The Duality of Time and Information in Concurrency and Branching Time

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1. Outline

1. Chu spaces
2. Applications to program semantics: schedules and their dual automata
3. Chu-like categories
4. Presheaf-like categories
5. Typed Chu spaces
6. Applications to philosophy: dualism, properties, qualia

2. Chu spaces: Motivation

- Duality. ($\mathbf{Chu}(\mathbf{Set}, K)$ is a self-dual category).
- Generality. (Every category of relational structures of total arity n is embeds concretely in $\mathbf{Chu}(\mathbf{Set}, 2^n)$.
- Real-world applications, as e.g. per the title.

3. Chu spaces: Definition

Given a set K, a **Chu space over** K is a triple (A, r, X) such that A and X are sets and r is a function $r : A \times X \to K$ defining an $A \times X$ matrix over K. That is, $r(a, x)$ is the element at row a and column x.

Duality. The dual of a Chu space (A, r, X) is the Chu space $(A, r, X)\perp = (X, r^{\smile}, A)$.

4. Uncertainty principle for Chu spaces

A Chu space over $2 = 0, 1$ cannot have:

1. an all-ones row and an all-zeroes column, and dually;
2. closure of rows under OR and columns under AND, and dually.

5. Extensionality

- Extensional: No duplicate rows
- Normal: No duplicate columns
- Biextensional: extensional and normal

6. Representation of lattice-like objects as biextensional Chu spaces

Set: A discrete Chu space (no omitted columns) s.t. $|K| \geq 2$.

CABAs: A codiscrete Chu space (no omitted rows). $|K| \geq 2$.

Remaining examples take $K = \mathbf{2} = \{0, 1\}$.

Poset: columns closed under arbitrary OR and AND.

Distributive lattice: rows closed under arbitrary OR and AND

Complete join-semilattice: rows and columns closed under arbitrary OR.

Complete meet-semilattice: rows and columns closed under arbitrary AND.

T0 topological space: columns closed under arbitrary OR and finite AND.

Finite-dimensional vector space over GF2: finite, rows and columns closed under XOR. Necessarily $2^n \times 2^n$. (Infinite-dimensional case need not be square.)

Many other examples.

7. Properties of Chu spaces

A property of a Chu space is a set of omitted columns.

The property *true* is the empty set.

The maximal property is the set of all omitted columns.

Remark. The properties of a Chu space form a CABA.

The only property of a set is *true* (the degenerate CABA).

The properties of any other structure form a nondegenerate CABA.

8. Chu transforms $(f, g) : (A, r, X) \to (B, s, Y)$

Biextensional case: An $A \times Y$ matrix whose rows come from (B, s, Y) and whose columns come from (A, r, X). Row a of (f, g) equals row $f(a)$ of $\mathcal{B}$ while column y equals column $g(y)$ of (A, r, X).

Image of $(f, g) : (f(A), s', Y)$ s.t. $s'(b', y) = s(b, y)$ for all b in $f(A)$ and all y in Y.

Theorem

Any property of a Chu space is a property of its image under a Chu transform.

Proof.

Any omitted column of the source must be omitted from the transform. Hence its image under f must be omitted from the subspace $f(A)$ of the target. □

General case (no extensionality needed): A Chu transform is a pair (f, g) where $f : A \to B, g : Y \to X$ s.t. $s(f(a), y) = r(a, g(y))$ for all a in A, y in Y (adjointness).

9. Comonoids

A **comonoid** is a Chu space (A, r, X) over $\mathbf{2}$ such that

(i) (The counit) its states include the all-zeroes and all-ones columns.

(ii) (The comultiplication, aka the crossword property) Given any $A \times A$ matrix whose rows and columns are drawn from the columns of (A, r, X), its main diagonal is a column of X.

A T1 comonoid is one s.t. the underlying poset of the rows is discrete.

10. Are there any hard problems about Chu spaces?

Problem (open for 20 years). Is every T1 comonoid discrete?

True for countable coomonoids (known in 1995).

In 2015 Pace Nielsen answered this in the negative with a counterexample of cardinality $\aleph_1$. Accepted by JLMS.

George M. Bergman, Pace P. Nielsen, *On Vaughan Pratt's crossword problem*, Journal of the London Mathematical Society, Volume 93, Issue 3, 1 June 2016, Pages 825845, https://doi.org/10.1112/jlms/jdw011.

11. Interpretation as schedules and their dual automata

Simplest case: $K = \mathbf{2} = \{0, 1\}$.

Define a schedule to be a poset, and an automaton to be a distributive lattice. Interpret the rows of a schedule as its events, with the order constraining their order in time. Interpret its columns as the possible states of its dual automaton.

12. True concurrency via transition T

$K = \{0, T, 1\} = \mathbf{3}$.

Motivation: refine before-after to before-during-after, T being the state of transition of an event.

A connected event is the discrete 1×3 Chu space 0T1 over **3**, having three states, before, during, and after. Here T is understood geometrically as an edge connecting 0 and 1.

Represent true concurrence of two events as the discrete two-point Chu space over **3**. Its dual automaton, 9×2, consists of the 4 quiescent states 00, 01, 10, 11, the 4 semi-quiescent states 0T, T0, 1T, T1, and the truly concurrent state TT.

Represent $mutex(a, b)$ (mutual exclusion) as the 2×8 Chu space over **3** whose dual 8×2 automaton omits the row TT.

Each schedule over 2 gives rise to many schedules over **3** according which states containing T are omitted.

This gives a denotational semantics for higher dimensional automata, with the dimension of a state being simply the number of T's in it.

13. Branching time via cancellation

Motivation: Prior denotational semantics of branching time were all higher order and obscure. before-during-after-cancelled, as in "the event has been cancelled", represented by state X.

Distinguish $ab + ac$ from $a(b + c)$ as having the additional property that one of b or c must be cancelled before a starts.

14. Set-like categories

Motivation: A new way of defining $\mathbf{Chu}(\mathbf{Set}, K)$.

A locally small category $\mathcal{C}$ is **set-like** when it contains an object $\mathbf{1}$ such that

(i) $|C(\mathbf{1}, \mathbf{1})| = 1$ ($\mathbf{1}$ is rigid).

(ii) (Extensionality) For any two morphisms $f, g : X \to Y$ s.t. $fx = gx$ for all $x : \mathbf{1} \to X$, $f = g$.

Interpretation: Each object X of $\mathcal{C}$ represents the set $\mathcal{C}(\mathbf{1}, X)$ of elements of X, while each morphism $f : X \to Y$ represents the function defined by its left action on the elements of X.

Take **Set** to be the maximal set-like category whose objects are all homsets of all locally small categories. (Maximality fills up the homsets. Circular, but all definitions of **Set** are inevitably circular in one way or another.)

15. Chu-like categories

A locally small category $\mathcal{C}$ is **Chu-like over** K when it contains objects $\mathbf{1} = 1!K$ and $\bot = K!1$ such that

(i) $\mathbf{1}$ and $\bot$ are rigid.

(ii) (Biextensionality) For any two morphisms $f, g : ArX \to BsY$, if $yfa = yga$ for all $a : \mathbf{1} \to ArX$ and all $y : BsY \to \bot$, then $f = g$.

Interpret K as $\mathcal{C}(\mathbf{1}, \bot)$, For any object ArX, interpret $\mathcal{C}(\mathbf{1}, ArX)$ as A, $\mathcal{C}(ArX, \bot)$ as X, and xa as $r(a, x)$ for all $a \in A, x \in X$.

Interpret every homomorphism $h : ArX \to BsY$ as the pair (f, g) where $f : A \to B$ is the left action of h on $\mathcal{C}(\mathbf{1}, ArX)$ and g is the right action of h on $\mathcal{C}(BSY, \bot)$. That is, $f(a) = ha$ and $g(y) = yh$.

Every homomorphism $h : ArX \to BsY$ satisfies adjointness as a consequence of associativity of composition, as displayed thus.

$$\mathbf{1} \xrightarrow{a} ArX \xrightarrow{h} BsY \xrightarrow{y} \bot$$

The Chu-like categories over K are the subcategories $\mathcal{C}$ of $\mathbf{Chu}(Set, K)$ defined analogously to the set-like categories.

A locally small category $\mathcal{C}$ is **Grph-like** when it contains objects V, E such that

(i) V and E are rigid;

(ii) $\mathcal{C}(V,E) = \{s,t\}$ and $\mathcal{C}(E,V)$ is empty;

(iii) (Extensionality) For any two morphisms $f,g : G \to H$, if $fv = gv$ for all $v : V \to G$, and $fe = ge$ for all $e : E \to G$, then $f = g$.

The **Grph**-like categories are the subcategories $\mathcal{C}$ of **Grph** that contain a graph V consisting of one vertex v and no edges, and a graph E consisting of one edge e and distinct vertices es, et, such that the subcategory is full on the carriers $\mathcal{C}(V,G)$ (the vertices of G) and $\mathcal{C}(E,G)$ (the edges of G).

17. Presheaf-like categories

These generalize **Grph**-like categories uniformly.

Given a small category $\mathcal{J}$, a presheaf-like category $\mathcal{C}$ is one for which $\mathcal{J}$ is a full subcategory of $\mathcal{C}$ and the morphisms of $\mathcal{C}$ are extensional on homsets $\mathcal{C}(j, X)$ for all $j \in ob(J)$.

18. Typed Chu spaces

These are a common generalization of Chu-like categories and presheaf-like categories.

Given a profunctor (distributor, (bi)module) $K : \mathcal{L} \not\rightarrow \mathcal{J}$ between two small categories $\mathcal{L}$ and $\mathcal{J}$, a typed-Chu-space-like category $\mathcal{C}$ is one for which K is a subcategory of $\mathcal{C}$ whose objects are those of $L = ob(\mathcal{L})$ and $J = ob(\mathcal{J})$ and whose morphisms are all morphisms $k : J \rightarrow L$ of K. (The notation $\mathcal{L} \not\rightarrow \mathcal{J}$ is per convention, though Peter Johnstone prefers $\mathcal{J} \not\rightarrow \mathcal{L}$, reasonably.)

This creates a framework in which the objects of $\mathcal{J}$ serve as the sorts of the points of a typed Chu space while the objects of $\mathcal{L}$ serve as the properties of its states. The adjointness chain generalizes as follows.

$$j' \xrightarrow{f} j \xrightarrow{a} A \xrightarrow{h} B \xrightarrow{y} \ell \xrightarrow{\pi} \ell'$$

19. Applications to philosophy

1. Cartesian dualism: A as body, X as mind, $r : A \times X \to K$ as a mathematical pituitary gland.
2. (Co)extensionality of properties via their *possible* states.
3. Qualia as the morphisms $k : j \to \ell$, e.g. the possible colors of a cat, the possible heights of buildings, ... Qualia are both points and states, expressing C.I.Lewis's intutions about the psychological vs. physical ambiguity of the nature of qualia.

THANK YOU