Undecidability of $\mathsf{FL_e}$ in the presence of structural rules

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4th SYSMICS Workshop
Chapman University
Orange, California

17 September 2018

Application 6. Residuated frames and (un)decidability

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Definition

A (commutative) **residuated lattice** is an algebraic structure $\mathbf{R} = (R, \vee, \wedge, \cdot, \backslash, /, 1)$, such that

- $(R, \vee, \wedge)$ is a lattice
- $(R, \cdot, 1)$ is a (commutative) monoid
- For all $x, y, z \in R$

  $$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y,$$

  where $\leq$ is the induced lattice order.

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(Commutative) residuated lattices form a variety, denoted by $(\mathcal{C})\mathcal{RL}$.
If $[\mathrm{r}]$ is a rule (axiom), then $(\mathcal{C})\mathcal{RL} + [\mathrm{r}]$ denotes the subvariety by adjoining $[\mathrm{r}]$.

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$[\mathrm{k}_n^m]$ denotes the knotted rule

$$\mathcal{RL} : \ (\forall x)\ x^n \leq x^m \quad \Bigg| \quad \mathsf{FL} : \ \frac{\Pi, \overbrace{X, ..., X}^{m}, \Sigma \vdash \psi}{\Pi, \underbrace{X, ..., X}_{n}, \Sigma \vdash \psi}$$

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  ∘ Proof theoretically, such axioms correspond to inference rules, e.g.,

$$x \leq x^2 \vee 1 \iff \frac{\Pi, X, X, \Sigma \vdash \psi \quad \Pi, \Sigma \vdash \psi}{\Pi, X, \Sigma \vdash \psi}$$

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  ∘ So we restrict our investigation to the commutative case.

Residuated frames

Definition [Galatos & Jipsen 2013]

A **residuated frame** is a structure $\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!/, 1)$, s.t.

- $(W, \circ, 1)$ is a monoid and W' is a set.
- $N \subseteq W \times W'$, called the *Galois relation,* and
- $\backslash\!\backslash : W \times W' \to W'$ and $/\!/ : W' \times W \to W'$ such that
- N is a **nuclear**, i.e. for all $u, v \in W$ and $w \in W'$,
  $$(u \circ v)\ N\ w \quad \text{iff} \quad u\ N\ (w \,/\!/\, v) \quad \text{iff} \quad v\ N\ (u \,\backslash\!\backslash\, w).$$

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Define $^{\triangleright} : \mathcal{P}(W) \to \mathcal{P}(W')$ and $^{\triangleleft} : \mathcal{P}(W') \to \mathcal{P}(W)$ via
$X^{\triangleright} = \{y \in W' : \forall x \in X,\ xNy\}$ for each $X \subseteq W$ and
$Y^{\triangleleft} = \{x \in W : \forall y \in Y,\ xNy\}$ for each $Y \subseteq W'$.
Then $(^{\triangleright}, ^{\triangleleft})$ is a Galois connection.
So γ_N defined via $X \xmapsto{\gamma_N} X^{\triangleright\triangleleft}$ is a closure operator on $\mathcal{P}(W)$.
Fact: N is nuclear iff γ_N is a nucleus.

Residuated frames cont.

Theorem [Galatos & Jipsen 2013]

$\mathbf{W}^+ := (\gamma_N[\mathcal{P}(W)], \cup_{\gamma_N}, \cap, \circ_{\gamma_N}, \backslash, /, \gamma_N(\{1\}))$,

$$X \cup_{\gamma_N} Y = \gamma_N(X \cup Y) \text{ and } X \circ_{\gamma_N} Y = \gamma_N(X \circ Y),$$

is a residuated lattice.

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∘ We can encode "desirable properties" we want a RL to satisfy in N.
∘ In particular, (simple) rules in the signature $\langle \vee, \cdot, 1 \rangle$ are preserved via $(-)^+$,

Rules in the signature $\langle \vee, \cdot, 1 \rangle$ and Linearization

Any equation $s = t$ in the signature $\langle \vee, \cdot, 1 \rangle$ is equivalent to some conjunction of **simple rules**.

$$(\mathsf{d}) \quad x_1 \cdots x_n \leq \bigvee_{j=1}^{m} x_1^{d_j(1)} \cdots x_n^{d_j(n)},$$

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- $x \leq y \iff x \vee y = y$
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  E.g., the rule
  $$(\forall u)(\forall v)\ u^2 v \leq u^3 \vee uv$$
  is equivalent to, via the substitution σ: $u \overset{\sigma}{\mapsto} x \vee y$ and $v \overset{\sigma}{\mapsto} z$,
  $$(\forall x)(\forall y)(\forall z)\ xyz \leq x^3 \vee x^2 y \vee xy^2 \vee y^3 \vee xz \vee yz$$

Simple rules and Residuated Frames

Let $\mathbf{W} = (W, W', N)$ be a residuated frame and (d) be the simple rule given by

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We say $\mathbf{W} \models [\mathsf{d}]$ iff for all $u_1, ..., u_n \in W$ and $v \in W'$, the following inference rule is satisfied

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Proposition [Galatos & Jipsen 2013]

All simple rules are preserved by $(-)^+$. In particular,

$$\mathbf{W} \models [\mathsf{d}] \text{ iff } \mathbf{W}^+ \models (\mathsf{d}).$$

The *Word Problem*

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We say $\mathcal{V}$ has an **undecidable word problem** if there exists a finite presentation $\langle X, E\rangle$ such that there is no algorithm deciding whether the q.e. ($\&E \implies s = t$) holds in $\mathcal{V}$ having $s, t \in T(X)$ as inputs.

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Or equivalently, there is a finitely presented algebra $\mathbf{A} \in \mathcal{V}$ generated by X such that the following set is undecidable:

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- undecidable word problem $\Rightarrow$ undecidable q.e. theory.

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  - **Increment register** r: $q \ {+r} \ q'$
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input configuration	inst.	output configuration
$\langle q; n_1, ..., n_i, ..., n_k\rangle$	$\xmapsto{q \; +r_i \; q'}$	$\langle q'; n_1, ..., n_i + 1, ..., n_k\rangle$
$\langle q; n_1, ..., n_i + 1, ..., n_k\rangle$	$\xmapsto{q \; -r_i \; q'}$	$\langle q'; n_1, ..., n_i, ..., n_k\rangle$
$\langle q; n_1, ..., 0, ..., n_k\rangle$	$\xmapsto{q \; 0r_i \; q'}$	$\langle q'; n_1, ..., 0, ..., n_k\rangle$

And-branching k-Counter Machines (k-ACM)

A k-ACM $M = (R_k, Q, P)$, as introduced by Lincoln, Mitchell, Scedrov, Shankar (1992), is a type of non-deterministic parallel-computing counter machine that has

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- and a finite set P of **instructions** p of the form:

  ◦ **Increment:** $q \leq^p q'r$
  ◦ **Decrement:** $qr \leq^p q'$
  ◦ **Fork:** $q \leq^p q' \vee q''$,

  where $q, q', q'' \in Q$ and $r \in R_k$.

ACM's continued

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where $C_1, \ldots, C_n$ are configurations.
- An instruction p is a function (relation) on ID's that can replace a *single* configuration C by an ID v, i.e.
$$C \vee u \leq^p v \vee u$$

Computations

We view computations as order relations on the free commutative semiring $\mathbf{A}_M = (A_M, \vee, \cdot, \bot, 1)$ generated by $Q \cup R_k$, where $M = (R_k, Q, P)$ is a k-ACM and

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- $(A_M, \vee, \bot)$ is a commutative monoid with identity $\bot = \bigvee \emptyset$,
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Each instruction $p \in P$ defines a relation $\leq^p$ closed under

$$\frac{u \leq^p v}{ux \leq^p vx}\ [\cdot] \qquad \text{and} \qquad \frac{u \leq^p v}{u \vee w \leq^p v \vee w}\ [\vee],$$

for $u, v, w, x \in A_M$.

Computations

We view computations as order relations on the free commutative semiring $\mathbf{A}_M = (A_M, \vee, \cdot, \bot, 1)$ generated by $Q \cup R_k$, where $M = (R_k, Q, P)$ is a k-ACM and

- $(A_M, \vee, \bot)$ is a commutative monoid with identity $\bot = \bigvee \emptyset$,
- $(A_M, \cdot, 1)$ is a commutative monoid with identity 1, and
- multiplication $(\cdot)$ distributes over "join" $(\vee)$.

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for $u, v, w, x \in A_M$. We define the **computation relation** $\leq_M$ to be the smallest $(\cdot, \vee)$-compatible preorder containing $\bigcup_{p \in P} \leq^p$.

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Example Machine

Let $M = M_{\mathrm{even}} := (\{r\}, \{q_0, q_1, q_f\}, \{p_1, p_2, p_3\})$, with instructions

$$q_0 r \leq^{p_1} q_1; \quad q_1 r \leq^{p_2} q_0; \quad q_0 \leq^{p_3} q_f \vee q_f.$$

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Undecidable Problem

Theorem [LMSS 1992]

There exists a 2-ACM M such that membership of the set $\{u \in \mathrm{ID}(M) : u \in \mathrm{Acc}(M)\}$ is undecidable.

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Let $M = (R_k, Q, P)$ be a k-ACM and $u \in \mathrm{ID}(M)$,

- We can define a quasi-equation $\mathrm{acc}_M(u)$ in the signature $\langle \vee, \cdot, 1\rangle$ via

$$\&P \implies u \leq q_f.$$

ACM's and Residuated Frames

Let $M = (R_k, Q, P)$ be a k-ACM and $W := (Q \cup R_k)^*$ be the free commutative monoid generated by $Q \cup R_k$.

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The frame $\mathbf{W_M}$

Inspired by Horčík (2015), we let $W' := W$ and define the relation $N_M \subseteq W \times W'$ via

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Lemma

$\mathbf{W}_M := (W, W', N_M)$ is a residuated frame, $\mathbf{W}_M^+ \in \mathcal{CRL}$, and there exists a valuation $\nu : \mathrm{Tm} \to W_M^+$ such that $\mathbf{W}_M^+, \nu \models \& P$.

ACM's and Residuated Frames cont.

Let M be a k-ACM and $\mathcal{V} \subseteq (\mathcal{C})\mathcal{RL}$ a variety.

Theorem

If $\mathbf{W}_M^+ \in \mathcal{V}$ then for all $u \in \mathrm{ID}(M)$,

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Corollary

Suppose membership of $\mathrm{Acc}(M)$ is undecidable. If $\mathbf{W}_M^+ \in \mathcal{V}$ then $\mathcal{V}$ has an undecidable word problem. In particular, $(\mathcal{C})\mathcal{RL}$ has an undecidable word problem since $\mathbf{W}_{\tilde{M}}^+ \in \mathcal{CRL}$, where $\tilde{M}$ is the machine from LMSS (1992).

Simple rules in k-ACM's and the relation $\leq_{\mathsf{d}M}$

Let $M = (R_k, Q, P)$ be a k-ACM. Given a simple rule, e.g. $(\mathsf{d}) : x \leq x^2 \vee x^4$, we add "ambient" instructions of the form

$$t \leq^{\mathsf{d}} t^2 \vee t^4 \;\; \left(\prod_{i=1}^{n} t_i \leq^{\bar{\mathsf{d}}} \bigvee_{j=1}^{m} \prod_{i=1}^{n} t_i^{d_j(i)} \right),$$

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- We denote this new machine by $\mathsf{d}M$.

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Lemma

Let $M = (R_k, Q, P)$ be a k-ACM and (d) a simple rule. Then $\mathbf{W}_{\mathsf{d}M} \models [\mathsf{d}]$, and therefore $\mathbf{W}_{\mathsf{d}M}^{+} \in \mathcal{CRL} + (\mathsf{d})$.

Admissibility of simple rules for a machine

Definition

Let M be a k-ACM and (d) be a d-rule. We say (d) is **admissible** in M if

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However, we will rephrase admissibility as the intermediate notions **register** and **state admissibility**.

Admissibility cont.

We define $\leq^{\bar{\mathsf{d}}}$ to be the "ambient" instruction, for each $x \in R_k^*$ $(x_1, ..., x_n \in R_k^*)$,

$$x \leq^{\bar{\mathsf{d}}} x^2 \vee x^4 \quad \left(\prod_{i=1}^n x_i \leq^{\bar{\mathsf{d}}} \bigvee_{j=1}^{m} \prod_{i=1}^n x_i^{d_j(i)} \right),$$

and define $\leq_{\bar{\mathsf{d}}M}$ as usual.

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We say (d) is **register (state) admissible in** M if $\mathrm{Acc}(M) = \mathrm{Acc}(\bar{\mathsf{d}}M)$ $(\mathrm{Acc}(\bar{\mathsf{d}}M) = \mathrm{Acc}(\mathsf{d}M))$.

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Theorem

Let M be a k-ACM and (d) a d-rule. Then (d) is *state*-admissible in M iff there is no substitution $\sigma : \mathrm{Var} \to \mathrm{Var}^*$ such that $\sigma[\mathsf{d}] \equiv x^k \leq x$ or $\sigma[\mathsf{d}] \equiv x^k \leq 1$.

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- However, $q_0 r^3 \in \mathrm{Acc}(\mathsf{d}M)$, witnessed by

$$q_0 r^3 = q_0 r^2 r \leq^{\mathsf{d}} q_0 r^2 r^2 \vee q_0 r^2 r^4 = q_0 r^4 \vee q_0 r^6 \in \mathrm{Acc}(M)$$

  since $q_0 r^4 \in \mathrm{Acc}(M)$ and $q_0 r^6 \in \mathrm{Acc}(M)$.

Goal

Given an ACM M and a d-rule (d), is it possible to construct a new ACM M' such that

$$(1)\ C \in \mathrm{Acc}(M) \iff \theta(C) \in \mathrm{Acc}(M')$$

(where $\theta : \mathrm{ID}(M) \to \mathrm{ID}(M')$ is some computable function), and

$$(2)\ (\mathsf{d}) \text{ is register-admissible in } M'?$$

And if so, under what conditions?

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Fact

For each $q \in Q$,

$$qr_1^{n_1} r_2^{n_2} \in \mathrm{Acc}(M) \iff qr_1^{K^{n_1}} r_2^{K^{n_2}} \in \mathrm{Acc}(M_K).$$

Detecting applications of $\leq^{\mathsf{d}}$

Observation

Consider a configuration where the contents of some register r is $n = s + t$, whereafter $\leq^{\mathsf{d}}$ is applied to t-many tokens, i.e.,

$$qr^n = qr^s r^t \leq^{\mathsf{d}} qr^s(r^{2t} \vee r^{4t}) = qr^{s+2t} \vee qr^{s+4t}$$

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- (d) does not entail k-mingle, therefore (d) is M_K admissible.

Undecidable quasi-equational theory for 1-variable d-rules

Let $\mathfrak{D}_1$ be the set of 1-variable d-rules defined via (d) $\in \mathfrak{D}_1$ iff

$$(\mathsf{d}) : x^n \leq \bigvee_{m \in X} x^m \text{ such that } n \in X \text{ or } |X \setminus \{0\}| \geq 2 \text{ for some finite } X \subseteq \mathbb{N}.$$

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Let $\Gamma \subset \mathfrak{D}_1$ be finite. Then then $\mathcal{CRL} + \Gamma$ has an undecidable quasi-equational theory.

- $\mathcal{CRL} + (x^n \leq x^m)$ has the FEP, and hence is decidable for any $n \neq m$.
- However, the decidability of $\mathcal{CRL} + (x^n \leq x^m \vee 1)$ remains open, for any $n \neq m > 0$.

The general case.

Let (d) be an n-variable d-rule. We define the set $\mathfrak{D}$ via (d) $\in \mathfrak{D}$ if there exists $K > 1$ such that:

For all $s, s' \in \mathbb{N}^n$, if there exists $\alpha, \alpha' \in \mathbb{N}$ such that $d \bullet s + \alpha$ and $d \bullet s' + \alpha'$ are powers of K for each $d \in$ d, then there exists $\bar{d} \in$ d such that $\bar{d} \bullet s = l_n \bullet s$ and $\bar{d} \bullet s' = l_n \bullet s'$,

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Theorem

For every (d) $\in \mathfrak{D}$ there exists a $K > 1$ such that (d) is admissible in M_K, for any 2-ACM M. Consequently, $(\mathcal{C})\mathcal{RL} +$ (d) has an undecidable quasi-equational theory.

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Known results for Equational Theory

$[\mathrm{k}_n^m]$ represents the knotted rule $x^n \leq x^m$

Undecidable Eq. Theory	Decidable Eq. Theory
	$\mathcal{RL}$
	$\mathcal{CRL}$
$\mathcal{RL} + [\mathrm{k}_n^m], 1 \leq n < m$	
	$\mathcal{CRL} + [\mathrm{k}_n^m]$
$\mathcal{CRL} + (?)$	

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Let $M = (R_k, Q, P)$ be a k-ACM.
We define the equation $\epsilon^n_M(u)$ in the signature $\langle \to, \vee, \cdot, 1 \rangle$ via

$$\epsilon^n_M(u) := u \cdot (1 \wedge \bigwedge_{p \in P} p^{\to})^n \leq q_f,$$

where $p^{\to} := C \to v$, where p is the instruction $C \leq v$, and $n \geq 1$.

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for some finite $X \subset \mathbb{Z}^+$. Then for all $u \in \mathrm{ID}(M)$,

$$\mathcal{V} \models \epsilon^n_M(u) \iff \mathcal{V} \models \mathrm{acc}_M(u) \iff u \in \mathrm{Acc}(M)$$

and hence $\mathcal{V}$ has an undecidable equational theory.

Revisiting the definition of $\mathfrak{D}$

- Membership of $(\mathsf{d}) \in \mathfrak{D}$ is foremost dependent upon whether there exists *very special* non-negative integral solutions to a system of equations determined by certain partitions of $\mathsf{d} = \{d_1, ..., d_m\} \subset \mathbb{N}^n$ viewed as affine subspaces $\mathbb{R}^n$.

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  which, in turn, is equivalent to the non-existence of a substitution $\sigma : \mathrm{Var} \to \mathrm{Var}^*$ such that $\sigma(\mathsf{d})$ is equivalent to a *non-redundant* **spine**, i.e.,
  $$\prod_{i=1}^n x_i^{\lambda(i)} \leq (1\vee)\, x_1^{\rho_1(1)} \vee x_1^{\rho_2(1)} x_2^{\rho_2(2)} \vee \cdots \vee \prod_{i=1}^n x_i^{\rho_n(i)}$$
  with $\lambda \neq \rho_n$.

Revisiting the definition of $\mathfrak{D}$ cont.

Fact

Suppose (d) implies some *non-redundant spine*, i.e.,

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with $\lambda \neq \rho_n$. Then for every injective function $\phi : \mathbb{N} \to \mathbb{N}$, there exists $s \in \mathbb{N}^n$ and $\alpha \in \mathbb{N}$ such that $d \bullet s + \alpha \in \phi[N]$ but $d \bullet s \neq l_n \bullet s$, for all $d \in \mathsf{d}$.

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For any $n \in \mathbb{N}$, (d) $\in \mathfrak{D}$ iff there is no substitution $\sigma : \mathrm{Var} \to \mathrm{Var}^*$ such that σ(d) is equivalent to a *non-redundant spine*.

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Open

What is the decidability of $\mathcal{CRL}$ with non-redundant spines? E.g., $x \leq 1 \vee x^2$.

Thank You!

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