Logical Foundations of Categorization Theory

4th SYSMICS Workshop: Duality in algebra and logic

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joint ongoing work with Willem Conradie, Peter Jipsen, Krishna Manoorkar, Sajad Nazari, Nachoem Wijnberg...

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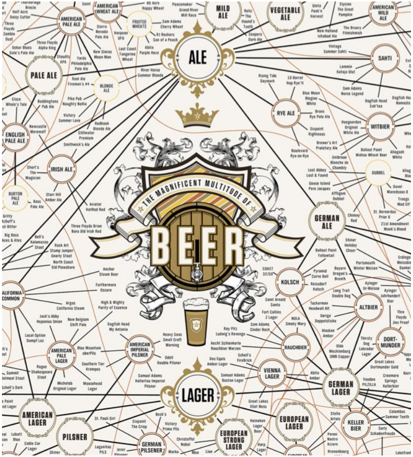
What is categorization?

From Wikipedia:
Categorization is the process in which ideas and objects are recognized, differentiated, and understood.
Ideally, a category illuminates a relationship between the subjects and objects of knowledge.
Categorization is fundamental in language, prediction, inference, decision-making and in all kinds of environmental interaction.



Overview and General Motivation

- **Truly interdisciplinary:** philosophy, cognition, social/management science, linguistics, AI.
- rapid development, **different approaches**;
- emerging **unifying perspective:** categories are dynamic in their essence; they shape and are shaped by processes of social interaction.
- **Data-driven** developments, both empirical and theoretical.
- However, what is **lacking**:
  - a **common ground** for the various approaches;
  - formal models addressing **dynamics** and connections with the processes of **social interaction**.
- **Research program:** logic as common ground; dynamics as starting point rather than outcome; systematic connection between dynamics and processes of social interaction.

Contrasting Views on Categorization

Classical (Aristotle)

- membership in a category *defined* by satisfaction of features.
- categorization: *deductive* process of reasoning with necessary and sufficient conditions;
- categories have sharp boundaries; no unclear cases.
- categories are represented equally well by each of its members.

Prototype (Rosch)

- some category-members *more central* than others (prototypes).
- categorization: *inductive* process of establishing *similarity* to prototype;
- categories have fuzzy boundaries; membership is graded.

Meanwhile, in logic...

Mathematical theory of LE-logics (LE: lattice expansions)

the integrated SYSMICS approach:

- algebraic and Kripke-style semantics;
- generalized Sahlqvist theory;
- semantic cut elimination, FMP;
- Goldblatt-Thomason theorem.

Can we make intuitive sense of LE-logics?

Basic lattice logic & main ideas

Language: $\mathcal{L} \ni \phi ::= p \in Prop \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi$

Lattice Logic: Set of $\mathcal{L}$-sequents $\phi \vdash \psi$

- containing:
  $p \vdash p \quad \bot \vdash p \quad p \vdash \top \quad p \vdash p \vee q \quad q \vdash p \vee q \quad p \wedge q \vdash p \quad p \wedge q \vdash q$
- closed under:
  $\frac{\phi \vdash \chi \quad \chi \vdash \psi}{\phi \vdash \psi} \quad \frac{\phi \vdash \psi}{\phi(\chi/p) \vdash \psi(\chi/p)} \quad \frac{\chi \vdash \phi \quad \chi \vdash \psi}{\chi \vdash \phi \wedge \psi} \quad \frac{\phi \vdash \chi \quad \psi \vdash \chi}{\phi \vee \psi \vdash \chi}$

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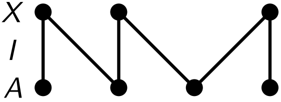
Challenge: Interpreting $\vee$ as 'or' and $\wedge$ as 'and' does not work, since 'and' and 'or' distribute over each other, while $\wedge$ and $\vee$ don't.
Proposal: Interpreting $\phi \in \mathcal{L}$ as **other entities** than sentences?
Examples: categories, concepts, theories, interrogative agendas.
The interpretation of $\vee$ and $\wedge$ in all these contexts is ok with failure of distributivity
Approach:

- Understand LE-logics as the logics of **these entities**;
- integrate LE-logics into more expressive logics capturing how these entities **interact** (e.g. with sentences, actions etc.).

Polarity-based semantics of LE-logics

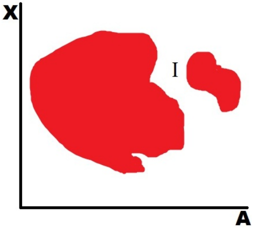
Formal contexts (A, X, I) are abstract representations of databases:



A: set of *Objects*

X: set of *Features*

$I \subseteq A \times X$. Intuitively, aIx reads: object a has feature x



Polarity-based semantics of LE-logics

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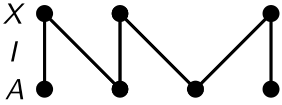

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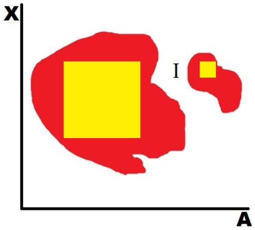

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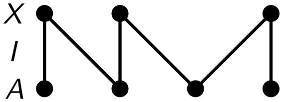

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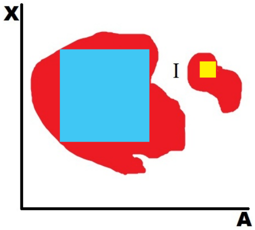

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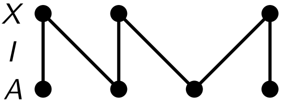

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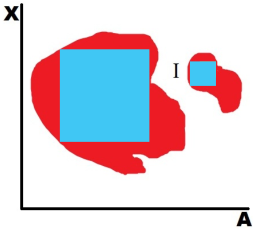
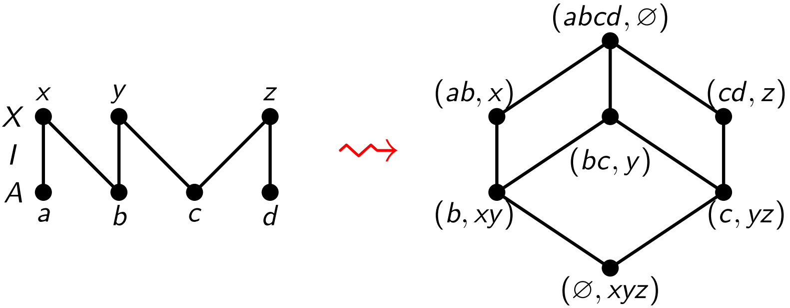


Formal concepts:
"rectangles" maximally contained in I

Complex algebras



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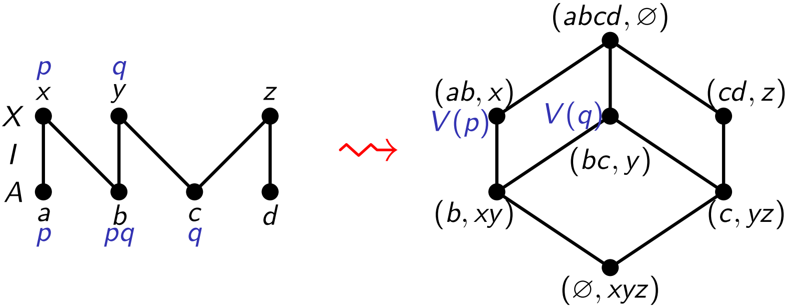
Formal contexts as $\mathcal{L}$-models



Let $\mathbb{P} = (A, X, I)$ and $\mathbb{P}^+$ be the complex algebra of $\mathbb{P}$.
Models: $\mathbb{M} := (\mathbb{P}, V)$ with $V : Prop \to \mathbb{P}^+$

$$V(p) := ([\![p]\!], (\![p]\!))$$

membership:	$\mathbb{M}, a \Vdash p$	iff	$a \in [\![p]\!]_{\mathbb{M}}$
description:	$\mathbb{M}, x \succ p$	iff	$x \in (\![p]\!)_{\mathbb{M}}$

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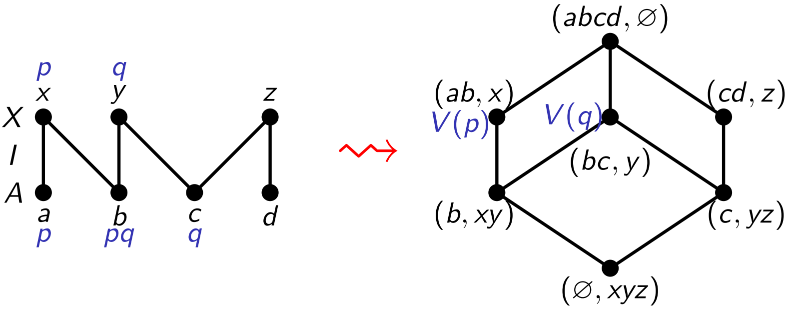
Formal contexts as $\mathcal{L}$-models



$$
\begin{array}{llll}
\mathbb{M}, a \Vdash \bot & \text{iff } \forall x(aIx) & \mathbb{M}, x \succ \bot & \text{always} \\
\mathbb{M}, a \Vdash \top & \text{always} & \mathbb{M}, x \succ \top & \text{iff } \forall a(aIx)
\end{array}
$$

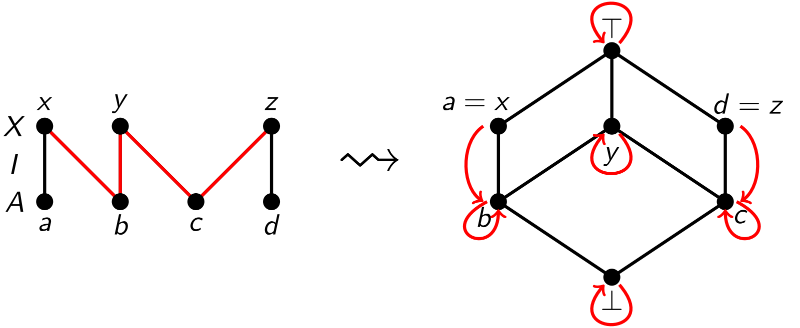
$$
\begin{array}{lll}
\mathbb{M}, a \Vdash \phi \wedge \psi & \text{iff} & \mathbb{M}, a \Vdash \phi \text{ and } \mathbb{M}, a \Vdash \psi \\
\mathbb{M}, x \succ \phi \wedge \psi & \text{iff} & \text{for all } a \in A \text{, if } \mathbb{M}, a \Vdash \phi \wedge \psi \text{, then } aIx \\
\mathbb{M}, a \Vdash \phi \vee \psi & \text{iff} & \text{for all } x \in X \text{, if } \mathbb{M}, x \succ \phi \vee \psi \text{, then } aIx \\
\mathbb{M}, x \succ \phi \vee \psi & \text{iff} & \mathbb{M}, x \succ \phi \text{ and } \mathbb{M}, x \succ \psi
\end{array}
$$

$$\mathbb{M} \models \phi \vdash \psi \;\text{ iff }\; [\![\phi]\!] \subseteq [\![\psi]\!] \;\text{ iff }\; (\![\psi]\!) \subseteq (\![\phi]\!)$$

Expanding the language with modal operators

Enriched formal contexts: $\mathbb{F} = (A, X, I, \{R_i \mid i \in Agents\})$

$R_i \subseteq A \times X$ and $\forall a((R^{\uparrow}[a])^{\downarrow\uparrow} = R^{\uparrow}[a])$ and $\forall x((R^{\downarrow}[x])^{\uparrow\downarrow} = R^{\downarrow}[x])$



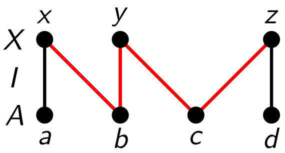
Language: $\mathcal{L}' \ni \phi ::= p \in Prop \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Box_i\phi$

$\Box_i\phi$: concept ϕ **according to** agent i

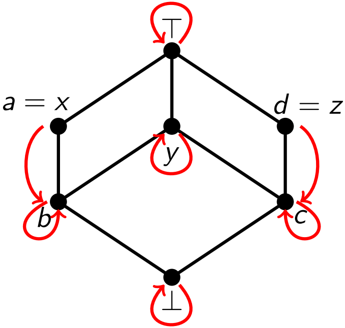
Logic:

- Additional axioms: $\top \vdash \Box_i\top \quad \Box_i\phi \wedge \Box_i\psi \vdash \Box_i(\phi \wedge \psi)$
- Additional rule: $\frac{\phi \vdash \psi}{\Box_i\phi \vdash \Box_i\psi}$

Interpretation of $\Box_i$-formulas on enriched formal contexts



$\rightsquigarrow$



$$V(\Box_i\phi) = \Box_i V(\phi) = (R_i^{\downarrow}[(\![\phi]\!)], (R_i^{\downarrow}[(\![\phi]\!)])^{\uparrow})$$

$\mathbb{M}, a \Vdash \Box_i\phi$ iff for all $x \in X$, if $\mathbb{M}, x \succ \phi$, then aR_ix

$\mathbb{M}, x \succ \Box_i\phi$ iff for all $a \in A$, if $\mathbb{M}, a \Vdash \Box_i\phi$, then aIx

Epistemic interpretation

'Factivity'

$\Box_i p \vdash p$ corresponds to

$$R_i \subseteq I$$

If agent i is aware that object a has feature x, then a has x 'objectively' (i.e. according to the database).

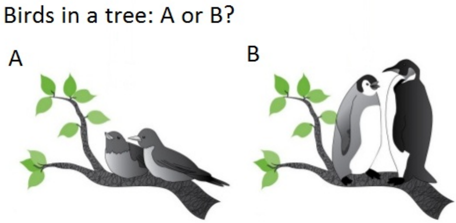
'Positive introspection'

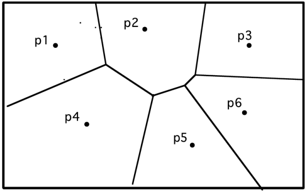
$\Box_i p \vdash \Box_i \Box_i p$ corresponds to $\forall x(R^{\downarrow}[x] \subseteq R^{\downarrow}[I^{\uparrow}[R^{\downarrow}[x]]])$, i.e.

$$R_i \subseteq R_i \,;R_i,$$

i.e. if agent i is aware that object a has feature x, then i must also be aware that a has all the features shared by all the objects which i is aware have feature x.

Core concept: Typicality





- in conceptual spaces, the **prototype** of a formal concept is defined as the **geometric center of that concept**;
- the closer (i.e. more similar) an object is to the prototype, the stronger its typicality.
- Advantage: **visually appealing**;
- Disadvantage: does not explain the **role of agents** in establishing the typicality of an object relative to a category.

Logical formalization of typicality

$i \in Agents$; let $S \ni s = i_1 \cdots i_n$ finite sequence of agents. Let

$$\mathcal{L}_C \ni \phi ::= p \in Prop \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Box_i \phi \mid C(\phi)$$

$$C(\phi) \text{ stands for } \bigwedge_{s \in S} \Box_s \phi$$

where for any $s \in S$, $\Box_s \phi := \Box_{i_1} \cdots \Box_{i_n} \phi$.
Hence $[\![C(\phi)]\!]$ can be understood as the set of **prototypes of** ϕ.

Interpretation of C-formulas on models

$$\begin{array}{lll} \mathbb{M}, a \Vdash C(\varphi) & \text{iff} & \text{for all } x \in X, \text{ if } \mathbb{M}, x \succ \varphi, \text{ then } aR_C x \\ \mathbb{M}, x \succ C(\varphi) & \text{iff} & \text{for all } a \in A, \text{ if } \mathbb{M}, a \Vdash C(\varphi), \text{ then } aIx, \end{array}$$

$R_C := \bigcap_{s \in S} R_s$, and $R_s \subseteq A \times X$ defined by induction on $s \in S$

- if $s = i$ then $R_s := R_i$;
- if $s = ti$, then $R_s^{\uparrow}[a] := R_t^{\uparrow}[I^{\downarrow}[R_i^{\uparrow}[a]]]$

Gradedness of non-typicality

if $a \notin [\![C(\phi)]\!]$ then

$$a \notin \bigcap_{s \in S} [\![\Box_s]\!]\phi = \bigcap_{s \in S} R_s^{\downarrow}[(\![\phi]\!)].$$

So a must fail the typicality test for some $s \in S$, and this failure can be more or less 'severe':
Definition: a is **at least as typical** as a member of ϕ than b is if

$$\{s \in S \mid b \in R_s^{\downarrow}[(\![\phi]\!)]\} \subseteq \{s \in S \mid a \in R_s^{\downarrow}[(\![\phi]\!)]\}.$$

Non epistemic interpretation: rough concepts

Conceptual approximation spaces: $\mathbb{F} = (A, X, I, R_\Box, R_\Diamond)$ with $R_\Box \subseteq A \times X$ and $R_\Diamond \subseteq X \times A$, I-compatible and s.t. $R_\blacksquare \,;\, R_\Box \subseteq I$.

Fact: $\mathbb{F} \models \Diamond p \vdash \Box p \quad$ iff $\quad R_\blacksquare \,;\, R_\Box \subseteq I$

$$
\begin{array}{lll}
\mathbb{M}, a \Vdash \Box(\varphi) & \text{iff} & \text{for all } x \in X \text{, if } \mathbb{M}, x \succ \varphi \text{, then } aR_\Box x \\
\mathbb{M}, x \succ \Box(\varphi) & \text{iff} & \text{for all } a \in A \text{, if } \mathbb{M}, a \Vdash \Box(\varphi) \text{, then } aIx, \\
\mathbb{M}, a \Vdash \Diamond\phi & \text{iff} & \text{for all } x \in X \text{, if } \mathbb{M}, x \succ \Diamond\phi \text{, then } aIx \\
\mathbb{M}, x \succ \Diamond\phi & \text{iff} & \text{for all } a \in A \text{, if } \mathbb{M}, a \Vdash \phi \text{, then } aR_\Diamond x.
\end{array}
$$

If (A, X, I) database and $R \subseteq A \times X$ I-compatible,

aIx stands for "object a has feature x"

aRx stands for "object a demonstrably has feature x"

If $R_\Box := R$ and $R_\Diamond := R^{-1}$, then

$[\![\Box\phi]\!] = \{a \in A \mid \forall x(x \succ \phi \Rightarrow aRx)\}$ **provable members** of ϕ.

$$(\![\Diamond\phi]\!) = \{x \in X \mid \forall a(a \Vdash \phi \Rightarrow aRx)\},$$

hence $[\![\Diamond\phi]\!] :=$ **possible members** of ϕ.