

# KAKUTANI DUALITY FOR GROUPS

*Based on a joint work V. Marra*

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The problem

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# Embedding spaces

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It is well known that

*every compact Hausdorff space  $X$  can be embedded in some hypercube  $[0, 1]^J$  for some index set  $J$ .*

Suppose that  $X$  is now endowed with a function  $\delta: X \rightarrow \mathbb{N}$ .

## Problem

Given a pair  $\langle X, \delta \rangle$ , is there a continuous embedding  $\iota: X \rightarrow [0, 1]^J$  in such a way that the **denominators** of the points in  $\iota[X]$  **agree with  $\delta$** ?

Let us assume that “agree” means that  $\delta(x) = \text{den}(\iota(x))$ .

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# Denominators

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Recall that  $\mathbb{N}$  forms a **complete lattice under the divisibility order**: the top being 0 and the bottom being 1.

Let  $J$  be a set and  $\bar{p} \in [0, 1]^J$ . If  $\bar{p} \in \mathbb{Q}^J$  we define its **denominator** to be the natural number

$$\text{den}(\bar{p}) = \text{lcd}\{p_i \mid i \in J\}$$

where **lcd** stands for **the least common denominator**. If  $\bar{p} \notin \mathbb{Q}^J$  we set  $\text{den}(\bar{p}) = 0$ .

1. A function  $f: [0, 1]^J \rightarrow [0, 1]$  **preserves** denominators if for any  $\bar{x} \in [0, 1]^J$ ,  $\text{den}(f(\bar{x})) = \text{den}(\bar{x})$ .
2. A function  $f: [0, 1]^J \rightarrow [0, 1]$  **respects** denominators if for any  $\bar{x} \in [0, 1]^J$ ,  $\text{den}(f(\bar{x})) \mid \text{den}(\bar{x})$ .

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# An easy counter-example

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Consider  $X = [0, 1]$  with its Euclidean topology and endow it with a constant  $\delta$ :

$$\forall x \in X \quad \delta(x) = 1.$$

The only points with denominator equal 1 in  $[0, 1]^J$  are the so-called **lattice points** i.e., points whose coordinates are either 0 or 1.

The only way  $\iota$  could agree with  $\delta$  is to send all points in one lattice point —**failing injectivity**— or by sending the points in different lattice points —**failing continuity**.

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# MV-algebras

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The above mentioned problem is crucial in the duality theory of **MV-algebras** —the equivalent algebraic semantics of Łukasiewicz logic.

An MV-algebra is a structure  $\langle A, \oplus, \neg, 0 \rangle$  such that

1.  $\langle A, \oplus, 0 \rangle$  is a commutative monoid,
2.  $\neg\neg x = x$ ,
3.  $\neg 0 \oplus x = \neg 0$
4.  $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ .

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## Example

The interval  $[0, 1]$  in the real numbers has a natural MV-structure given by the **truncated sum**  $x \oplus y = \min\{x + y, 1\}$  and  $\neg x = 1 - x$ . The importance of this structure comes from the fact that **it generates the whole variety of MV-algebras**.

# MV-algebras and compact spaces

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Theorem (Marra, S. 2012)

*Semisimple MV-algebras with their homomorphisms form a category that is dually equivalent to the category of compact Hausdorff spaces embedded in some hypercube, with  $\mathbb{Z}$ -maps among them.*

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Definition

For  $I, J$  arbitrary sets, a map from  $\mathbb{R}^I$  into  $\mathbb{R}^J$  is called  $\mathbb{Z}$ -map if it is continuous and piecewise (affine) linear map, where each (affine) linear piece has integer coefficients.

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Remark

Since every  $\mathbb{Z}$ -map  $f$  acts on each point as an linear function with integer coefficients, it respect denominators i.e.,

$$\text{den}(f(x)) \mid \text{den}(x).$$

# Mundici's functor

An **abelian  $l$ -group with order unit** (**ul-group**, for short), is a partially **ordered Abelian group**  $G$  whose order is a **lattice**, and that possesses an element  $u$  such that

for all  $g \in G$ , there exists  $n \in \mathbb{N}$  such that  $(n)u \geq g$ .

The functor  $\Gamma$  that takes an ul-group  $\langle G, +, -, 0, u \rangle$  to its **unital interval**  $[0, u]$  with operation  $\oplus$  and  $\neg$  defined as follows:

$$x \oplus y = \min\{u, x + y\} \quad \text{and} \quad \neg x = u - x,$$

is **full, faithful, and dense** hence it has a quasi-inverse  $\Xi$  and

## Theorem (Mundici 1986)

*The pair  $\Gamma, \Xi$  gives an equivalence of categories between the category of MV-algebras with their morphisms, and the category of ul-groups with ordered group morphisms preserving the order unit.*

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# Norm induced by the order unit

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## Definition

Let  $(G, u)$  be a  $ul$ -group. The order unit  $u$  induces a seminorm  $\| \cdot \|_u$  defined as follows:

$$\|g\|_u := \inf \left\{ \frac{p}{q} \in \mathbb{Q} \mid p, q \in \mathbb{N}, q \neq 0 \text{ and } q|g| \leq pu \right\}$$

The seminorm  $\| \cdot \|_u: G \rightarrow \mathbb{R}^+$  is in fact a norm if, and only if,  $G$  is archimedean. **Any semisimple MV-algebra  $A$  inherits a norm** from its enveloping (archimedean) group  $\Xi(A)$ .

## Definition

An **norm-complete MV-algebra** is a semisimple MV-algebra which is Cauchy-complete w.r.t. its induced norm.

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# Kakutani duality

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## Theorem (Kakutani-Yosida duality 1941)

A *unital real vector lattice*  $(V, u)$  is isomorphic to  $(C(X), 1)$  for some *compact Hausdorff space*  $X$ , if, and only if,  $V$  is *Archimedean and norm-complete* (with respect to the norm  $\| \cdot \|_u$  induced by the unit).

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## Question

What if we want to substitute *ul-group* for *real vector lattice* in the above statement?

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## Remark

An answer was already given by Stone: compact Hausdorff spaces correspond to Archimedean, complete and *divisible* *ul-groups*.

# Denominator preserving maps

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## Theorem (Goodearl-Handelman 1980)

Let  $X$  be a compact Hausdorff space. For each  $x \in X$  choose  $A_x$  to be either  $A_x = \mathbb{R}$  or  $A_x = (\frac{1}{n})\mathbb{Z}$ . Then, the algebra of functions

$$\{f \in C(X) \mid f(x) \in A_x \text{ for all } x \in X\},$$

is a norm-complete  $ul$ -group and every such a group can be represented in this way.

As a corollary we obtain

## Corollary

The norm-completion of the algebra of  $\mathbb{Z}$ -maps is given by all continuous maps which respect denominators.

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# A duality for norm-complete MV-algebras

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## The category $\mathbf{MV}$

Let  $\mathbf{MV}$  be the category whose objects are **semisimple MV-algebras** and arrows are MV-homomorphisms.

## The category $\mathbf{A}$

Let  $\mathbf{A}$  be the category whose objects are **pairs  $\langle X, \delta \rangle$** , where  $X$  is a **compact Hausdorff space** and  $\delta$  is a map from  $X$  into  $\mathbb{N}$ . An arrow between two objects  $\langle X, \delta \rangle$  and  $\langle Y, \delta' \rangle$  is a continuous map  $f: X \rightarrow Y$  that **respects denominators**, i.e.,

$$\delta'(f(x)) \mid \delta(x).$$

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# A duality for norm-complete MV-algebras

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## The functor $\mathcal{L}$

Let  $\mathcal{L}: \mathbb{A} \rightarrow \mathbf{MV}$  be the assignment that associates to every object  $\langle X, \delta \rangle$  in  $\mathbb{A}$  the MV-algebra

$$\mathcal{L}(\langle X, \delta \rangle) := \{g \in \mathbf{C}(X) \mid \forall x \in X \quad \text{den}(g(x)) \mid \delta(x)\},$$

and to any  $\mathbb{A}$ -arrow  $f: \langle X, \delta \rangle \rightarrow \langle Y, \delta' \rangle$  the MV-arrow that sends each  $h \in \mathcal{L}(\langle Y, \delta' \rangle)$  into the map  $h \circ f$ .

# A duality for norm-complete MV-algebras

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## The functor $\mathcal{M}$

Let  $\mathcal{M}: \mathbf{MV} \rightarrow \mathbb{A}$  be the assignment that associates to each MV-algebra  $A$ , the pair  $\langle \text{Max}(A), \delta_A \rangle$ , where  $\text{Max}(A)$  is maximal spectrum of  $A$  and, for any  $\mathfrak{m} \in \text{Max}(A)$ ,

$$\delta_A(\mathfrak{m}) := \begin{cases} n & \text{if } A/\mathfrak{m} \text{ has } n+1 \text{ elements} \\ 0 & \text{otherwise.} \end{cases}$$

Let also  $\mathcal{M}$  assign to every MV-homomorphism  $h: A \rightarrow B$  the map that sends every  $\mathfrak{m} \in \mathcal{M}(B)$  into its inverse image under  $h$ , in symbols  $\mathcal{M}(h)(\mathfrak{m}) = h^{-1}[\mathfrak{m}] \in \text{Max}(A)$ .

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# A duality for norm-complete MV-algebras

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## Theorem

The functors  $\mathcal{L}$  and  $\mathcal{M}$  form a *contravariant adjunction*.

So, what is left to do in order to find a duality is to characterise the fixed points on each side.

It is quite easy to see the the fixed points on the algebraic side are exactly the norm-complete MV-algebras.

What are the fixed points on the topological side?

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# A-normal spaces

## Definition

An object  $\langle X, \delta \rangle$  in  $A$  is said **a-normal** (for arithmetically normal) if for any pair of points  $x, y \in X$  such that  $x \neq y$ ,

1. if  $\delta(y) \neq 0$ , then, letting  $d := \frac{1}{\delta(y)}$ , there exists a family of open sets  $\{O_q \mid q \in (0, d) \cap \mathbb{Q}\}$
2. if  $\delta(y) = 0$ , then there are infinitely many  $d \in [0, 1]$  such that for each of those there exists a family of open sets  $\{O_q \mid q \in (0, d) \cap \mathbb{Q}\}$

the families  $\{O_p\}$  are such for any  $p, q \in (0, d) \cap \mathbb{Q}$  and  $n \in \mathbb{N}$

1.  $p < q$  implies  $\{x\} \subseteq O_p \subseteq \overline{O_p} \subseteq O_q \subseteq \overline{O_q} \subseteq \{y\}^c$ .
2.  $\delta^{-1}[\{n\}] \subseteq \bigcup \{O_p \mid \text{den}(p) \mid n\}$ .

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# A-normal spaces

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## Theorem

*For any set  $I$ , the  $a$ -space  $\langle [0, 1]^I, \text{den} \rangle$  is  $a$ -normal.*

## Lemma (A-normality is weakly hereditary)

*If an  $a$ -space  $\langle X, \delta \rangle$  is  $a$ -normal, then so are all its closed  $a$ -subspaces.*

## Theorem

*An  $a$ -space  $\langle X, \delta \rangle$  is  $a$ -normal if, and only if, there exist a set  $I$  and an  $a$ -iso from  $X$  into an  $a$ -subspace of  $\langle [0, 1]^I, \text{den} \rangle$ .*



# Sketch of the proof

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The key step in the proof is to show that there are enough **good** functions to separate points:

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## Theorem

Let  $\langle X, \delta \rangle$  be an  $a$ -normal space. For any pair of distinct points  $x, y \in X$ ,

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1. if  $\delta(y) \neq 0$ , then there exists a **denominator respecting, continuous function**  $f: X \rightarrow [0, 1]$  such that  $f(x) = 0$  and  $f(y) = \frac{1}{\delta(y)}$
2. if  $\delta(y) = 0$ , then there are infinitely many  $d \in [0, 1]$  such that for each of them there is a **denominator respecting, continuous function**  $f: X \rightarrow [0, 1]$  such that  $f(x) = 0$  and  $f(y) = d$

# Sketch of the proof

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Then we can use

## Theorem (Kelley's Embedding Lemma)

Let  $X$  and  $Y$  be topological spaces and  $\mathcal{F}$  be a family of functions from  $X$  to  $Y$ . Suppose that all functions in  $\mathcal{F}$  are *continuous* and that they *separate points*. Then the evaluation map  $\text{ev}: X \rightarrow Y^{\mathcal{F}}$  given by

$$\text{ev}(x) = (f(x))_{f \in \mathcal{F}}$$

is continuous and injective.

# Sketch of the proof

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It is immediate to see that if all functions in  $\mathcal{F}$  respect denominators, then so does  $\text{ev}$ . Additionally, if  $x \in X$ , then

1. if  $\delta(x) \neq 0$ , then the value  $\frac{1}{\delta(x)}$  is attained by some  $f$  on  $x$ , so the function  $\text{ev}$  actually **preserves**  $\delta(x)$ ;
2. if  $\delta(x) = 0$ , then there infinitely  $a$ -maps  $f$  that on  $x$  attain infinitely different values, hence  $\text{den}(\text{ev}(x)) = 0$ .

# Kakutani duality, for groups

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## Corollary

The category of norm-complete archimedean  $ul$ -groups is **dually equivalent** to the full subcategory of  $\mathbb{A}$  given by all  **$a$ -normal spaces**.

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## Corollary

The category of norm-complete MV-algebras is **dually equivalent** to the full subcategory of  $\mathbb{A}$  given by all  **$a$ -normal spaces**.

Thanks for your attention!