\rightarrow y)\rightarrow((y\rightarrow x)\rightarrow x)$$
$$1=(\neg y\rightarrow \neg x)\rightarrow(x\rightarrow y)$$
$$1=(\neg y\rightarrow y)\rightarrow y$$ (Consequentia Mirabilis)

Boole

the Łukasiewicz axiom $((x \to y) \to y) = ((y \to x) \to x)$

the intriguing basic tautology

$$((x \to y) \to y) \to ((y \to x) \to x)$$

in boolean logic says $x \vee y = y \vee x$

the Łukasiewicz axiom $((x \to y) \to y) = ((y \to x) \to x)$

the intriguing basic tautology

$$((x \to y) \to y) \to ((y \to x) \to x)$$

in boolean logic says $x \vee y = y \vee x$

in Łukasiewicz logic this has a deeper meaning:
it says that implication is continuous

some nontrivial results ...

THEOREM (Rose-Rosser 1958**)**
Syntactic tautologies coincide with semantic tautologies.

some nontrivial results ...

THEOREM (Rose-Rosser 1958)
Syntactic tautologies coincide with semantic tautologies.

COROLLARY (D.M., Theoretical Computer Science, 1987)
The tautology problem of Łukasiewicz logic is coNP-complete.

some nontrivial results ...

THEOREM (ROSE-ROSSER 1958)
Syntactic tautologies coincide with semantic tautologies.

COROLLARY (D.M., Theoretical Computer Science, 1987)
The tautology problem of Łukasiewicz logic is coNP-complete.

For very recent computational applications, see the paper
D.M. "WORD PROBLEMS IN ELLIOTT MONOIDS",
Advances in Mathematics, 335 (2018) 343-371.
DOI 10.1016/j.aim.2018.07.015

... and an embarassing fact

DEFINITION A formula ψ is a *syntactic consequence* of a set Θ of formulas if it is obtainable from the syntactic tautologies and Θ via finitely many applications of modus ponens.

... and an embarassing fact

DEFINITION A formula ψ is a *syntactic consequence* of a set Θ of formulas if it is obtainable from the syntactic tautologies and Θ via finitely many applications of modus ponens.

DEFINITION A formula ψ is a *(Bolzano-Tarski) semantic consequence* of a set Θ of formulas every valuation giving value 1 to all formulas in Θ also gives value 1 to ψ.

... and an embarassing fact

DEFINITION A formula ψ is a *syntactic consequence* of a set Θ of formulas if it is obtainable from the syntactic tautologies and Θ via finitely many applications of modus ponens.

DEFINITION A formula ψ is a *(Bolzano-Tarski) semantic consequence* of a set Θ of formulas every valuation giving value 1 to all formulas in Θ also gives value 1 to ψ.

FACT *Syntactic consequences differ from (Bolzano-Tarski) semantic consequences.*

We will explain the reasons for this failure. By refining the Bolzano-Tarski paradigm, we will fix the problem

The role of prime ideals

My failure to prove the completeness in 1958 using MV-algebras was a disappointment to me at that time. I tried that year and even after I left Cornell, but with no success. My mistake was in trying to pound the thing out by sticking to maximal ideals.

...

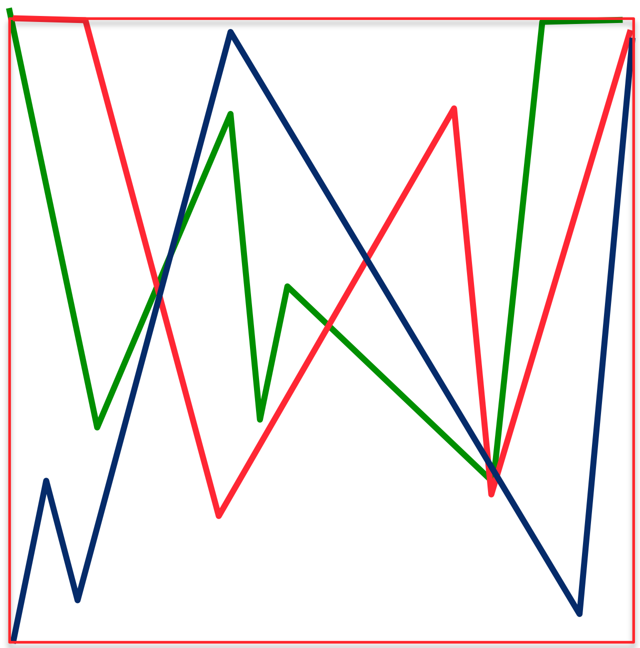
But a lucky break occurred when Dana Scott realized, with far-reaching insight, that there is a notion of prime ideals in MV-algebras (a notion I had not considered until then).

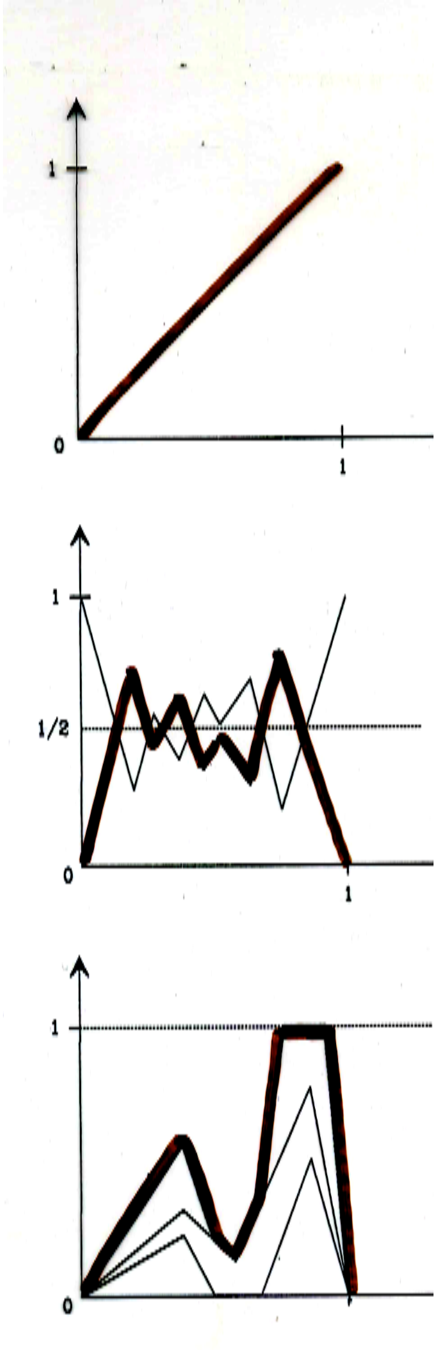
C. C. CHANG, *The writing of the MV-algebras*, Studia Logica, 61 (1998) 3-6.

free MV-algebras and their prime spectral spaces

The free one-generator MV-algebra F_1

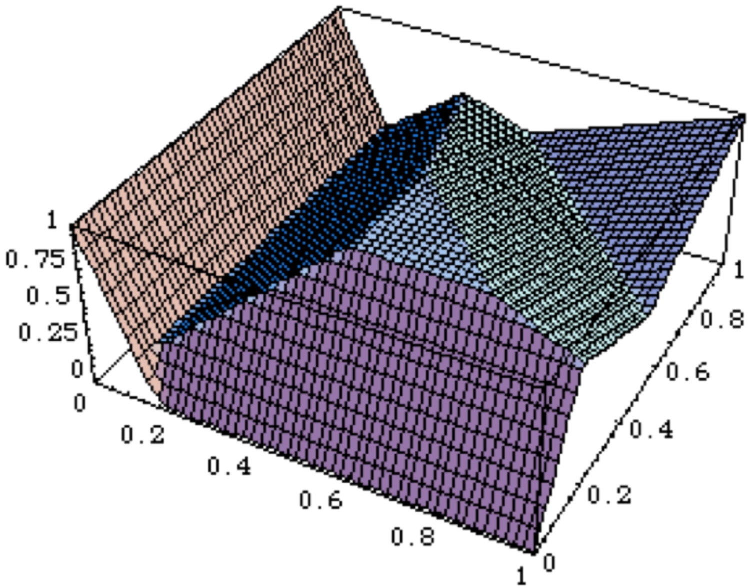
F_1 *consists of all piecewise linear continuous* [0,1]-*valued functions with integer coefficients, defined over* [0,1]. (One-variable McNaughton functions)

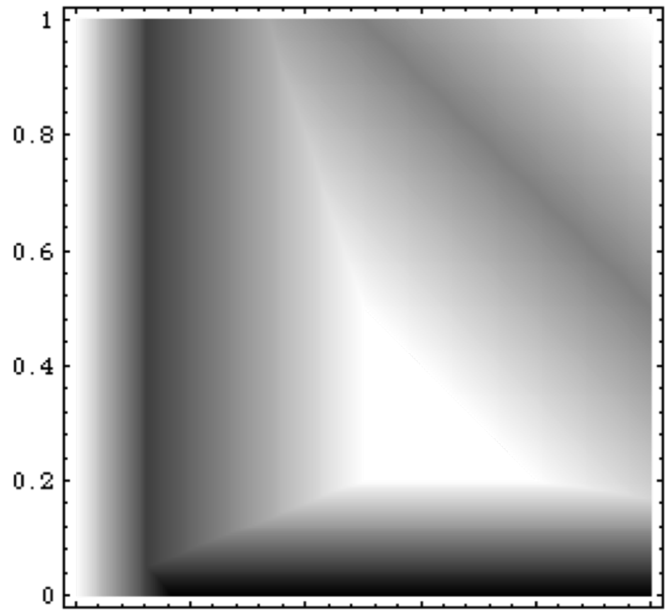




The free two-generator MV-algebra F_2

F_2 *consists of all piecewise linear continuous* $[0,1]$-*valued functions with integer coefficients, defined over the* $[0,1]^2$.





Piecewise linearity ensures that all directional derivatives exist

folklore on prime ideals of MV-algebras

DEFINITION An ideal J of A is *prime* if A/J is totally ordered.

folklore on prime ideals of MV-algebras

DEFINITION An ideal J of A is *prime* if A/J is totally ordered.

FACT *All ideals containing a prime ideal occur in a chain. Thus above any prime ideal there is precisely one* maximal.

folklore on prime ideals of MV-algebras

DEFINITION An ideal J of A is *prime* if A/J is totally ordered.

FACT *All ideals containing a prime ideal occur in a chain. Thus above any prime ideal there is precisely one* maximal.

FACT *The prime ideals above a* minimal *prime of any n-generator MV-algebra form a chain of length* $\leq n$.

folklore on prime ideals of MV-algebras

DEFINITION An ideal J of A is *prime* if A/J is totally ordered.

FACT *All ideals containing a prime ideal occur in a chain. Thus above any prime ideal there is precisely one* maximal.

FACT *The prime ideals above a* minimal *prime of any n-generator MV-algebra form a chain of length* $\leq n$.

FACT *Every ideal in the intersection of the primes above it.*

folklore on prime ideals of MV-algebras

DEFINITION An ideal J of A is *prime* if A/J is totally ordered.

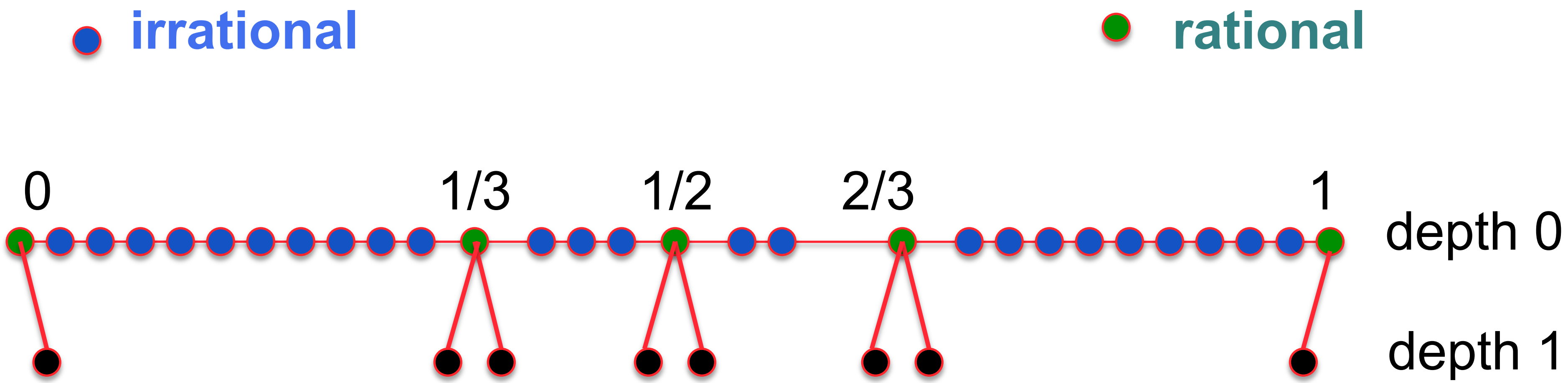
FACT *All ideals containing a prime ideal occur in a chain. Thus above any prime ideal there is precisely one* maximal.

FACT *The prime ideals above a* minimal *prime of any n-generator MV-algebra form a chain of length* $\leq n$.

FACT *Every ideal in the intersection of the primes above it.*

(This innocent looking fact is to the effect the quotient operation $f \rightarrow f/P$ for prime ideals fully controls the deductive closure operation, and foreshadows a complete semantic consequence.)

the prime ideals of the free MV-algebra F_1



the prime ideals of free MV-algebras

(M.Busaniche, D.M., Ann. Pure Appl. Logic 147, 2007)

Prime ideals of depth 0 are just maximal ideals, and are in one-one correspondence with points of $[0,1]^n$

the prime ideals of free MV-algebras

(M.Busaniche, D.M., Ann. Pure Appl. Logic 147, 2007)

Prime ideals of depth 0 are just maximal ideals, and are in one-one correspondence with points of $[0,1]^n$

Each prime ideal P of Krull depth 1 is below a single maximal M, and uniquely determines a direction d in $\mathbf{R}^n$

the prime ideals of free MV-algebras

(M.Busaniche, D.M., Ann. Pure Appl. Logic 147, 2007)

Prime ideals of depth 0 are just maximal ideals, and are in one-one correspondence with points of $[0,1]^n$

Each prime ideal P of Krull depth 1 is below a single maximal M, and uniquely determines a direction d in $\mathbf{R}^n$

Each prime ideal P' of depth 2 is below a single prime P of depth 1 and uniquely determines a direction $d' \perp d$ in $\mathbf{R}^n$

the prime ideals of free MV-algebras

(M.Busaniche, D.M., Ann. Pure Appl. Logic 147, 2007)

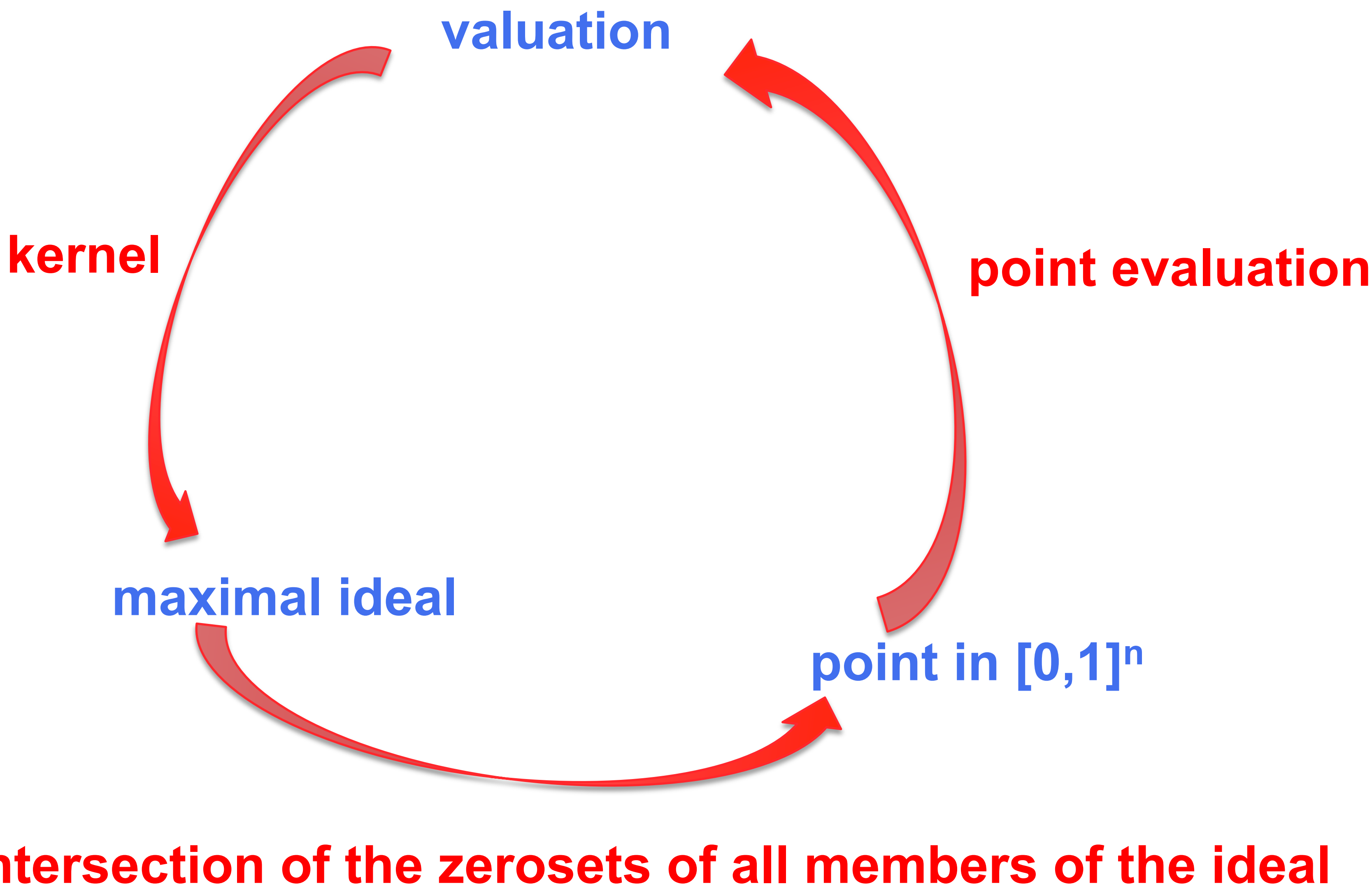
Prime ideals of depth 0 are just maximal ideals, and are in one-one correspondence with points of $[0,1]^n$

Each prime ideal P of Krull depth 1 is below a single maximal M, and uniquely determines a direction d in $\mathbf{R}^n$

Each prime ideal P' of depth 2 is below a single prime P of depth 1 and uniquely determines a direction $d' \perp d$ in $\mathbf{R}^n$

THEOREM *The prime ideals of the free MV-algebra* F_n *are labelled by points of* $[0,1]^n$ *together with orthonormal bases of* $\mathbf{R}^n$

Krull depth 0



Krull depth 1

x point in $[0,1]^n$ together with a unit vector d in R^n

all functions f
with f(x)=0, and
$\partial f(x)/\partial d = 0$

the only zeroset
and zerodirection
common to all
functions in the
prime ideal

depth 1 prime ideal

Krull depth 2

point x in $[0,1]^n$ and a pair of unit vectors $d \perp d'$ in R^n

all functions f with f(x)=0, ∂(x)/∂d =0, and ∂(y)/∂d' =0 for all y in x+εd

the only zeroset, zerodirection and perpendicular zerodirection common to all members of the ideal

depth 2 prime ideal

the prime ideals of free MV-algebras

(M.Busaniche, D.M., Ann. Pure Appl. Logic 147, 2007)

the prime ideals of free MV-algebras

(M.Busaniche, D.M., Ann. Pure Appl. Logic 147, 2007)

a McNaughton function f belongs to a prime ideal of depth 0 (i.e., a maximal M, in correspondence with a point x) iff f vanishes at x

the prime ideals of free MV-algebras

(M.Busaniche, D.M., Ann. Pure Appl. Logic 147, 2007)

a McNaughton function f belongs to a prime ideal of depth 0 (i.e., a maximal M, in correspondence with a point x) iff f vanishes at x

f belongs to a prime ideal P immediately below M iff f belongs to M **and** $\partial f(x)/\partial d = 0$ with d the direction associated to P

the prime ideals of free MV-algebras
(M.Busaniche, D.M., Ann. Pure Appl. Logic 147, 2007)

a McNaughton function f belongs to a prime ideal of depth 0 (i.e., a maximal M, in correspondence with a point x) iff f vanishes at x

f belongs to a prime ideal P immediately below M iff f belongs to M **and** $\partial f(x)/\partial d = 0$ with d the direction associated to P

f belongs to a prime ideal P' immediately below P iff f belongs to P **and** $\partial f(x+\varepsilon d)/\partial d' = 0$ for all small $\varepsilon > 0$, where $d' \perp d$, is the direction associated to P'. ... ...

the prime ideals of free MV-algebras

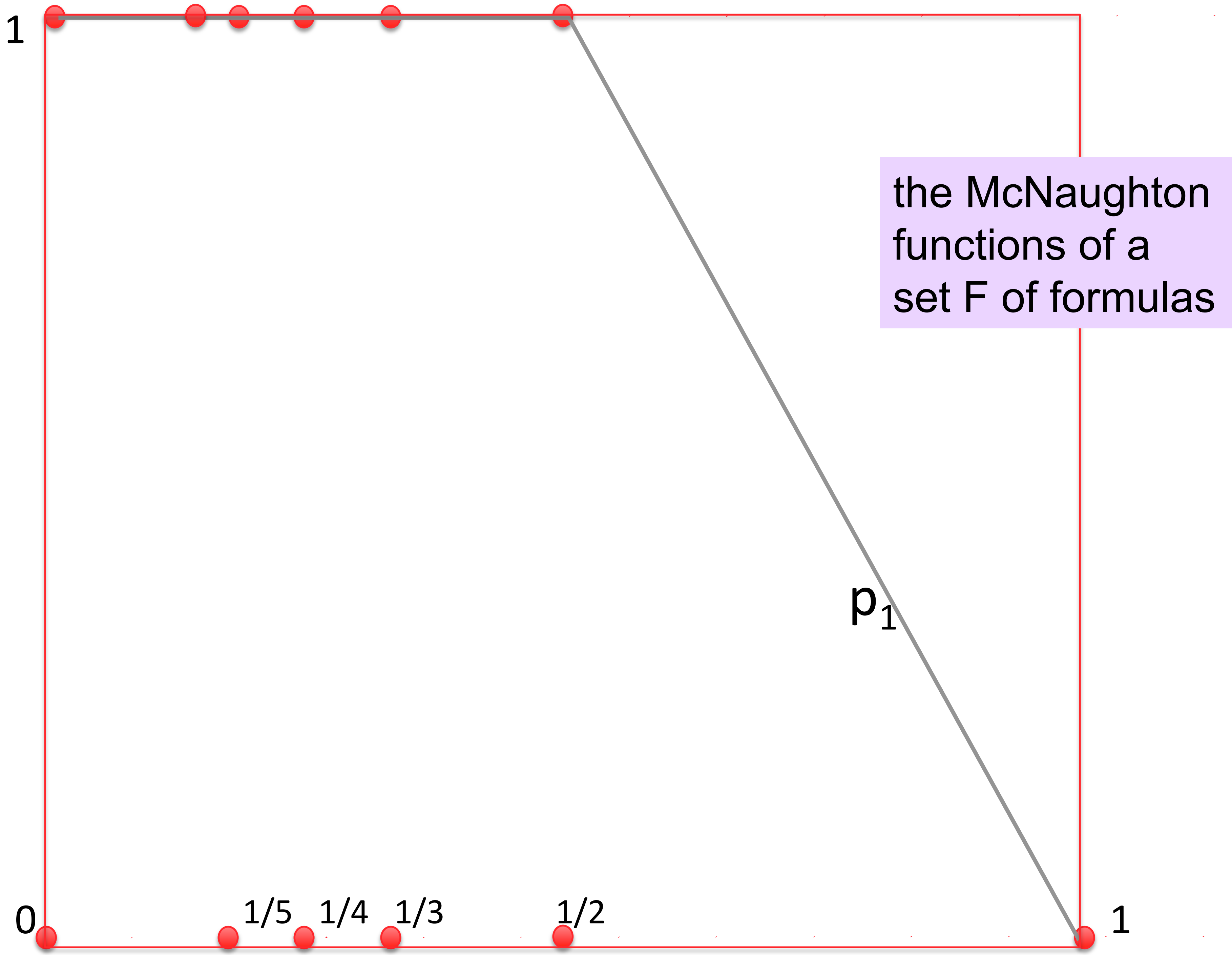
(M.Busaniche, D.M., Ann. Pure Appl. Logic 147, 2007)

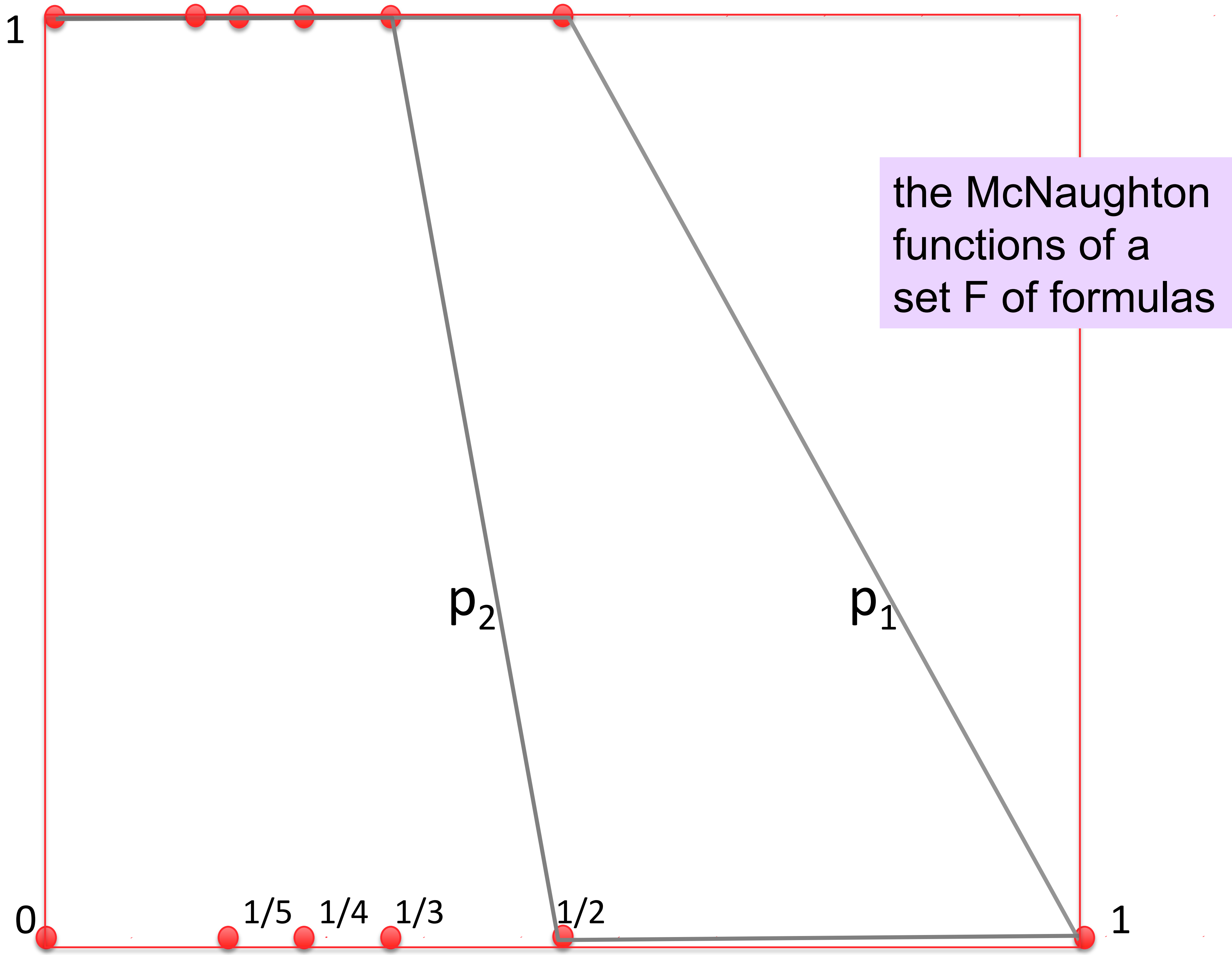
a McNaughton function f belongs to a prime ideal of depth 0 (i.e., a maximal M, in correspondence with a point x) iff f vanishes at x

f belongs to a prime ideal P immediately below M iff f belongs to M **and** $\partial f(x)/\partial d = 0$ with d the direction associated to P

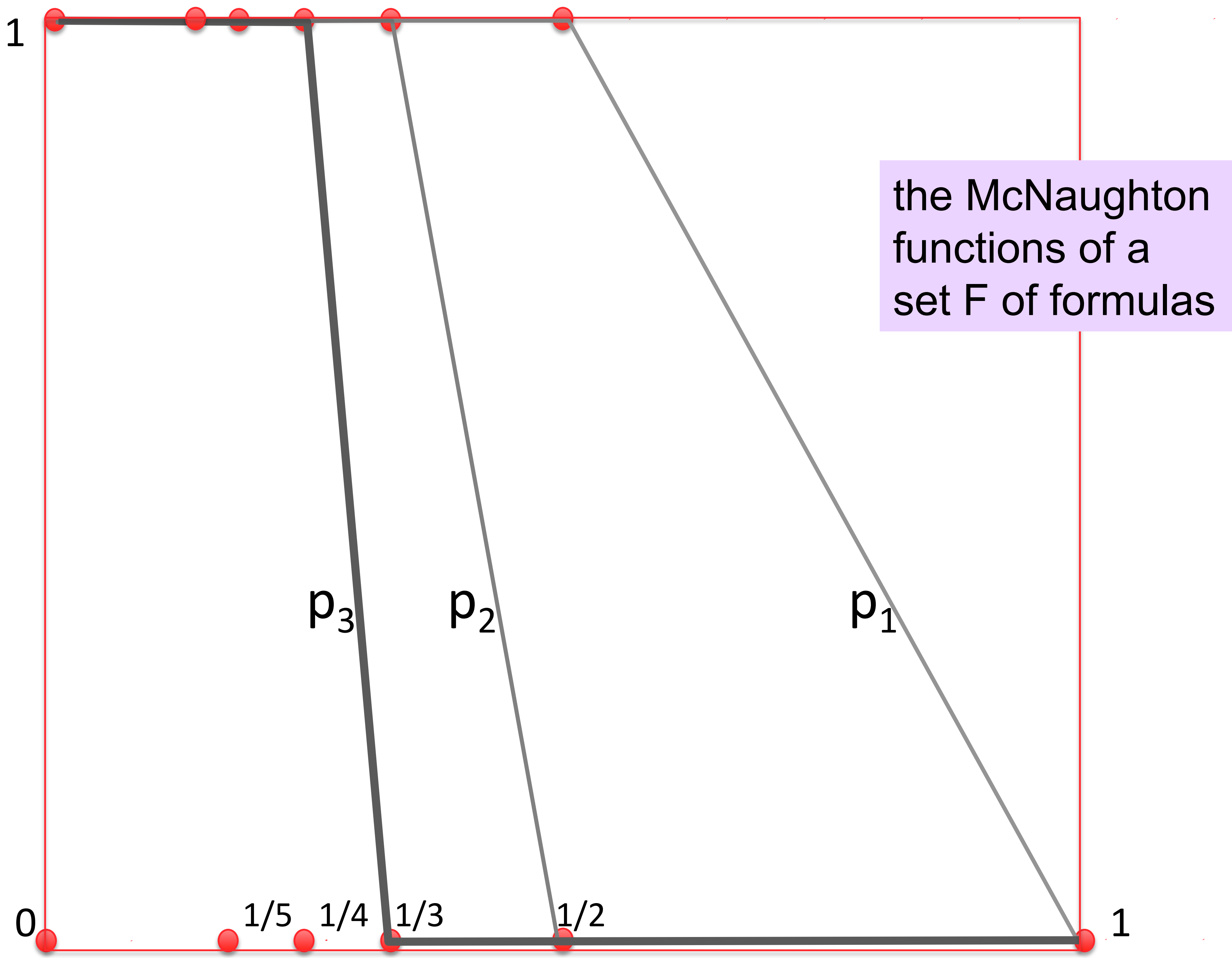
f belongs to a prime ideal P' immediately below P iff f belongs to P **and** $\partial f(x+\varepsilon d)/\partial d' = 0$ for all small $\varepsilon > 0$, where $d' \perp d$, is the direction associated to P'. ... ...

this shows that the quotient operation $f \rightarrow f/P$ for prime ideals P has a **quantitative** content, generalizing the quotient operation $f \rightarrow f/M$ for M a maximal, which amounts to a classical **evaluation** a one-dimensional example explaining the difference between syntactic consequence and Bolzano-Tarski consequence in infinite-valued Łukasiewicz logic

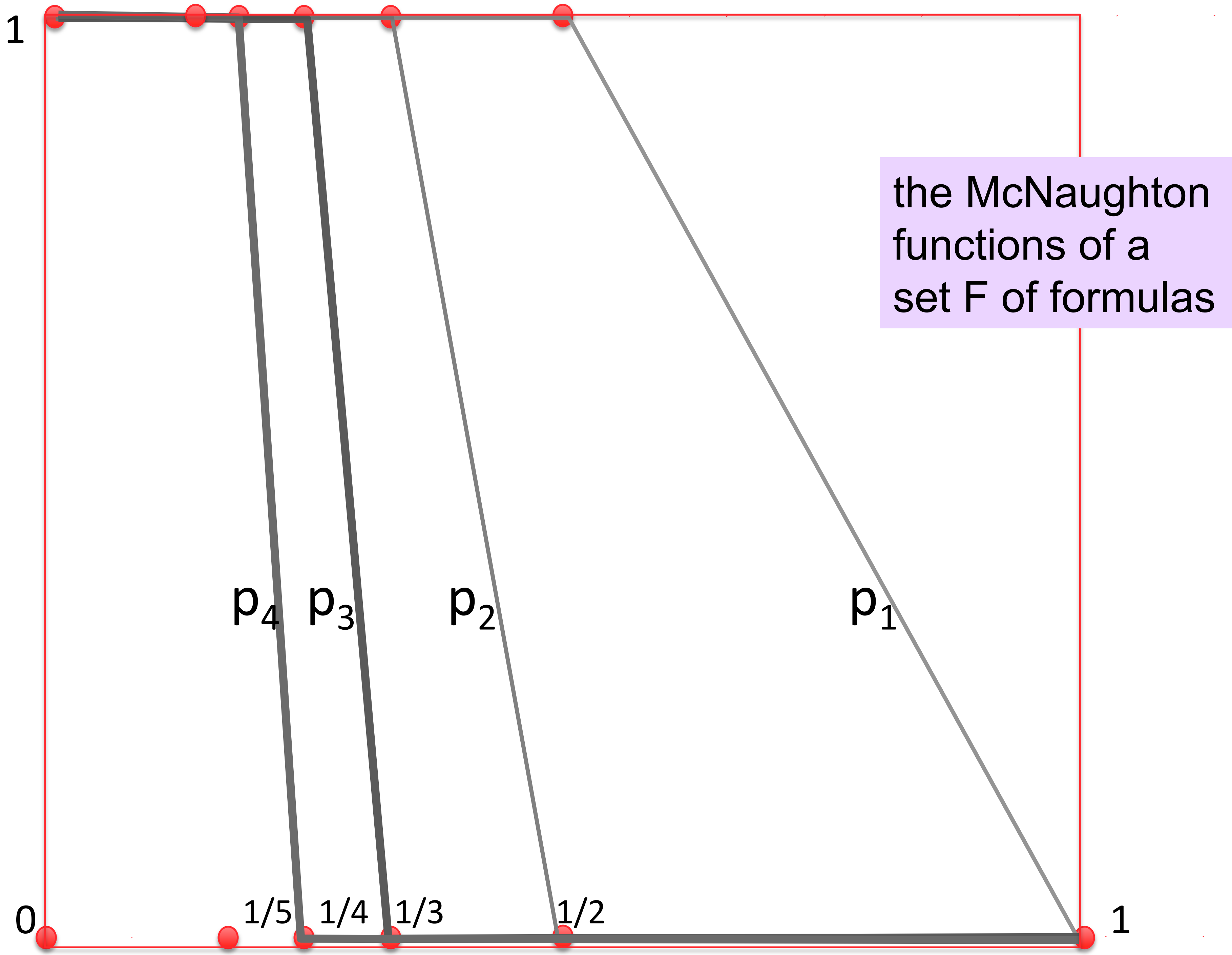

1
0
1/5
1/4
1/3
1/2
1
p1
the McNaughton
functions of a
set F of formulas

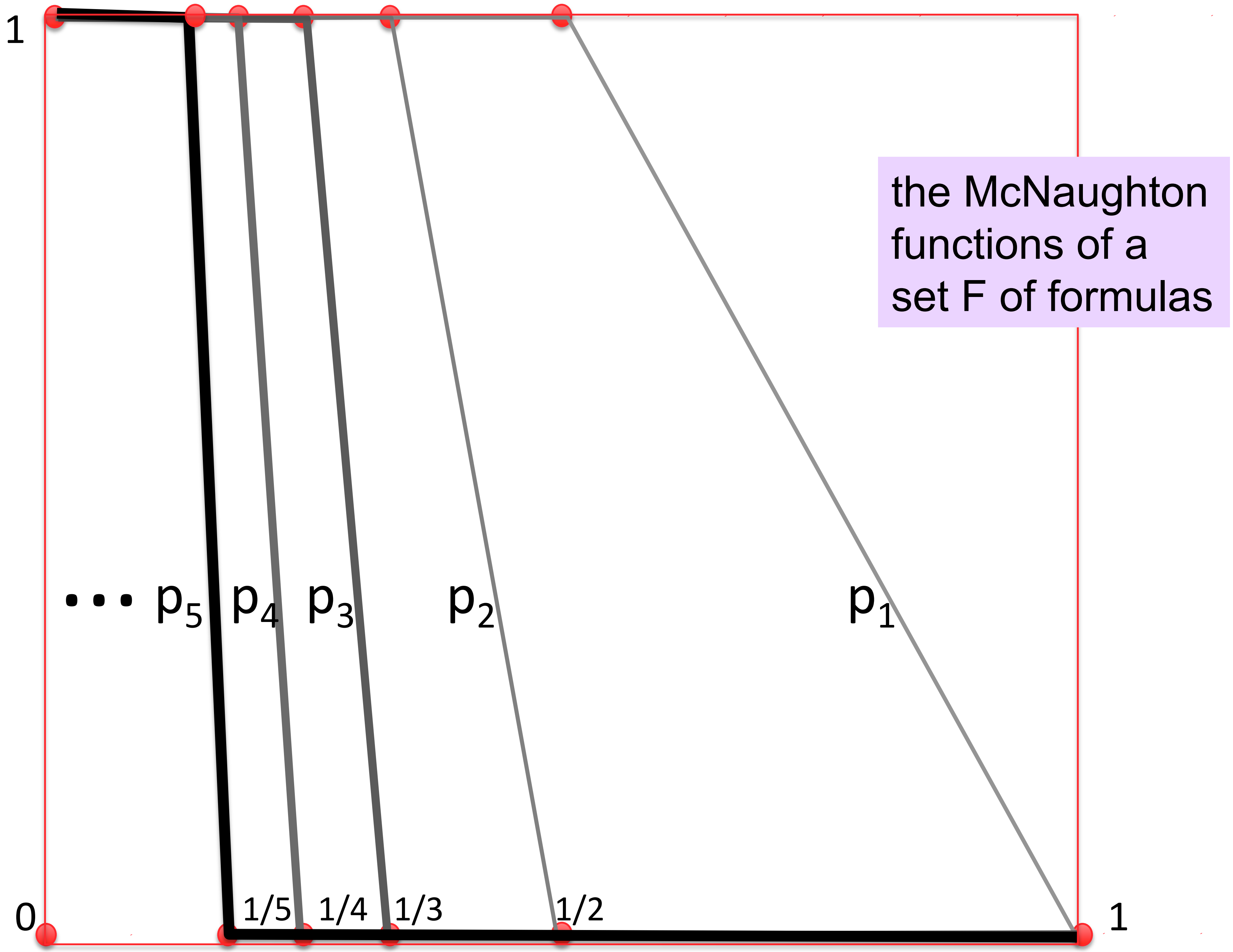


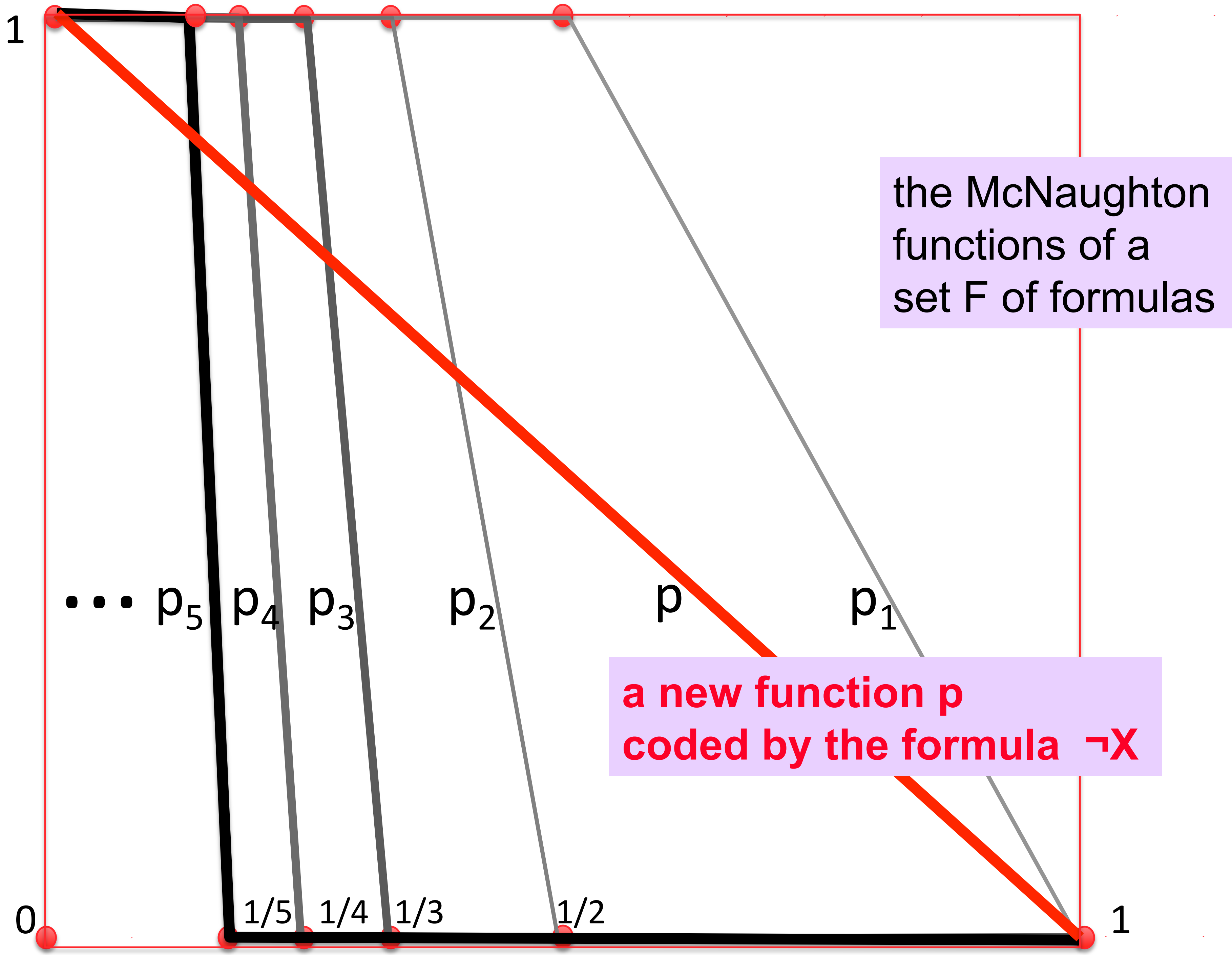
1
0
1/5
1/4
1/3
1/2
1
p2
p1
the McNaughton functions of a set F of formulas



1
0
1/5
1/4
1/3
1/2
1
p_3
p_2
p_1
the McNaughton functions of a set F of formulas


1
0
1/5
1/4
1/3
1/2
1
p_4
p_3
p_2
p_1
the McNaughton functions of a set F of formulas


1
0
1
the McNaughton functions of a set F of formulas
... p_5 p_4 p_3 p_2 p_1
1/5
1/4
1/3
1/2



1
0
1
1/5
1/4
1/3
1/2
• • • p_5 p_4 p_3 p_2 p p_1
the McNaughton functions of a set F of formulas
a new function p coded by the formula ¬X

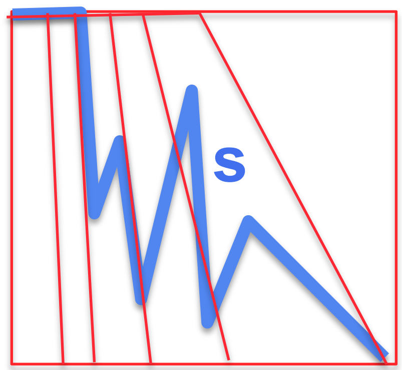
beyond the Bolzano-Tarski paradigm

$F=\{p_1, p_2,...\}$ is a set of McNaughton functions $p_i=1$ on a right neighbourhood of the origin 0

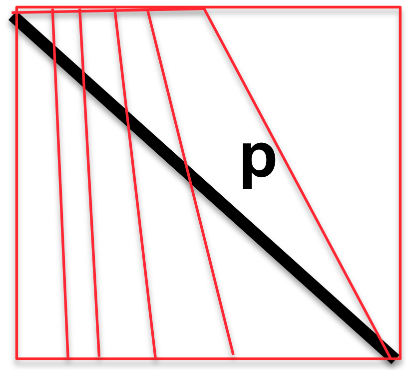
s is a stable consequence of F, because s=1 over a right neighbourhood of 0. Remarkably, s is also a syntactic consequence of the p_i

p is not a stable consequence of F, because p is not equal to 1 on a neighbourhood of 0.

However, p is a Bolzano-Tarski consequence of F, because t(0)= 1.



stable



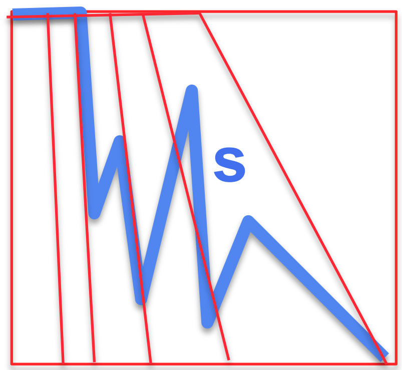
Bolzano-Tarski
not stable

beyond the Bolzano-Tarski paradigm

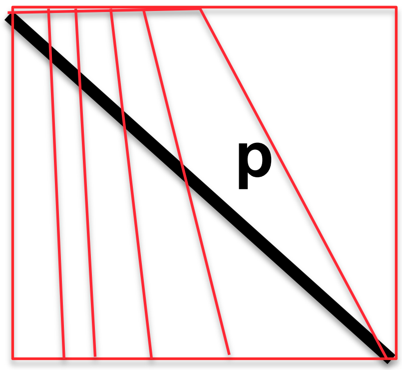
p belongs to all maximal ideals to which each p_i belongs, but fails to belong to the prime ideal of all McNaughton functions which have zero derivative at the origin. All p_i have 0 derivative at the origin.

Valuations induced by maximal ideals are not sufficient to check consequence

One needs a richer class of valuations, also taking into account the stability properties common to all premises



stable



Bolzano-Tarski
not stable

refining the Bolzano-Tarski paradigm in infinite-valued Łukasiewicz logic

the dynamics of a material point
is controlled by position and speed

the dynamics of a material point
is controlled by position and speed

the dynamics of a rigid body is
controlled by more parameters:
e.g., angular speed along each axis.

the dynamics of a material point
is controlled by position and speed

the dynamics of a rigid body is
controlled by more parameters:
e.g., angular speed along each axis.

the deductive closure of a set *F* in Łukasiewicz logic needs:
—pointwise evaluation of f over the modelset of F
—evaluation of derivatives along all pairwise orthogonal
directions associated to the prime ideals containing *F*.

classical consequence (static evaluation)

f is a ***Bolzano-Tarski consequence*** *of a set F of premises if for every model (= valuation) v,*

$$p(v) = 1 \quad \textit{for all } p \textit{ in } F,$$

$$\Downarrow$$

$$f(v)=1$$

stable consequence (perturbative evaluation)

f *is a* ***stable consequence*** *of a set F of premises if for every valuation* v *and perturbation* $\mathrm{d}v$ *in the valuation space,*

$$p(v) = 1 \quad \textit{and} \quad p(v+\mathrm{d}v) = 0 \quad \textit{for all} \quad p \quad \textit{in } F$$

$$\Downarrow$$

$$f(v+\mathrm{d}v) = 1$$

stable consequence (quantitative evaluation)

f *is a* ***stable consequence*** *of a set* F *of premises if for every valuation* v *and direction* d *in the valuation space,*

$$p(v) = 1 \quad and \quad \partial p(v)/\partial d = 0 \quad for\ all \quad p \quad in\ F$$

$$\Downarrow$$

$$f(v) = 1 \quad and \quad \partial f(v)/\partial d = 0$$

derivatives arise from the classification of prime ideals

the completeness theorem

THEOREM (D.M. 2013, Outstanding Contributions to Logic, 6, to Petr Hàjek)

The following conditions are equivalent for f:

- f *is a stable consequence of a set* F *of premises*
- f *is a syntactic consequence of* F, *i.e.,* f *is obtainable by finitely many applications of modus ponens from* (F $\cup$ *the tautologies)*
- f *belongs to each prime ideal that contains* F.

the completeness theorem

THEOREM (D.M. 2013, Outstanding Contributions to Logic, 6, to Petr Hàjek)

The following conditions are equivalent for f:

- f *is a stable consequence of a set* F *of premises*
- f *is a syntactic consequence of* F, *i.e.,* f *is obtainable by finitely many applications of modus ponens from* ($F \cup$ *the tautologies*)
- f *belongs to each prime ideal that contains* F.

COROLLARY *Stable consequence is* ***finitary****: if* f *is a stable consequence of* F *then* f *is a stable consequence of a finite subset of* F

stable consequence and differential evaluations

Differential valuations make sense because the valuation space is a cube $[0,1]^n$, and formulas code $[0,1]$-valued functions over $[0,1]^n$ having all directional **derivatives**.

stable consequence and differential evaluations

Differential valuations make sense because the valuation space is a cube $[0,1]^n$, and formulas code $[0,1]$-valued functions over $[0,1]^n$ having all directional **derivatives**.

Geometrically, stable consequence states that whevever all premises are true under a **perturbation** of their models, so is the conclusion.

stable consequence and differential evaluations

Differential valuations make sense because the valuation space is a cube $[0,1]^n$, and formulas code $[0,1]$-valued functions over $[0,1]^n$ having all directional **derivatives**.

Geometrically, stable consequence states that whevever all premises are true under a **perturbation** of their models, so is the conclusion.

Algebraically, s is a stable consequence of a set F of premises iff s is a member of every **prime** ideal $P \supseteq F$ of the free algebra.

stable consequence and differential evaluations

Differential valuations make sense because the valuation space is a cube $[0,1]^n$, and formulas code $[0,1]$-valued functions over $[0,1]^n$ having all directional **derivatives**.

Geometrically, stable consequence states that whevever all premises are true under a **perturbation** of their models, so is the conclusion.

Algebraically, s is a stable consequence of a set F of premises iff s is a member of every **prime** ideal P⊇F of the free algebra.

In the boolean fragment, stable consequence = Bolzano-Tarski consequence, stating that s is a member of every **maximal** ideal P containing F. The valuation space $\{0,1\}^n$ is totally disconnected.

a recipe for semantics to meet syntax in **other logics**

one may single out a class $\mathcal{K}$ of congruences having the following **irreducibility** property: every congruence is an intersection of congruences in the class $\mathcal{K}$. In general, maximals won't do.

a recipe for semantics to meet syntax in **other logics**

one may single out a class $\mathcal{K}$ of congruences having the following **irreducibility** property: every congruence is an intersection of congruences in the class $\mathcal{K}$. In general, maximals won't do.

one may then look for the **quantitative/algebraic** content of the quotient map $f \rightarrow f/k$ for k a congruence in the special class $\mathcal{K}$. This is what one expects from a **valuation**.

a recipe for semantics to meet syntax in **other logics**

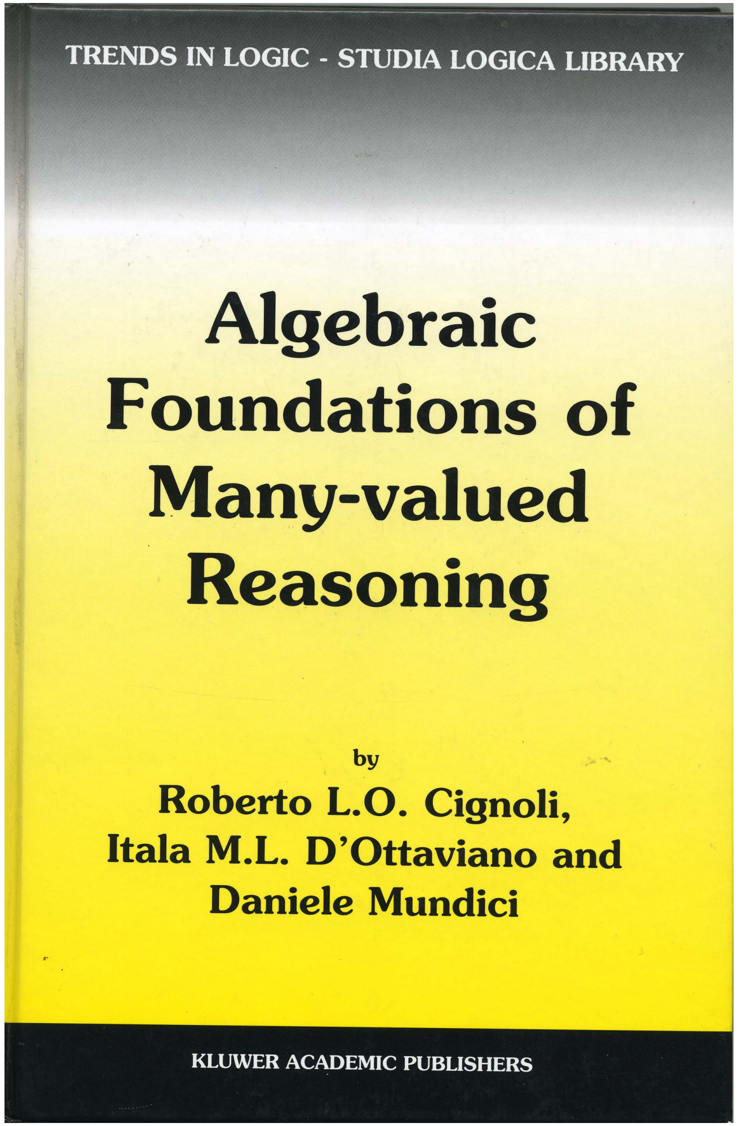
one may single out a class $\mathcal{K}$ of congruences having the following **irreducibility** property: every congruence is an intersection of congruences in the class $\mathcal{K}$. In general, maximals won't do.

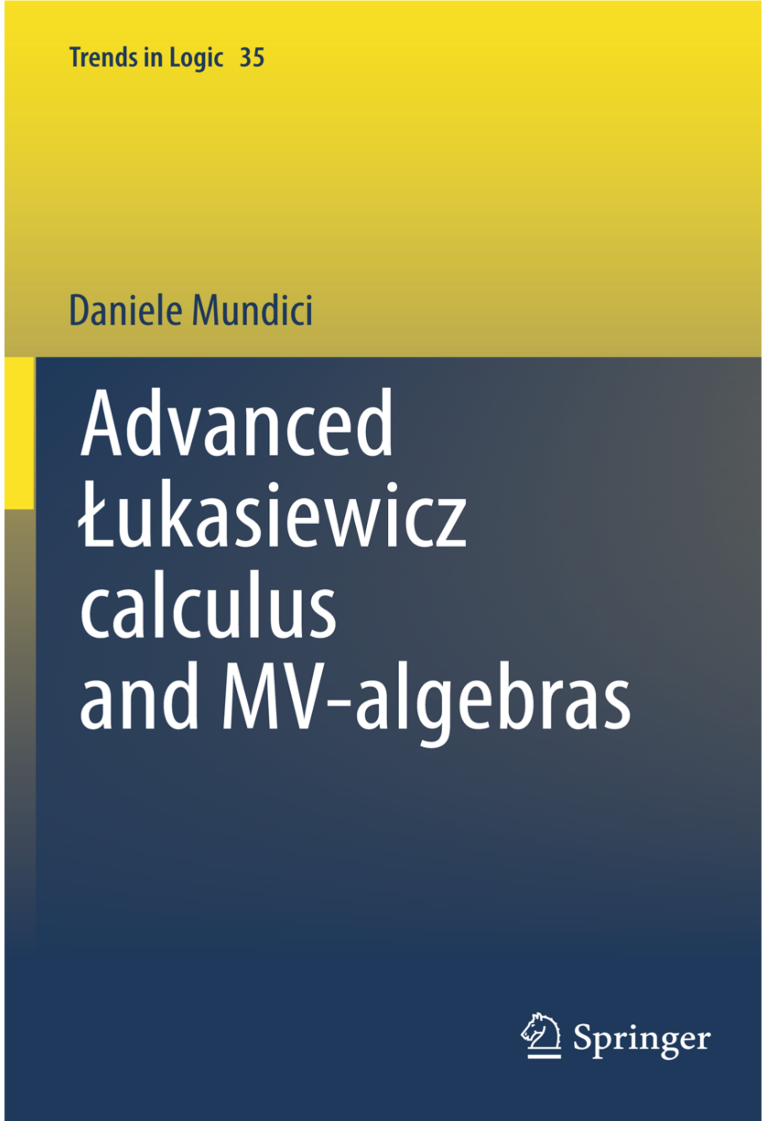
one may then look for the **quantitative/algebraic** content of the quotient map $f \to f/k$ for k a congruence in the special class $\mathcal{K}$. This is what one expects from a **valuation**.

Traditional candidates for $\mathcal{K}$: maximal ideals (in the lucky case of boolean algebras and finite-valued logics), prime ideals, finitely subdirectly irreducible ideals or congruences,...

THANK YOU

2000





2011